Exploring small fusion rings and tensor categories

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In collaboration with Gert Vercleyen

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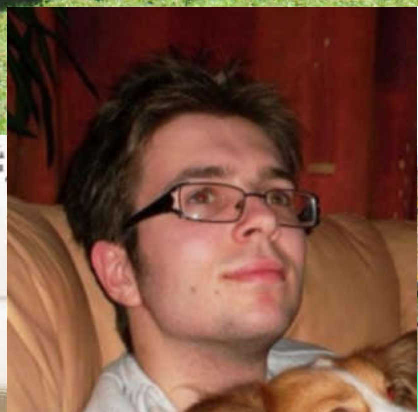
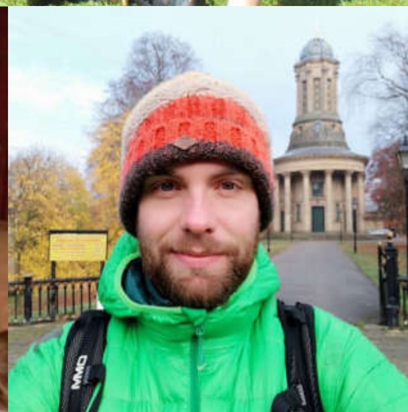
Science Foundation Ireland sfi For what's next
IRISH RESEARCH COUNCIL An Chomhairle um Thaighde in Éirinn
Maynooth University National University of Ireland Maynooth

The Group in Maynooth (and many alumni)


and Aaron


And Babatunde


and Gert!


and Alex

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IRC GOIPG Scholarships

Finding and Solving Anyon Models

Joost Slingerland, DIAS (Dublin)
Dublin, September 2007

A little bit of history...

Based on work with
Parsa Bonderson, Kirill Shtengel
Many thanks to
Zhenghan Wang, Tobias Hagge, Alexei Kitaev, ... +Frank Verstraete!

- Bonderson, JKS, Shtengel. PRL 97, 016401 (2006)
- Bonderson, JKS, Shtengel. PRL 98, 070401 (2007)
- Bonderson, JKS, Shtengel, arXiv:0707.4206 (2007)
- This talk mostly Bonderson, JKS, in preparation.

Overview

- Fusion models/Anyon models

  Abstractions of low energy effective models for systems with topological excitations.
  Describe only the lowest states in each topological sector and their fusion/braiding.
- Finding Fusion rules
- Solving the Pentagon/Hexagon equations
- Some results on classification/Hall states

Goals

- Algebraically solve specific anyon models

  (corresponding to non-Abelian Hall states, rotating Bose condensates,...)
- Make a "catalogue" of the simplest anyon models

  i.e. Tables of fusion rules, quantum dimensions, topological spins, S-matrices...
  USES:
  - look-up tables for constants in experimental predictions (think interferometry, tunneling)
  - challenges/suggestions for building local models
  - design of gates for TQC
  - (counter)examples for the theory of fusion models/anyon models
  - useful in topological symmetry breaking program (with Sander Bais)
  - ... (suggestions welcome!)

Can do better now!
Anyon Wiki +
Mathematica package
In development

- Find new anyon models (ribbon categories) www.thphys.nuim.ie/AnyonWiki

Main Page

Welcome to AnyonWiki

The idea is to collect practical information for people that work with anyons and related mathematical structures here. Hopefully this will be accessible to non-experts, such as physicists working on topologically ordered systems.

Places to go

Some pages under construction

- Fusion rules
- Fusion ring
- List of small multiplicity-free fusion rings
- F-symbols
- R-symbols
- Pentagon equation
- Hexagon equation
- Quantum dimension
- Frobenius-Schur indicator
- Modular matrices
- Monodromy matrix

The main feature of the Wiki at the moment

(Hopefully I can show this live)

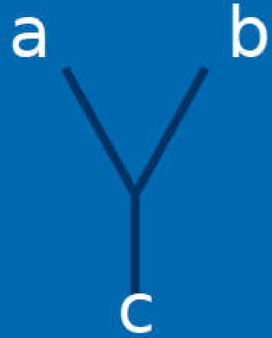
List of small multiplicity-free fusion rings

The list below contains all multiplicity free fusion rings up to 6 particles. Clicking on a name will redirect you to the page of the respective fusion ring. For more iformation on the formal names $\mathrm{FR}^{r,n}_{m}$ see the page on formal fusion ring names.

Names	Rank	Number Selfdual Particles	Number Non-zero Structure Constants	$\mathcal{D}^2_{PF}$	Commutative	Group	Categorifiable To Fusion Category	Categorifiable To Braided Fusion Category	Categorifiable To Modular Fusion Category	Categorifiable To Unitary Fusion Category
$\mathrm{FR}^{1,0}_{1}$: Trivial	1	1	1	1.	True	True	True	True	True	True
$\mathrm{FR}^{2,0}_{1}$: $\mathbb{Z}_2 \cong \mathrm{SU}(2)_1$	2	2	4	2.	True	True	True	True	True	True
$\mathrm{FR}^{2,0}_{2}$: $\mathrm{Fib} \cong \mathrm{PSU}(2)_3$	2	2	5	3.618	True	False	True	True	True	True
$\mathrm{FR}^{3,0}_{1}$: $\mathrm{Ising} \cong \mathrm{SU}(2)_2$	3	3	10	4.	True	False	True	True	True	True
$\mathrm{FR}^{3,0}_{2}$: $\mathrm{Rep}(\mathrm{D}_3) \cong \mathrm{PSU}(2)_4$	3	3	11	6.	True	False	True	True	False	True
$\mathrm{FR}^{3,0}_{3}$: $\mathrm{PSU}(2)_5$	3	3	14	9.296	True	False	True	True	True	True
$\mathrm{FR}^{3,2}_{1}$: $\mathbb{Z}_3 \cong \mathrm{SU}(3)_1$	3	1	9	3.	True	True	True	True	True	True
$\mathrm{FR}^{4,0}_{1}$: $\mathbb{Z}_2 \times \mathbb{Z}_2$	4	4	16	4.	True	True	True	True	True	True
$\mathrm{FR}^{4,0}_{2}$: $\mathrm{SU}(2)_3 \cong \mathrm{Fib} \times \mathbb{Z}_2$	4	4	20	7.236	True	False	True	True	True	True
$\mathrm{FR}^{4,0}_{3}$: $\mathrm{Rep}(\mathrm{D}_5) \cong \mathrm{SO}(5)_2/\mathbb{Z}_2$	4	4	22	10.	True	False	True	True	False	True
$\mathrm{FR}^{4,0}_{4}$: $\mathrm{PSU}(2)_6 \cong \mathrm{HI}(\mathbb{Z}_2)$	4	4	24	13.657	True	False	True	True	False	True
$\mathrm{FR}^{4,0}_{5}$: $\mathrm{Fib} \times \mathrm{Fib}$	4	4	25	13.090	True	False	True	True	True	True
$\mathrm{FR}^{4,0}_{6}$: $\mathrm{PSU}(2)_7$	4	4	30	19.234	True	False	True	True	True	True
$\mathrm{FR}^{4,2}_{1}$: $\mathbb{Z}_4 \cong \mathrm{SU}(4)_1$	4	2	16	4.	True	True	True	True	True	True
$\mathrm{FR}^{4,2}_{2}$: $\mathrm{Potts} \cong \mathrm{TY}(\mathbb{Z}_3)$	4	2	18	6.	True	False	True	False	False	True
$\mathrm{FR}^{4,2}_{3}$: $\mathrm{Fib}(\mathbb{Z}_3)$	4	2	19	8.303	True	False	False	False	False	False
$\mathrm{FR}^{4,2}_{4}$: Pseudo $\mathrm{PSU}(2)_6$	4	2	24	13.657	True	False	True	True	False	True
$\mathrm{FR}^{5,0}_{1}$: $\mathrm{Rep}(\mathrm{D}_4) \cong \mathrm{TY}(\mathbb{Z}_2 \times \mathbb{Z}_2)$	5	5	28	8.	True	False	True	True	False	True

Fusion Rules and Fusion Algebras

Fusion Rules



$$a \times b = \sum_c N_c^{ab} c$$

Here N_c^{ab} is the number of states for two particles with topological charges a,b such that the overall charge is c (We will take $N_c^{ab} < 2$.) (Not in this talk!)

There is a unit/vacuum/chargeless sector 1, with $N_b^{a1} = N_b^{1a} = \delta_{ab}$

For each charge, a there is a conjugate charge $\hat{a}$ with $N_1^{a\hat{a}} = N_1^{\hat{a}a} = 1$

Also, should have $N_c^{ab} = N_b^{\hat{a}c} = N_{\hat{a}}^{b\hat{c}} = N_{\hat{c}}^{\hat{b}\hat{a}} = N_{\hat{b}}^{a\hat{c}} = N_a^{\hat{b}c}$

and most importantly, Associativity (order of fusion doesn't matter)

$$\Sigma_e N_e^{ab} N_d^{ec} = \Sigma_f N_f^{bc} N_d^{af}$$

Now can do a brute force search for fusion rules with C types of charges

But: Search space grows like 2^{C^3} (argl! Can only do up to C=5....)

$M^{C\wedge 3}$ with multiplicity up to M-1....

Smarter, but still slow....

Searching for Fusion Algebras

The fusion algebra may have one-dimensional representations λ_j , i.e.

$$a \to \lambda_j(a) \in \mathbb{C} \qquad \lambda_j(a)\lambda_j(b) = \sum_c N_c^{ab} \lambda_j(c)$$

In 2 or more dimensions (or with self-dual particles), have $a \times b = b \times a$
so may hope for C such representations.

Now note: (λ_j) is an eigenvector of $(N^a)_c^b$ with eigenvalue $\lambda_j(a)$

Using this can reduce the search space for fusion rules
(hopefully to size of order 2^{C^2}) $M^{C\wedge 2}$ with multiplicity up to M-1....

fix one fusion matrix and find the eigenvalues and eigenvectors.
If they give a full set of 1-D reps, we can reconstruct the other fusion matrices!

This idea comes from Gepner and Kapustin 1995.
They used it to find *modular* theories up to C=6. (S-matrix diagonalises fusion)
We want a much more general class theories, including
- *Non-modular* theories (for theories with fermions, such as QHE)
- Non-braided theories
- Non-commutative theories with $a \times b \neq b \times a$ (for 1D systems, doubling)

Still Searching for Fusion Rings – less cute(?) but actually faster and more flexible.

First, break the permutation symmetry of the problem:

- Order the particles so selfdual ones come before non-selfdual ones and the nsd appear in pairs.
  (i.e. Fix the duality stucture – also good for implementing the pivotal identities ($N^{ab}{}_c = N^{\bar{a}c}{}_b$ etc.)
- Order the selfdual and non-selfdual ptcls by the sum of the entries in their fusion matrices,

So add these inequalities to the requirements on N: $\sum_{\substack{b=1\\c=1}}^{r} [N_i]_b^c \leq \sum_{\substack{b=1\\c=1}}^{r} [N_{i+1}]_b^c$

Then run a backtracking algorithm filling in the constraints

(*)Find a constraint on N with the fewest possible number of variables.
  Call the set of variables V_1.
  Find all constraints that depend only on the variables in V_1
  Call that set of equations C_1
(**) Assume all variables in V_1 known and repeat (*) on the remaining variables and constraints to obtain V_2 and C_2.
Iterate to find more V_i and C_i until all variable are exhausted.
(***) Fill in variables like this: if you get all the way down you have a Fusion Ring

```
for V[1,1] in 0:m, V[1,2] in 0:m,...,V[1,-1] in 0:m
  if( all constr in C[1] are verified )
  for V[2,1] in 0:m, V[2,2] in 0:m,...,V[2,-1] in 0:m
    if( all constr in C[2] are verified )
    ...
      for V[k,1] in 0:m, V[k,2] in 0:m,...,V[k,-1] in 0:m
        if( constraints in C[k] are verified )
        saveSol( { V[1,1], V[1,2], ..., V[k,-1] } )
```

Note:
- No need to fill in an entire N-matrix
- This will also work (better) with extra constraints

Introduction to “FusionRings”, a Mathematica Package for Fusion Rings

```
<< FusionRings`
```

```
?FusionRingList
?FRL
```

FusionRingList is a list of all saved FusionRing objects.

Shorthand for FusionRingList.

```
Fibring = FRL[[3]]
```

```
FusionRing[Fib]
```

```
SymbolicMultiplicationTable[Fibring] // TableForm
```

TableForm=

ψ[1]	ψ[2]
ψ[2]	ψ[1] + ψ[2]

```
MatrixForm /@ MultiplicationTable[Fibring]
```

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right\}$$

```
QDs[Fibring]
```

$$\left\{1, \frac{1}{2}\left(1+\sqrt{5}\right)\right\}$$

```
NNonZeroStructureConstants[Fibring]
```

```
5
```

```
Names /@ Take[FRL, 9]
```

$\{\{\text{Trivial}\}, \{\mathbb{Z}_2, SU(2)_1, TY(\mathbb{Z}_1)\}, \{\text{Fib}, PSU(2)_3, HI(\mathbb{Z}_1)\}, \{\text{Ising}, SU(2)_2, TY(\mathbb{Z}_2)\}, \{\text{Rep}(D_3), PSU(2)_4, \text{Rep}(S_3)\}, \{PSU(2)_5\}, \{\mathbb{Z}_3, SU(3)_1\}, \{\mathbb{Z}_2\times\mathbb{Z}_2\}, \{SU(2)_3, \text{Fib}\times\mathbb{Z}_2\}\}$

```
WhichDecompositions[FRL[[9]]]
```

$\{\{\text{FusionRing}[\mathbb{Z}_2], \text{FusionRing}[\text{Fib}]\}\}$

```
SubFusionRings[FRL[[9]]]
```

$\{\langle|\text{SubSet}\to\{1,2\}, \text{SubFusionRing}\to\text{FusionRing}[\mathbb{Z}_2]|\rangle, \langle|\text{SubSet}\to\{1,4\}, \text{SubFusionRing}\to\text{FusionRing}[\text{Fib}]|\rangle\}$

Fusion rings found (so far)

```
Length[FRL]
```

28 451

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p]], {p, 1, 9}]
```

{1, 17, 161, 2787, 16 711, 5502, 2160, 970, 142}

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && Mult[r] == q]], {p, 1, 9}, {q, 1, 16}] // MatrixForm
```

MatrixForm=

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	3	4	6	5	9	6	10	12	9	10	20	9	13	16	25
10	17	24	45	55	81	92	137	151	186	238	291	246	340	349	525
16	37	82	134	209	336	477	733	1463	1794	2283	3049	1300	1323	1550	1925
39	154	384	872	533	872	976	1672	0	0	0	0	0	0	0	0
43	319	562	1236	0	0	0	0	0	0	0	0	0	0	0	0
96	874	0	0	0	0	0	0	0	0	0	0	0	0	0	0
142	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 2: $\psi \times \psi = 1 + n\,\psi$

Fusion Ring numbers by rank, multiplicity and number of non-self-dual particles

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 0 && Mult[r] == q]], {p, 1, 9}, {q, 1, 16}]
```

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	4	6	5	9	6	10	12	9	10	20	9	13	16	25
6	15	22	42	53	79	89	134	149	183	236	287	244	338	346	522
10	28	67	115	192	310	448	696	1432	1751	2241	2979	1242	1275	1498	1843
20	102	286	703	313	532	611	1085	0	0	0	0	0	0	0	0
18	194	208	474	0	0	0	0	0	0	0	0	0	0	0	0
38	428	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 2 && Mult[r] == q]], {p, 3, 9}, {q, 1, 16}]
```

MatrixForm=

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	2	3	2	2	3	3	2	3	2	4	2	2	3	3
5	8	15	19	17	26	29	37	31	43	42	70	58	48	52	82
11	39	82	152	202	312	342	557	0	0	0	0	0	0	0	0
17	91	277	642	0	0	0	0	0	0	0	0	0	0	0	0
30	276	0	0	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 3: only Z_3

Rank 4: single equation in 3 variables:

$N(2,2,3)^2 + N(2,3,3)^2 = 1 + N(2,3,3) + N(2,2,3)\,N(3,3,3)$

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 4 && Mult[r] == q]], {p, 5, 9}, {q, 1, 16}]
```

MatrixForm=

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	13	16	17	18	28	23	30	0	0	0	0	0	0	0	0
7	32	76	120	0	0	0	0	0	0	0	0	0	0	0	0
18	136	0	0	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 5, mult 2:
Representation ring of smallest odd order non-Abelian group.
Dimensions: {1,1,1,3,3} (order 21)

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 6 && Mult[r] == q]], {p, 7, 9}, {q, 1, 16}]
```

MatrixForm=

1	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	34	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Rank 7, mult 2:
Both have dimensions: {1,1,1,3,3,3,3}.
But there is only one non-Abelian group of order 39.... mult 3: dims {1,1,1,1,1,5,5}

```
Table[Length[Cases[FRL, r_ /; Rank[r] == p && NNSD[r] == 8 && Mult[r] == q]], {p, 9, 9}, {q, 1, 16}]
```

MatrixForm=

2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Z_9 and $Z_3 \times Z_3$

The “Million Dollar” Question:

Which of our rings correspond to fusion categories?

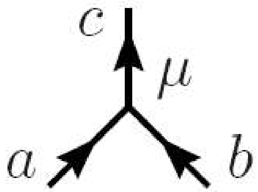
Or even complete Anyon Models (UBTCs) or TQFTs (MTCs)?

(These questions are easier to answer, in the negative at least)

Fusion Category Basics

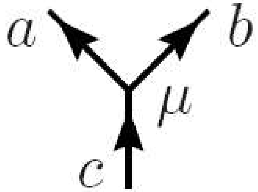
Fusion/splitting histories correspond to bra/ket states in a Hilbert space, can build up multiparticle states, inner products, operators etc.

$$(d_a d_b d_c)^{-1/4}$$



$$= \langle a, b; c, \mu | \in V^c_{ab}$$

$$(d_a d_b d_c)^{-1/4}$$



$$= | a, b; c, \mu \rangle \in V^{ab}_c$$

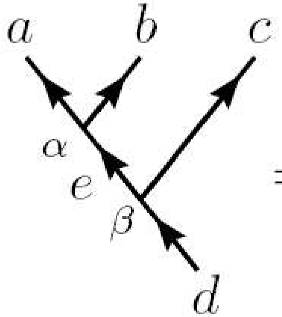
Dimensions of these spaces:

$$N^{ab}_c$$

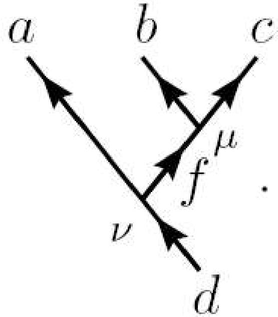
Fusion rules:

$$a \times b = \sum_c N^{ab}_c c$$

Recoupling, F-matrix / F-symbols – Associativity at the level of states.
F-symbols fully determine the category. F-moves allow for reduction of diagrams



$$= \sum_{f,\mu,\nu} \left[F^{abc}_d\right]_{(e,\alpha,\beta)(f,\mu,\nu)}$$



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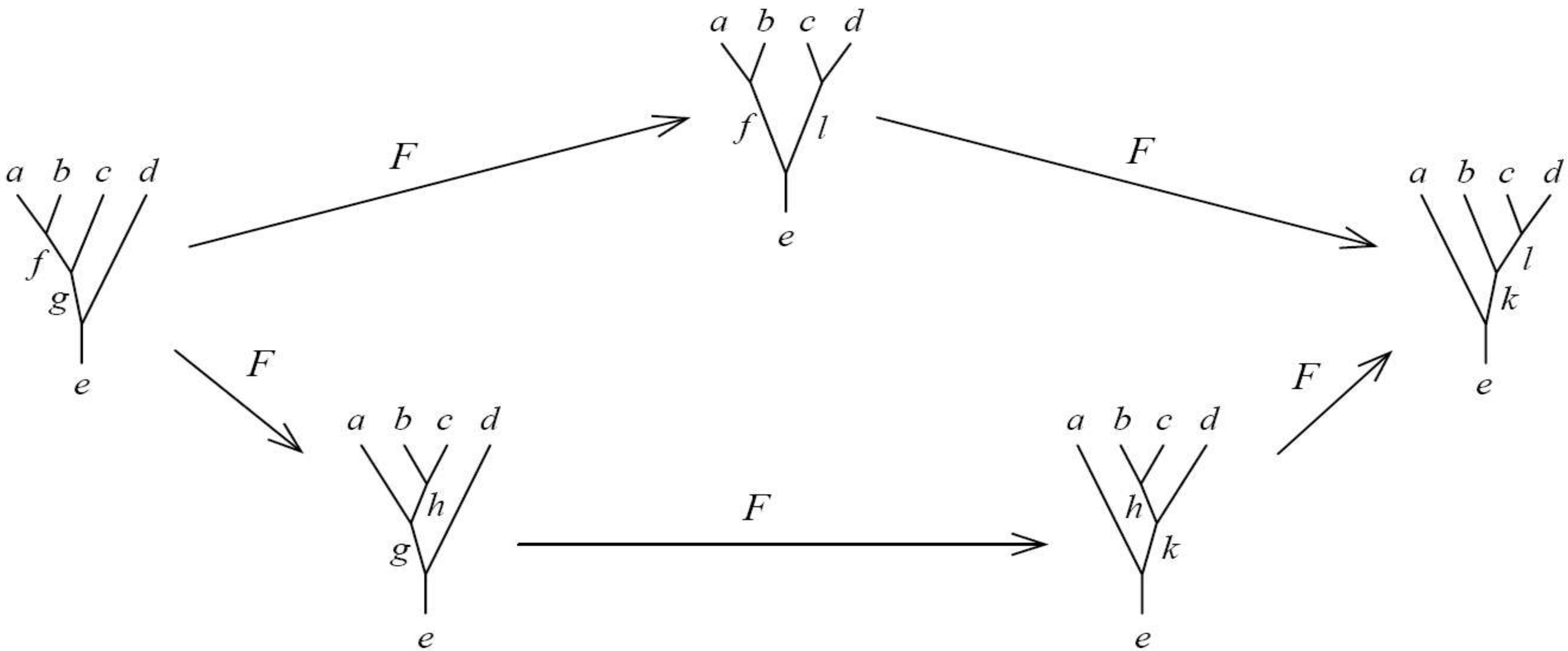
The Pentagon Equations – Consistency condition for combining local F-moves



$$\left[F_e^{fcd}\right]_{gl}\left[F_e^{abl}\right]_{fk} = \sum_h \left[F_g^{abc}\right]_{fh}\left[F_e^{ahd}\right]_{gk}\left[F_k^{bcd}\right]_{hl}$$

Can write down Pentagons knowing only the fusion rules, then try to solve for F-symbols.

We mostly assume fusion multiplicities are < 2 ("no multiplicities"), so no vertex labels.
For this case we now have a solver that is in principle algorithmic, though with terrible scaling.
(We are preparing to build a solver with multiplicities)

Properties of the pentagon and hexagon equations

- Third order polynomial equations in many variables
- Many more equations than variables: solutions do not always exist
- "Gauge" freedom (basis choices) gives parameter families of equivalent solutions

Ocneanu rigidity:
only discrete solutions can exist modulo gauge freedom

Once gauge freedom is fixed the equations can (in principle)
be solved algorithmically, using Groebner bases (uses rigidity)

- This scales very badly with the number of variables)
- However, the number of variables can be drastically reduced by using that many equations are linear in at least one variable

Gauge fixing with good properties, solution of the pentagon/hexagon etc. for Multiplicity Free theories was implemented in a Mathematica program starting 2005/6 (runs on laptop).

However, problem with zero F-symbols... (next slide)

Fixing the Gauge

Different choice of basis vectors in splitting spaces gauge transforms F and R

$$\left[F^{abc}_{d}\right]'_{ef} = \frac{u^{af}_{d}\, u^{bc}_{f}}{u^{ab}_{e} u^{ec}_{d}} \left[F^{abc}_{d}\right]_{ef} \qquad \left[R^{ab}_{c}\right]' = \frac{u^{ba}_{c}}{u^{ab}_{c}} R^{ab}_{c}.$$

Basic Idea

Can set $\left[F^{abc}_{d}\right]'_{ef} = c \neq 0$ by choice of one u, e.g. $u^{ab}_{e} = \dfrac{u^{af}_{d}\, u^{bc}_{f} \left[F^{abc}_{d}\right]}{u^{ec}_{d}\, c}$

Now eliminate u from the equations for the F', choose a different F-symbol and repeat. Iterate until all u's are eliminated.

Problem: What if $\left[F^{abc}_{d}\right]_{ef} = 0$? (...next slide!)

Note: This is -much- more complicated with multiplicities

Gauge Fixing Tambara-Yamagami theories

TY THEORY $a, b, c \ldots \in G$ (GROUP) AND EXTRA: σ

FUSION RULES
$$\begin{cases} a \times b = ab & \text{(GROUP MULT)} \\ a \times \sigma = \sigma \times a = \sigma \\ \sigma \times \sigma = \sum_{a \in G} a \end{cases}$$

F-SYMBOLS:

$[F^{abc}_{abc}]_{ab,bc} \rightarrow$ WRITE F^{abc}

SIM $F^{\sigma bc}, F^{a\sigma c}, F^{ab\sigma}, F^{\sigma\sigma c}_d, F^{\sigma b\sigma}_d, F^{a\sigma\sigma}_d$

FINALLY $[F^{\sigma\sigma\sigma}_{\sigma}]_{ef}$

GAUGING:

$$F^{abc} \longrightarrow \frac{u^{b,c}\, u^{a,bc}}{u^{a,b}\, u^{ab,c}} F^{abc}$$

$$F^{ab\sigma} \longrightarrow \frac{u^{a\sigma}\, u^{b\sigma}}{u^{a,b}\, u^{ab,\sigma}} F^{ab\sigma}$$

→ CAN CHOOSE THESE TO MAKE $F^{ab\sigma} = 1$

$$F^{\sigma b\sigma}_1 \longrightarrow \frac{u^{b\sigma}\, u^{\sigma\sigma}_e}{u^{\sigma b}\, u^{\sigma\sigma}_e}$$

→ CAN CHOOSE TO MAKE $F^{\sigma b\sigma}_1 = 1$

$F^{\sigma\sigma c}_c = 1$ (USING $u^{\sigma\sigma}_c$)

$F^{a\sigma\sigma}_a = 1$ (using $u^{a\sigma}$) ALL FIXED!

TY Solution in that gauge

$$
\mathrm{TY}\begin{cases}
F^{h_1 h_2 h_3} = 1 \\
F^{\sigma\sigma h_1}_{h_2} = F^{h_1 \sigma\sigma}_{h_2} = F^{\sigma h_1 h_2}_{\sigma} = F^{h_1 h_2 \sigma}_{\sigma} = 1 \\
F^{h_1 \sigma h_2} = F^{\sigma h_1 \sigma}_{h_2} = \chi(h_1, h_2) \rightarrow \text{SYMM BICHAR NONDEGEN.} \\
\left[F^{\sigma\sigma\sigma}_{\sigma}\right]_{h_1 h_2} = \pm \frac{1}{\sqrt{|H|}} \bar{\chi}(h_1, h_2)
\end{cases}
$$

Note: G must be Abelian

Finding zero F-symbols

Old trick: Can actually find absolute values of the F-symbols in many small examples, using unitarity and sum-free pentagon equations (sum over h has one term)

$$\sum_e \left| \left[F_d^{abc}\right]_{ef} \right|^2 = \sum_f \left| \left[F_d^{abc}\right]_{ef} \right|^2 = 1$$

$$\left| \left[F_c^{a\bar{a}c}\right]_{1f} \right|^2 = \frac{d_f}{d_a d_c} \quad \text{and} \quad \left| \left[F_a^{ab\bar{b}}\right]_{e1} \right|^2 = \frac{d_e}{d_a d_b}.$$

$$\left| \left[F_e^{fcd}\right]_{gl} \right|^2 \left| \left[F_e^{abl}\right]_{fk} \right|^2 = \left| \left[F_g^{abc}\right]_{fh} \right|^2 \left| \left[F_e^{ahd}\right]_{gk} \right|^2 \left| \left[F_k^{bcd}\right]_{hl} \right|^2 .$$

Now can set F'=|F| during gauge fixing procedure (i.e. c=|F|).

Then the resulting gauge has good properties

- If F-matrices can be unitary, they will be
- If F-matrices can be real, they will be

Unfortunately, this trick does not always work (can't always fix |F|s).
Now have new code that finds a finite number of sets of F-symbols which might be zero (can say more).
Then we try solving with each possible set equal to zero.
The new code is algorithmic (but can be slow) and no longer uses unitarity.

Excluding existence of Modular/Braided/just tensor categories

Without directly attacking the pentagon etc.

Modular Tensor Categories have a modular S matrix which
- simultaneously diagonalizes the fusion matrices
- is symmetric and unitary
- satisfies $S^2=C$ (charge conjugation, so $S^4=1$)

Also, there is a diagonal T-matrix with $(ST)^3=S^2$ (we ignore this here)

It's easy to check if such a matrix exists for any set of fusion rules.
- Find the fusion characters
  (diagonalize a random linear combination of the fusion matrices)
- Reorder and renormalize them in all possible orders
- See if one of the orders satisfies the requirements on S above

Non-Modular, but still Braided Tensor Cats
- must have a symmetric subcategory ("transparent objects", Mueger!)
- such symmetric cats are representation cats of finite groups (Dopplicher-Roberts)

We can easily identify absence of subrings which are representation rings of groups
(Then not braided, if not modular)

Obstructions to just tensor cat-structure also in recent work of Liu/Palcoux/Wu

Numbers of Rings with S-matrices, by multiplicity, rank and number of non-self-duals

1	2	2 1	4 1	2 1 1	6 1 3	3 2 0 1	9 2 2 1	6 2 2 2 2	Multiplicity free
0	1	1 0	4 0	2 1 0	9 0 1	5 1 2 0	13 0 2 0	0 0 0 0 0	M=2
0	1	0 0	2 0	1 0 0	2 0 1	0 0 0 0	0 0 0 0	0 0 0 0 0	M=3
0	1	0 0	3 0	0 1 0	3 0 1	0 0 0 0	0 0 0 0	0 0 0 0 0	
0	1	0 0	2 0	0 0 0	0 1 1	0 0 0 0	0 0 0 0	0 0 0 0 0	
0	1	0 0	3 0	0 0 0	1 1 1	0 0 0 0	0 0 0 0	0 0 0 0 0	

There are cases where S exists, but there is no MTC

- Some trivial (rank 2, most of the others)

- Some less trivial (next slide.....)

"Pseudo-Modular" Tensor Categories?

A section from my 2006/7 notes:

B.24 SU(2)$_4$ like fusion rules without hexagon solutions

$$\begin{array}{|ccccc|}
\psi_0 & \psi_1 & \psi_2 & \psi_3 & \psi_4 \\
\psi_1 & \psi_0 & \psi_2 & \psi_4 & \psi_3 \\
\psi_2 & \psi_2 & \psi_0+\psi_1+\psi_2 & \psi_3+\psi_4 & \psi_3+\psi_4 \\
\psi_3 & \psi_4 & \psi_3+\psi_4 & \psi_1+\psi_2 & \psi_0+\psi_2 \\
\psi_4 & \psi_3 & \psi_3+\psi_4 & \psi_0+\psi_2 & \psi_1+\psi_2
\end{array} \qquad (70)$$

These fusion rules are similar to those for $SU(2)_4$, but the particles ψ_3 and ψ_4 (with quantum dimension $\sqrt{3}$) are not self-dual, but dual to each other. There are 2 solutions to the pentagon, forming 1 mirror pair, tabulated below. Both solutions are unitary and both are invariant under the automorphism of the fusion rules that exchanges ψ_3 with ψ_4. There are no solutions to the hexagon equations.

unitary			$\mathcal{D} = 2\sqrt{3}$		
	ψ_0	ψ_1	ψ_2	ψ_3	ψ_4
d	1	1	2	$\sqrt{3}$	$\sqrt{3}$
κ	1	1	1	0	0

(71)

```
Chop[2 Sqrt[3] modulist512[[3, 2, 1]]] // MatrixForm
```

latrixForm=

"S-matrix":

$$\begin{pmatrix}
1. & 1. & 2. & 1.73205 & 1.73205 \\
1. & 1. & 2. & -1.73205 & -1.73205 \\
2. & 2. & -2. & 0 & 0 \\
1.73205 & -1.73205 & 0 & 0.-1.73205\,i & 0.+1.73205\,i \\
1.73205 & -1.73205 & 0 & 0.+1.73205\,i & 0.-1.73205\,i
\end{pmatrix}$$

118 non-Abelian Fusion Rings (so far)

```
Table[Length[Select[nonabrings, (Rank[#] == p && Mult[#] == m) &]], {m, 1, 8}, {p, 6, 9}]
```

latrixForm=

2	3	19	26
2	5	40	0
1	6	0	0
1	6	0	0
1	0	0	0
2	0	0	0
2	0	0	0
2	0	0	0

By mult/rank

2 0	3 0 0	13 3 3	11 12 3 0	M=1
2 0	5 0 0	30 5 5	0 0 0 0	M=2
1 0	6 0 0	0 0 0	0 0 0 0	M=3
1 0	6 0 0	0 0 0	0 0 0 0	
1 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	
2 0	0 0 0	0 0 0	0 0 0 0	

By mult/rank split by number of non-self-dual particles (2,4,6,8)

Rank 6 non-Abelian Fusion Rings

Multiplicity free:

- We have the smallest non-Abelian group D_3 (or S_3)
- The Haagerup-Izumi theory HI(Z_3), fusion rules:

$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$	$\psi[6]$
$\psi[2]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[4]$	$\psi[3]$
$\psi[3]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]$	$\psi[4]$
$\psi[4]$	$\psi[2]+\psi[3]+\psi[4]+\psi[6]$	$\psi[2]+\psi[3]+\psi[4]+\psi[5]$	$\psi[1]+\psi[2]+\psi[3]+\psi[4]$	$\psi[3]$	$\psi[2]$
$\psi[5]$	$\psi[3]$	$\psi[4]$	$\psi[2]$	$\psi[6]$	$\psi[1]$
$\psi[6]$	$\psi[4]$	$\psi[2]$	$\psi[3]$	$\psi[1]$	$\psi[5]$

With multiplicity:

- simple generalizations of Haagerup, with n times (psi[3]+psi[4]+psi[5])

$\psi[1]$	$\psi[2]$	$\psi[3]$	$\psi[4]$	$\psi[5]$	$\psi[6]$
$\psi[2]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[4]$	$\psi[3]$
$\psi[3]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$\psi[2]$	$\psi[4]$
$\psi[4]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[6]$	$5\psi[2]+5\psi[3]+5\psi[4]+\psi[5]$	$\psi[1]+5\psi[2]+5\psi[3]+5\psi[4]$	$\psi[3]$	$\psi[2]$
$\psi[5]$	$\psi[3]$	$\psi[4]$	$\psi[2]$	$\psi[6]$	$\psi[1]$
$\psi[6]$	$\psi[4]$	$\psi[2]$	$\psi[3]$	$\psi[1]$	$\psi[5]$

Four others without subgroups, one of them particularly interesting at multiplicity two... (next slide)

Note: All other non-Abelian rings we have found (at rank>6) have a group as a subring

The rank 6 Fusion Hecke algebra at multiplicity 2

ψ[1]	ψ[2]	ψ[3]	ψ[4]	ψ[5]	ψ[6]
ψ[2]	ψ[1] + ψ[2]	ψ[6]	ψ[4] + ψ[5]	ψ[4]	ψ[3] + ψ[6]
ψ[3]	ψ[5]	ψ[1] + ψ[3]	ψ[4] + ψ[6]	ψ[2] + ψ[5]	ψ[4]
ψ[4]	ψ[4] + ψ[6]	ψ[4] + ψ[5]	ψ[1] + ψ[2] + ψ[3] + 2 ψ[4] + ψ[5] + ψ[6]	ψ[3] + ψ[4] + ψ[5] + ψ[6]	ψ[2] + ψ[4] + ψ[5] + ψ[6]
ψ[5]	ψ[3] + ψ[5]	ψ[4]	ψ[2] + ψ[4] + ψ[5] + ψ[6]	ψ[4] + ψ[6]	ψ[1] + ψ[3] + ψ[4]
ψ[6]	ψ[4]	ψ[2] + ψ[6]	ψ[3] + ψ[4] + ψ[5] + ψ[6]	ψ[1] + ψ[2] + ψ[4]	ψ[4] + ψ[5]

The elements $\mathbf{2}$ and $\mathbf{3}$ generate Fibonacci subrings. Elements $\mathbf{5} = \mathbf{3} \times \mathbf{2}$ and $\mathbf{6} = \mathbf{2} \times \mathbf{3}$ can be regarded as couples of Fibonacci particles and $\mathbf{4} = \mathbf{2} \times \mathbf{3} \times \mathbf{2} = \mathbf{3} \times \mathbf{2} \times \mathbf{3}$ as a tripple of Fibonacci particles. Indeed, the above fusion ring is completely determined by the generators $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$ together with the relations $\mathbf{2}^2 = \mathbf{1} + \mathbf{2}, \mathbf{3}^2 = \mathbf{1} + \mathbf{3}$, and $\mathbf{2} \times \mathbf{3} \times \mathbf{2} = \mathbf{3} \times \mathbf{2} \times \mathbf{3}$. The structure of the ring thus corresponds to that of a Hecke algebra with generators $\mathbf{2}$ and $\mathbf{3}$.

Q: Does anyone already know if there is a corresponding fusion category?

The rank 7 non-Abelian Fusion Rings

Quantum dimensions of all 20 rings below

```
FullSimplify[ToRadicals[QuantumDimensions /@ nonabsevens]] // MatrixForm
```

MatrixForm=

1	1	1	1	$\sqrt{6}$	1	1
1	1	1	1	3	1	1
1	Root$[1 - \#1 - 4\#1^2 + \#1^3 \&, 3]$	Root$[1 - \#1 - 4\#1^2 + \#1^3 \&, 3]$	Root$[1 - \#1 - 4\#1^2 + \#1^3 \&, 3]$	Root$[9 - 6\#1 - 3\#1^2 + \#1^3 \&, 3]$	1	1
1	1	1	1	$1 + \sqrt{7}$	1	1
1	2	2	2	3	1	1
1	3	$3 + \sqrt{13}$	$3 + \sqrt{13}$	$3 + \sqrt{13}$	1	1
1	$2 + \sqrt{6}$	$2 + \sqrt{6}$	$2 + \sqrt{6}$	$3 + \sqrt{6}$	1	1
1	3	4	4	4	1	1
1	$4 + \sqrt{19}$	$4 + \sqrt{19}$	$4 + \sqrt{19}$	$5 + \sqrt{19}$	1	1
1	1	1	1	$\frac{1}{2}\left(3 + \sqrt{33}\right)$	1	1
1	$3 + \sqrt{15}$	$4 + \sqrt{15}$	$4 + \sqrt{15}$	$4 + \sqrt{15}$	1	1
1	3	$\frac{1}{2}\left(9 + \sqrt{97}\right)$	$\frac{1}{2}\left(9 + \sqrt{97}\right)$	$\frac{1}{2}\left(9 + \sqrt{97}\right)$	1	1
1	Root$[9 + 3\#1 - 6\#1^2 + \#1^3 \&, 3]$	Root$[1 - 4\#1 - \#1^2 + \#1^3 \&, 3]$	Root$[1 - 4\#1 - \#1^2 + \#1^3 \&, 3]$	Root$[1 - 4\#1 - \#1^2 + \#1^3 \&, 3]$	1	1
1	Root$[3 + 5\#1 - 6\#1^2 + \#1^3 \&, 3]$	Root$[3 + 5\#1 - 6\#1^2 + \#1^3 \&, 3]$	Root$[3 + 5\#1 - 6\#1^2 + \#1^3 \&, 3]$	Root$[9 - 24\#1 - 5\#1^2 + \#1^3 \&, 3]$	1	1
1	$2\left(3 + \sqrt{10}\right)$	$2\left(3 + \sqrt{10}\right)$	$2\left(3 + \sqrt{10}\right)$	$7 + 2\sqrt{10}$	1	1
1	$5 + \sqrt{34}$	$6 + \sqrt{34}$	$6 + \sqrt{34}$	$6 + \sqrt{34}$	1	1
1	Root$[2 - 4\#1 - 14\#1^2 + \#1^3 \&, 3]$	Root$[2 - 4\#1 - 14\#1^2 + \#1^3 \&, 3]$	Root$[2 - 4\#1 - 14\#1^2 + \#1^3 \&, 3]$	Root$[27 + 3\#1 - 11\#1^2 + \#1^3 \&, 3]$	1	1
1	1	1	1	$2 + \sqrt{10}$	1	1
1	Root$[1 - \#1 - 13\#1^2 + \#1^3 \&, 3]$	Root$[1 - \#1 - 13\#1^2 + \#1^3 \&, 3]$	Root$[1 - \#1 - 13\#1^2 + \#1^3 \&, 3]$	Root$[18 - 6\#1 - 6\#1^2 + \#1^3 \&, 3]$	1	1
1	3	$2\left(3 + \sqrt{10}\right)$	$2\left(3 + \sqrt{10}\right)$	$2\left(3 + \sqrt{10}\right)$	1	1

TY(D_3) not cat

Fib(D_3)

...

All have Z_3

A nice class of Fusion Rings (many non-Abelian, at least some cats)

Let G be a finite group, H a normal subgroup of G.

We take simple objects labeled by the elements g of G

With g1 x g2 = g1 . g2

These act on a single orbit of further objects, labeled by elements of G/H
There are |G|/|H| of these, notated as gH
(g is some element of the coset gH).

We have a further element g0H of G/H and an automorphism
A: G/H → G/H,satisfying

A(g0H)=g0H

$A^{-1}(gH) = (g0H)^{-1}$ A(gH) g0H

We then define:

g1 x g2 = g1 . g2 (as before)

g1 x g2H= (g1.g2)H

g1H x g2 = g1H . A(g2 H)

g1H x g2H = L (g1H . A(g2H) . g0H) sum_{h in H} h

Where L(gH) is an arbitrary element of the coset gH inside G.
(the choice of lift into G doesn't matter)

This captures many of the
Non-Abelian rank 8 theories
(with G=D_3, H=Z_3)

Computer Solutions of a non-Abelian fusion category based on D3, with two non-group objects

```
656 possibilities for the outer labels on the F-symbol (=number of 'F-matrices')

648 F-symbols with only one fusion channel

455  F-symbols with only one fusion channel and no vac on the three upper lines

Number of F-symbols not fixed after the gauging step:  477

fs[psi[7], psi[6], psi[7], psi[6], psi[5], psi[5]] → a[1]
fs[psi[7], psi[7], psi[6], psi[6], psi[2], psi[3]] → a[2]
fs[psi[7], psi[7], psi[7], psi[7], psi[2], psi[1]] → a[3]
```

$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to -\frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(3i-\sqrt{3}\right)$
$a[3] \to \frac{1}{6}\left(3i+\sqrt{3}\right)$	$a[2] \to \frac{1}{\sqrt{3}}$	$a[1] \to \frac{1}{6}\left(-3i+\sqrt{3}\right)$

From prev slide:

G=D3, H=Z3, A=Id, g0H = H

Easily solved by computer

16 unitary solutions up to gauge

(some probably equivalent under
Fusion automorphism)

Expect to have Full Analytic solution soon :)
(Similar to solving Tambara-Yamagami)

Outlook/Wishlist

- Publish current data, make software version 0.1 available.
    Practical Issue: Make the Wiki work properly (pictures, file upload etc.)

- Include category information (F-symbols, R-symbols) in the database and tools

- Generate category information where not known using pentagon/hexagon solver
    Generalize the solver to deal with multiplicities!

- Find "Natural" constructions of the various non-Abelian Fusion Rings
    e.g. as boundary theories

- Implement "standard" constructions in Mathematica package,
    e.g. Drinfeld Center/Double, Anyon Condensation...

- Implement diagram reduction, ... quantities for physics modeling

- Look at various other classes of objects
    higher cats?

- Referencing (ouch!),
- Interface with other software
    (GAP, Kac), databases (nLab, VOA catalogue)...

- **Get community involved :) → Thanks for listening!**

Multiplicity Free Fusion Rings
up to rank 6

(Known) MTCs above the lines

Rank	self-dual		2 non-selfdual		4 non-self-dual	
1	trivial					
2	$\mathbb{Z}_2 \cong \mathbf{SU(2)_1}$					
	$\mathrm{Fib} \cong \mathbf{SO(3)_3}$					
3	$\mathrm{Ising} \cong \mathbf{SU(2)_2}$		$\mathbb{Z}_3 \cong \mathbf{SU(3)_1}$			
	$\mathbf{SO(3)_5}$					
	$\mathbf{D_3} \cong \mathbf{SO(3)_4}$					
4	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$\mathbb{Z}_4 \cong \mathbf{SU(4)_1}$			
	$\mathbf{SU(2)_3} \cong \mathbf{Fib} \times \mathbb{Z}_2$					
	$\mathrm{Fib} \times \mathrm{Fib}$					
	$\mathbf{SO(3)_7}$					
	$\mathbf{D_5} \cong \mathbf{SO(5)_2/\mathbb{Z}_2}$		$\mathbf{Ising}(\mathbb{Z}_3)$			
	$\mathbf{SO(3)_6}$		$\mathbf{Fib}(\mathbb{Z}_3)$			
			$\mathbf{pseudo\ SO(3)_6}$			
5	$\mathbf{SU(2)_4}$				$\mathbb{Z}_5 \cong \mathbf{SU(5)_1}$	
	$\mathbf{SO(3)_9}$					
	$\mathbf{D_4} \cong \mathbf{Ising}(\mathbb{Z}_2 \times \mathbb{Z}_2)$		$\mathbf{Ising}(\mathbb{Z}_4)$			
	$\mathbf{Fib}(\mathbb{Z}_2 \times \mathbb{Z}_2)$		$\mathbf{Fib}(\mathbb{Z}_4)$			
	$\mathbf{D_7} \cong \mathbf{SO(7)_2/\mathbb{Z}_2}$		$\mathbf{pseudo\ SU(2)_4}$			
	$\mathbf{5^0_4}$		$\mathbf{pseudo\ S_4}$			
	$\mathbf{S_4}$		$\mathbf{pseudo\ 5^0_7}$			
	$\mathbf{SO(3)_8}$					
	$\mathbf{5^0_7}$					
	$\mathbf{5^0_8}$					
6	$\mathbf{SU(2)_2} \times \mathbb{Z}_2$				$\mathbb{Z}_6 \cong \mathbf{SU(6)_1}$	
	$\mathbf{SU(2)_2 \times Fib}$				$\mathbf{SU(3)_2} \cong \mathbf{Fib} \times \mathbb{Z}_3$	
	$\mathbf{SU(2)_5} \cong \mathbf{SO(3)_5} \times \mathbb{Z}_2$					
	$\mathbf{SO(3)_5 \times Fib}$					
	$\mathbf{SO(5)_2}$					
	$\mathbf{SO(3)_{11}}$					
	$\mathbf{D_6} \cong \mathbf{SO(3)_4} \times \mathbb{Z}_2$		$\mathbf{F(D_3)}$		$\mathbf{Ising}(\mathbb{Z}_5)$	
	$\mathbf{6^0_2}$		$(\mathbf{SU(2)_2} \times \mathbb{Z}_4)\vert_{\mathbf{even}}$		$\mathbf{Fib}(\mathbb{Z}_5)$	
	$\mathbf{SO(3)_4 \times Fib}$		$\mathbf{6^2_3}$		$\mathbf{MR_6}$	
	$\mathbf{D_9}$		$\mathbf{T_{12}}$		$\mathbf{6^4_4}$	
	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$		$\mathbf{6^2_5}$		$\mathbf{6^4_5}$	
	$\mathbf{6^0_6}$		$\mathbf{6^2_6}$		$\mathbf{6^4_6}$	
	$\mathbf{6^0_7}$		$\mathbf{Haagerup}\vert_{\mathbf{even}}$			
	$\mathbf{6^0_8}$		$\mathbf{pseudo\ SO(5)_2}$			
	$\mathbf{6^0_9}$		$\mathbf{6^2_9}$			
	$\mathbf{6^0_{10}}$		$\mathbf{6^2_{10}}$			
	$\mathbf{SO(3)_{10}}$		$\mathbf{6^2_{11}}$			
	$\mathbf{6^0_{12}}$					
	$\mathbf{6^0_{13}}$					
	$\mathbf{6^0_{14}}$					