Violation of bulk-edge correspondence in a hydrodynamic model

Gian Michele Graf
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Mathematical Picture Language Seminar
Harvard University
Tuesday, December 15, 2020

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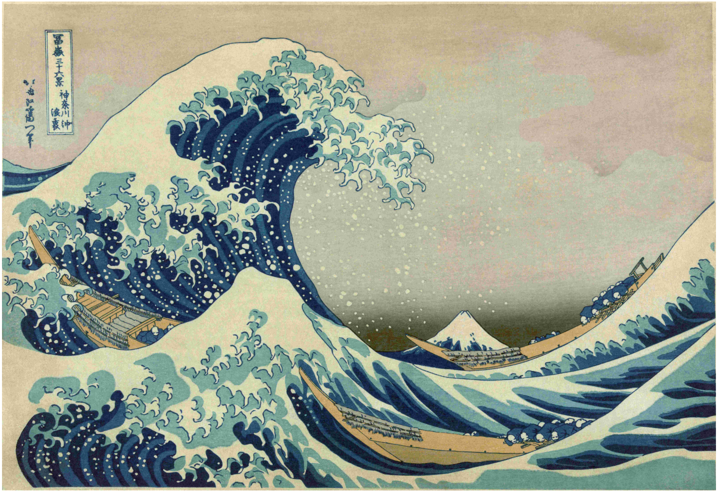
based on joint work with Hansueli Jud, Clément Tauber (to appear in CMP)
inspired by and at variance with P. Delplace et al. (Science 2017)

Outline

The Great Wave off Kanagawa



(by K. Hokusai, ∼1831)

Bulk-edge correspondence as a principle

- A topological insulator (infinitely extended, a.k.a. bulk) has an excitation gap above the ground state

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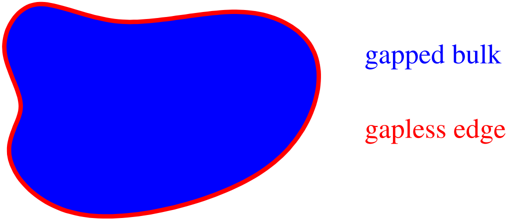

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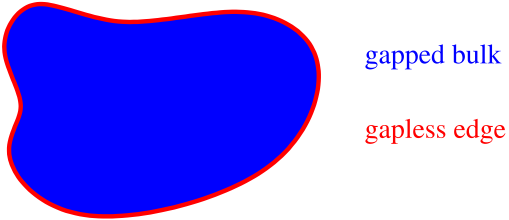

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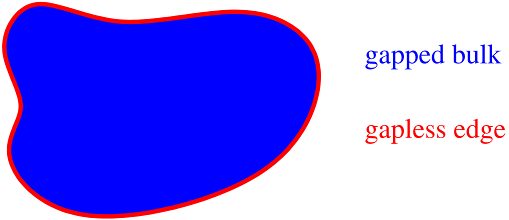
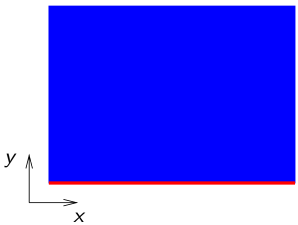


- The topological properties of bulk and edge match
- This talk: A counterexample: A topological bulk model not matched by its edge

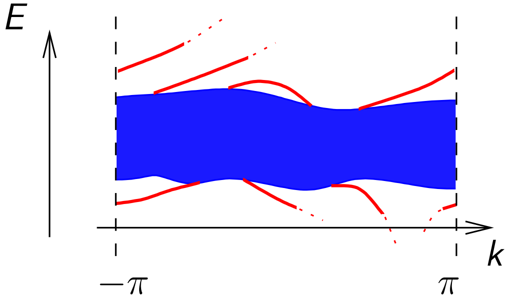
Bulk-edge correspondence as a precise statement

Half-plane geometry



(discrete periodic in x and ~~y~~)

Single-particle spectrum



- E: energy
- k: longitudinal momentum (good quantum number) on circle
- Bulk spectrum and edge spectrum resolved in k

Bulk-edge correspondence as a precise statement

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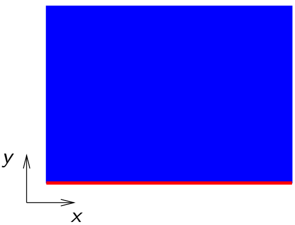

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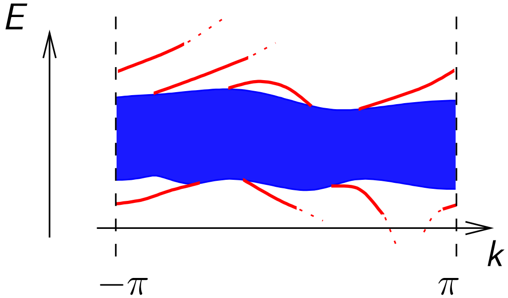

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Bulk-edge correspondence as a precise statement

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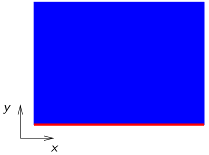

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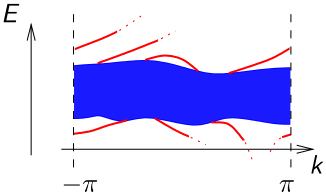

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- $(\mathcal{N}_\pm = 0$ if no further band above/below)

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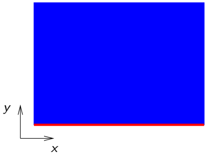

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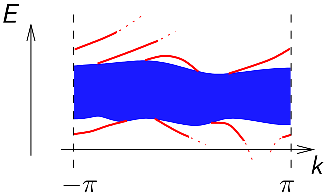

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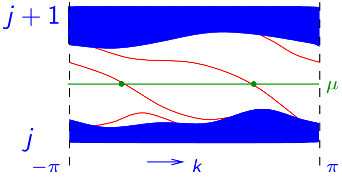
Theorem (Hatsugai)

$$\mathrm{ch}(P) = \mathcal{N}_+ - \mathcal{N}_-$$

where $\mathrm{ch}(P)$ is the Chern number of the bulk energy band (i.e. of the bundle P having eigenspaces as fibers)

Bulk-edge correspondence: A restatement



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$$\mathcal{N}_{j+1,-} = \mathcal{N}_{j.+} = \mathcal{N}^\sharp$$

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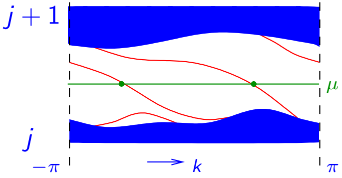

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and (bulk-edge correspondence)

$$\mathcal{N}^\sharp = \sum_{j' \le j} \mathrm{ch}(P_{j'}) \qquad (=: \text{bulk index})$$

by $\mathrm{ch}(P_{j'}) = \mathcal{N}_{j',+} - \mathcal{N}_{j',-}$ (telescoping sum).

The model (take it or leave it)

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Incompressible, shallow water equations:

$$\frac{\partial \eta}{\partial t} = -h \underline{\nabla} \cdot \underline{v}$$

$$\frac{\partial \underline{v}}{\partial t} = -g \underline{\nabla} \eta - f \underline{v}^{\perp} - \nu \Delta \underline{v}^{\perp}$$

- fields (dynamical):
  • velocity $\underline{v} = \underline{v}(x, y)$ • height above average $\eta = \eta(x, y)$
- parameters: gravity g, average depth h, angular velocity $f/2$, odd viscosity ν
- notation: $\underline{v}^{\perp}$ is $\underline{v}$ rotated by $\pi/2$

A quick derivation

Starting point: mass conservation/momentum balance, shallow water approximation

- Typical wavelength $\gg$ average depth h
- Horizontal velocity components $\underline{v} = \underline{v}(x, y)$; height $z = z(x, y)$
- Volume density ρ (incompressible); area density ρz

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- Euler equations

$$\frac{D}{Dt}(\rho z) = -\rho z \operatorname{div} \underline{v}\,, \qquad \rho z \frac{D\underline{v}}{Dt} = \underline{F} + \underline{\nabla} \cdot \underline{\underline{\sigma}}$$

  with body forces $\underline{F} = \underline{F}_{\mathrm{G}} + \underline{F}_{\mathrm{C}}$ (gravity & Coriolis)

$$\underline{F}_{\mathrm{G}} = -\underline{\nabla} U\,, \qquad U = \frac{1}{2}\rho g z^2\,, \qquad \underline{F}_{\mathrm{C}} = \rho z f \underline{v}^{\perp}$$

  and stress tensor $\underline{\underline{\sigma}} = 2\mu z D(\underline{v}^{\perp})$ (odd viscosity)
- Notation (shear velocity) $D(\underline{u})$: traceless symmetrized gradient of $\underline{u}$
- **Remark.** Usual shear viscosity $\underline{\underline{\sigma}} = 2\mu D(\underline{v})$ results in $\underline{\nabla} \cdot \underline{\underline{\sigma}} = -\mu \Delta \underline{v}$

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- Set $z = h + \eta$ and linearize in η, $\underline{v}$.
- Result:

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with $\nu = \mu h/\rho$; after rescaling $g, h = 1$

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- Purpose of ν: Ultraviolet completion

The model (restated)

Equations of motion

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In Hamiltonian form ($\underline{v} =: (u, v)$, $p_x := -\mathrm{i}\partial/\partial x$)

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi$$

$$\psi = \begin{pmatrix} \eta \\ u \\ v \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & p_x & p_y \\ p_x & 0 & \mathrm{i}(f - \nu \underline{p}^2) \\ p_y & -\mathrm{i}(f - \nu \underline{p}^2) & 0 \end{pmatrix} = H^*$$

The model as a spin 1 bundle

By translation invariance (momentum $\underline{k} \in \mathbb{R}^2$), H reduces to fibers

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where $\vec{S}$ is an irreducible spin 1 representation

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \, , \qquad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \, , \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}$$

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Eigenvalues

$$\omega_0(\underline{k}) = 0\,, \qquad \omega_\pm(\underline{k}) = \pm|\vec{d}(\underline{k})| = \pm(\underline{k}^2 + (f - \nu \underline{k}^2)^2)^{1/2}$$

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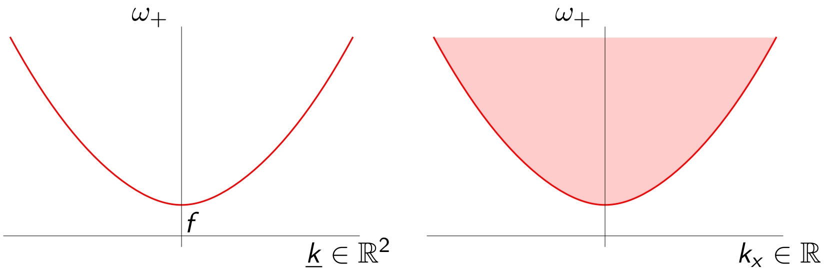

Left: ω_+ as a function of $\underline{k}$
Right: projected along k_y as a function of k_x

Remark: Gap above $\omega_0(\underline{k}) = 0$ is $f > 0$

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Eigenvectors (only ω_+):
Same as for $\vec{e} \cdot \vec{S}$ with $\vec{e} = \vec{d}/|\vec{d}|$, denoted

$$|\vec{e}\rangle = |\vec{e}, j = 1, m = 1\rangle\ , \qquad \underline{k} \mapsto \vec{e}(\underline{k})$$

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Remarks.

- The compactification of $\mathbb{R}^2$ is S^2.
- $\vec{e}(\underline{k}) \mapsto (0, 0, -\operatorname{sgn} \nu)$ as $\underline{k} \to \infty$ by $\vec{d}(\underline{k}) = (k_x, k_y, f - \nu \underline{k}^2)$
- $\vec{e} : \mathbb{R}^2 \to S^2$ extends to a continuous map $S^2 \to S^2$ wrapping the sphere once

Lemma. Let $f\nu > 0$. The line bundle $P_+^{(1)} = |\vec{e}\rangle\langle\vec{e}|$ defined by $\vec{e}(\underline{k})$ on S^2 has Chern number

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Proof. If $\vec{S}$ were a spin-$\frac{1}{2}$ representation, then

$$\operatorname{ch}(P_+^{(1/2)}) = \deg(\vec{e}) = +1$$

Now $P_+^{(1)} = P_+^{(1/2)} \otimes P_+^{(1/2)}$, so $\operatorname{ch}(P_+^{(1)}) = 1 + 1$ $\square$

Topological phenomena at interfaces

$f > 0$ (< 0) on northern (southern) hemisphere

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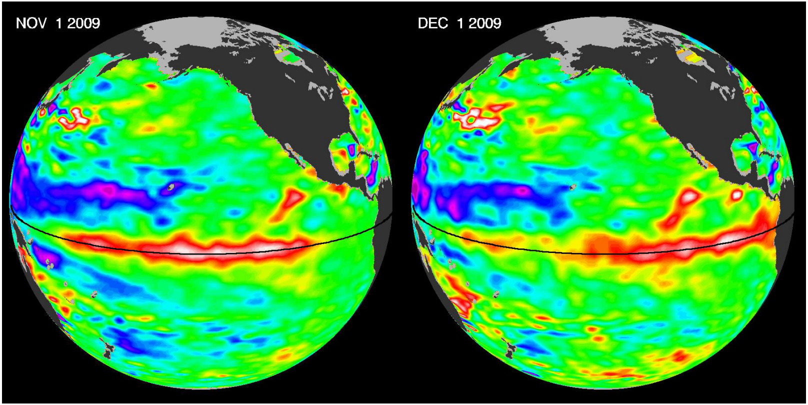
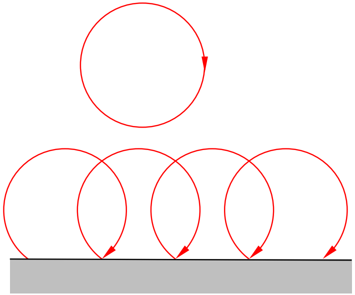


(Source: NASA)

The role of the coast



The figure illustrates the clockwise motion of both a particle in a magnetic field and of a wave in presence of a Coriolis force.

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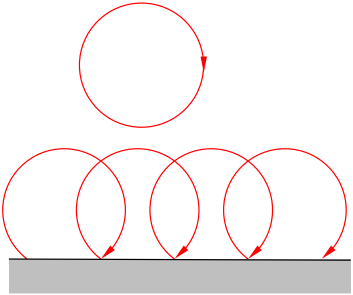

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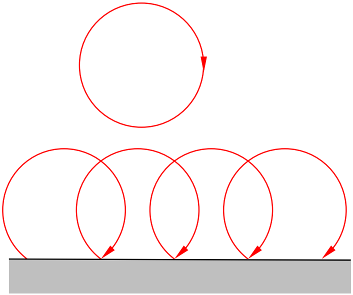

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Halperin's work led to the far reaching bulk-edge correspondence.

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- The edge index is associated with those states

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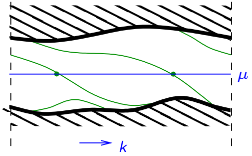

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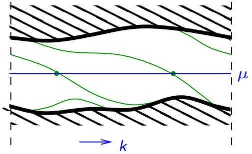

- The edge index is the (signed) number of edge states traversing the gap

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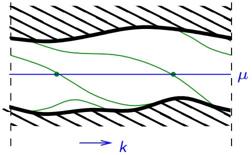

- The edge index is the (signed) number of edge states traversing the gap
- Bulk-edge correspondence: The edge index is the same as the bulk index of all bands below (or above) the gap, regardless of boundary conditions

Sighting land: The edge model

Sea restricted to upper half-space $y > 0$.
Boundary condition on field (η, u, v) at $y = 0$ (parametrized by $a \in \mathbb{R}$):

$$v = 0\,, \qquad \partial_x u + a\partial_y v = 0$$

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- A pair of boundary conditions defines self-adjoint operator H_a
- Other self-adjoint boundary condition exist (not considered here)

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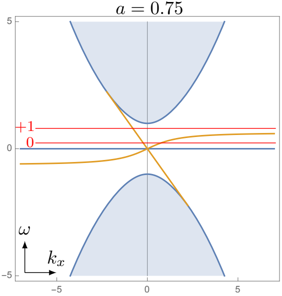

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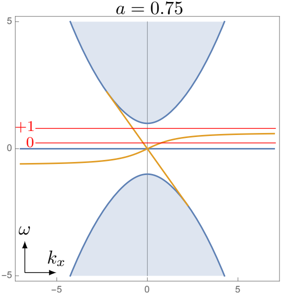

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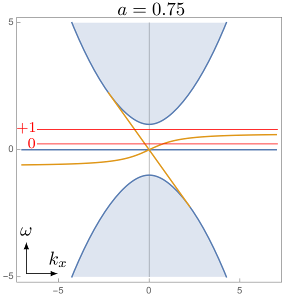

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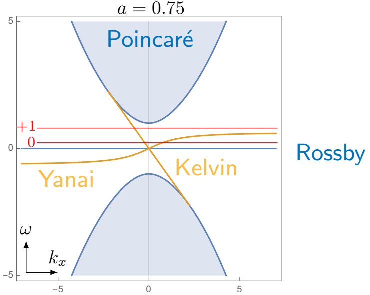

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Bulk-edge correspondence?

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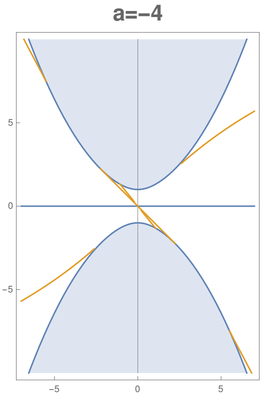

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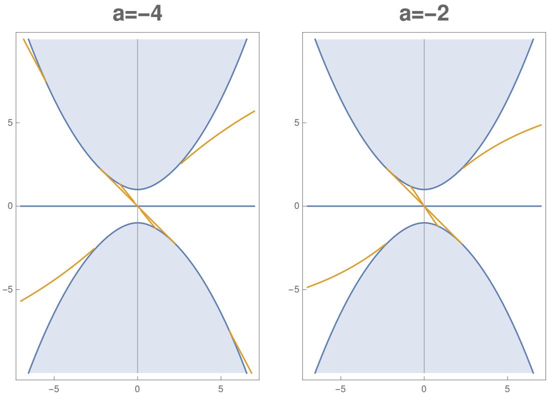

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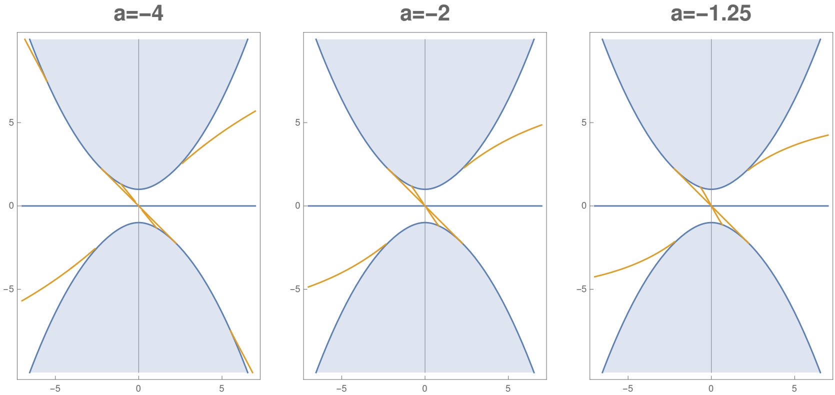

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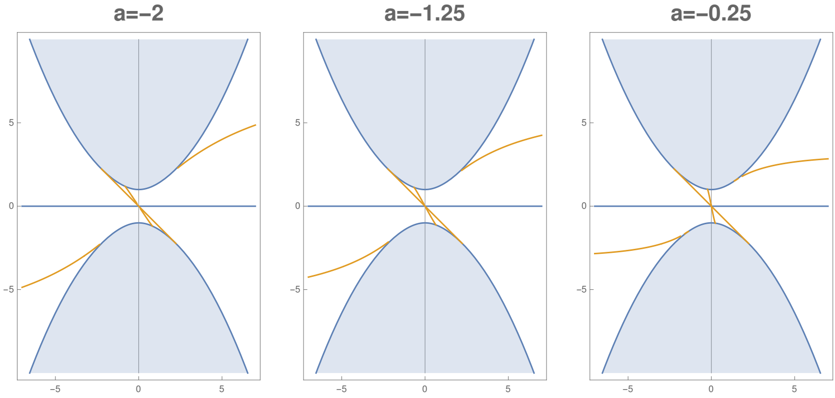

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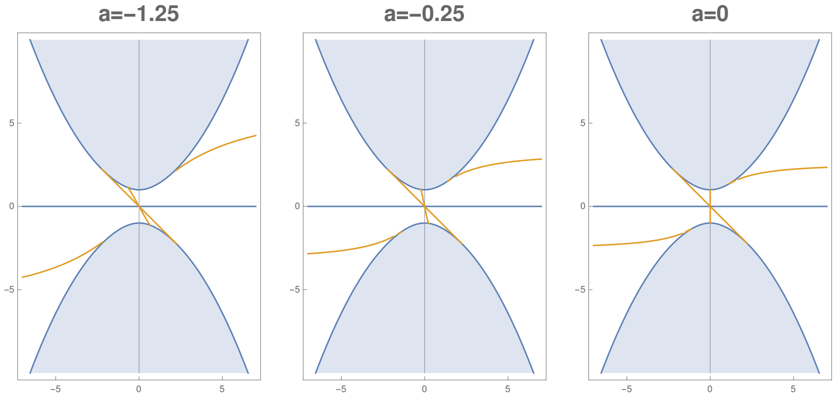

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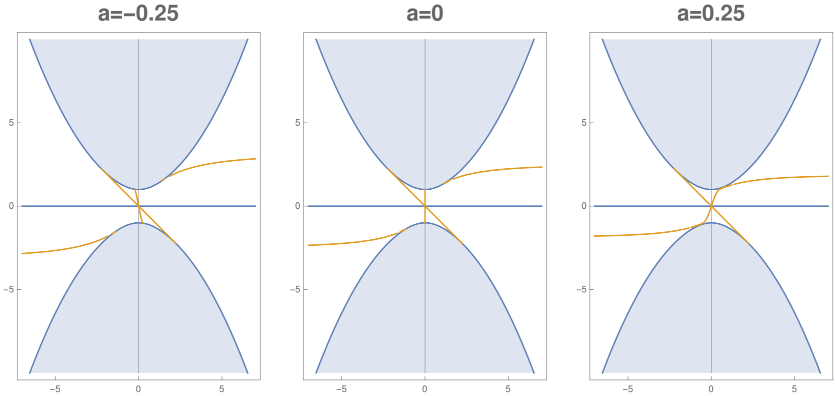

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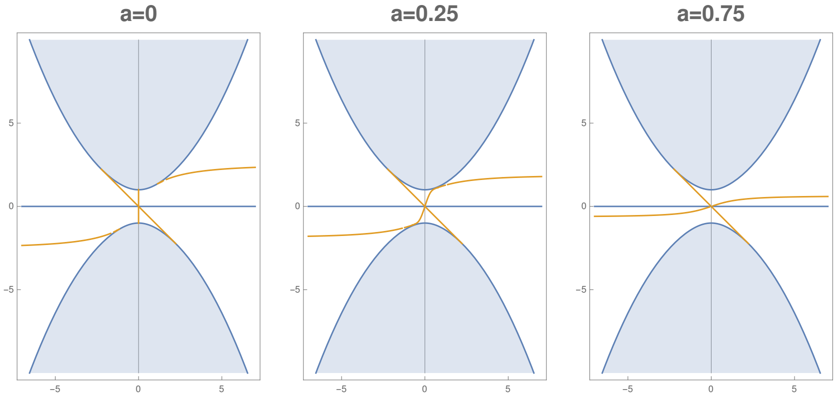

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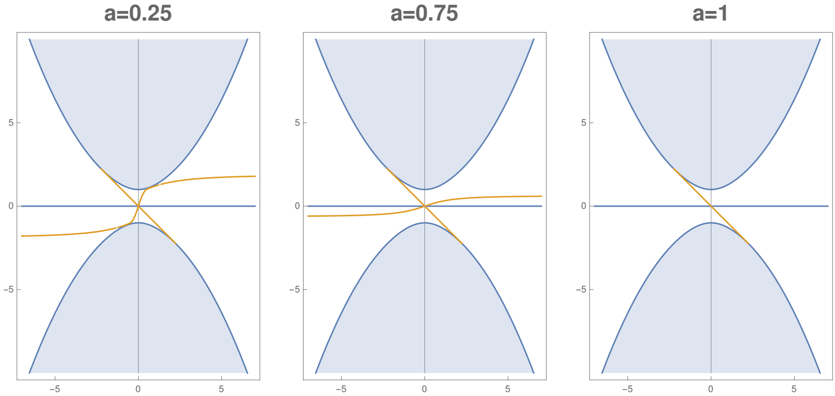

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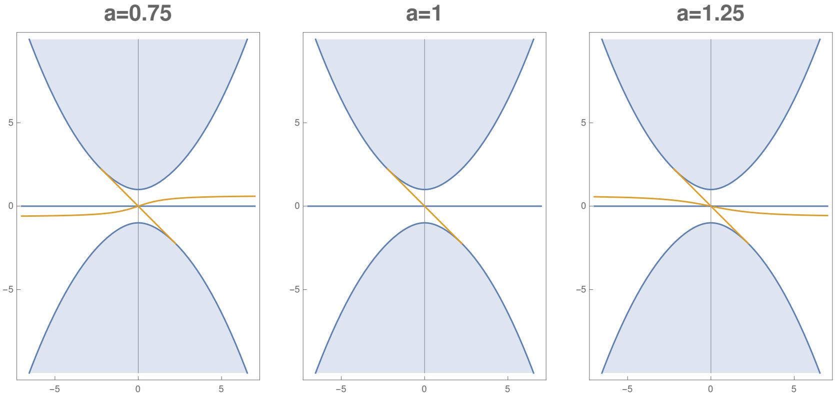

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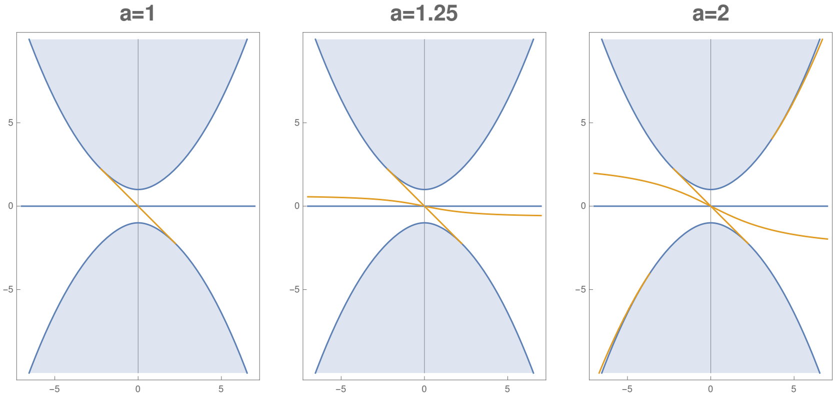

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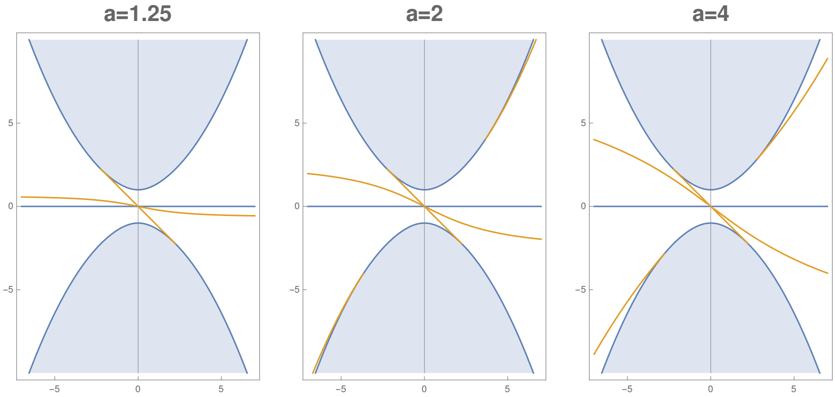

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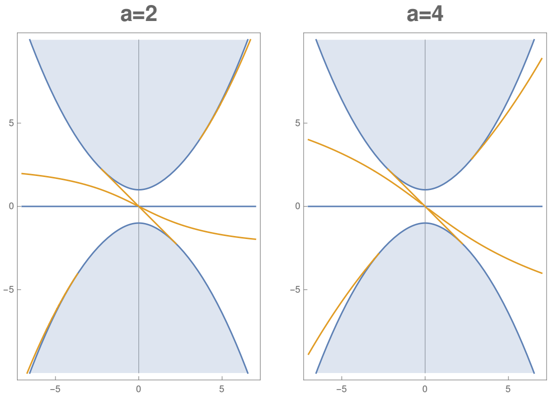

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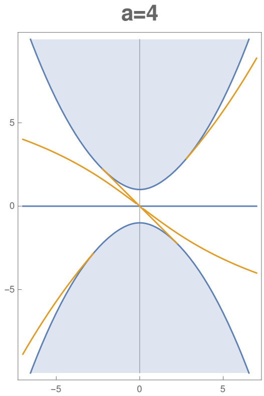

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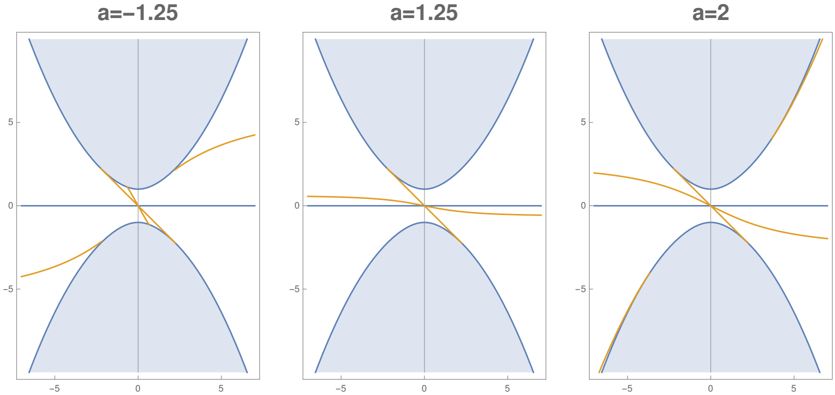
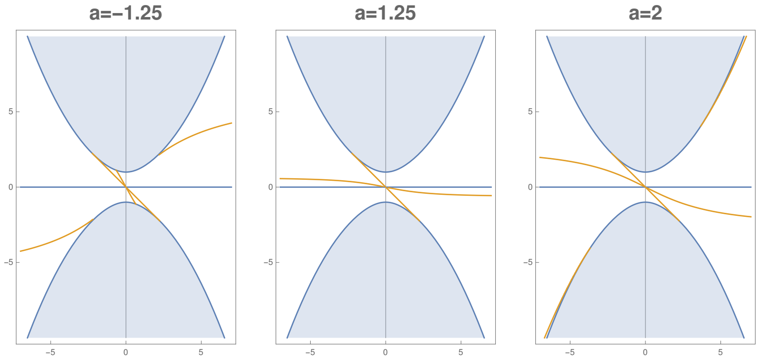


- Kelvin waves are seen in all cases
- Bulk-edge correspondence is violated!
- There are edge states never merging with a band
- There are edge states “merging at infinity”

Bulk-edge correspondence?



Theorem. (Violation of correspondence) As a function of the boundary parameter a, the edge index takes the values

$$\mathcal{N}_- = \begin{cases} 2 & (a < -\sqrt{2}) \\ 3 & (-\sqrt{2} < a < 0) \\ 1 & (0 < a < \sqrt{2}) \\ 2 & (a > \sqrt{2}) \end{cases}$$

Recall: The bulk index is $\mathrm{ch}(P) = 2$.

What is Bulk-edge correspondence (BEC) here

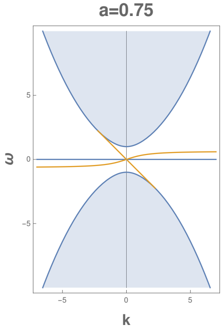
Statement (desired, à la Hatsugai)

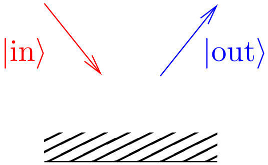
$$\mathcal{N}_- = \mathrm{ch}(P) \qquad (= 2)$$

where

- $\mathcal{N}_-$ the signed number of eigenstates emerging from the upper band
- P is the eigenbundle of the upper band and $\mathrm{ch}(P)$ its Chern number

Relation to scattering





Scattering of waves at the shore (“from inside the bulk”)

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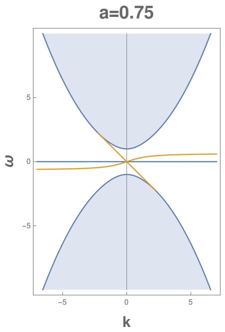

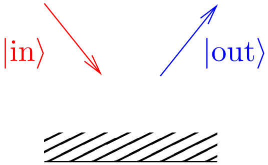

Scattering of waves at the shore ("from inside the bulk") defines scattering map

$$S : |\mathrm{in}\rangle \mapsto |\mathrm{out}\rangle$$

and scattering phase $S(k, \omega) = \langle \mathrm{in}|\mathrm{out}\rangle$ (k: longitudinal momentum, $\omega > \omega_+(k)$)

An attempt to prove BEC

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Relation can be split in two (Porta, G.):

$$\mathrm{ch}(P) = \mathcal{N}(S)$$
$$\mathcal{N}(S) = \mathcal{N}_- \qquad \text{(Levinson theorem)}$$

where

- $S = S(k, \omega_+(k) + \varepsilon)$, $(k \in S^1)$ is the scattering phase slightly above threshold $\omega_+(k)$ $(\varepsilon > 0)$
- $\mathcal{N}(f)$ winding number of $f : S^1 \to S^1$.

What goes wrong?

Is it?

$$\mathrm{ch}(P) = \mathcal{N}(S)$$

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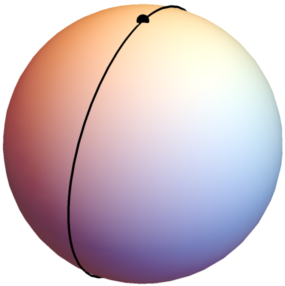
The sphere of momenta $\underline{k}$ with

- marked point $\underline{k} = \infty$
- meridian $k_y = 0$



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The hemispheres of incoming ($k_y < 0$) and outgoing ($k_y > 0$) states:

$$|\psi_{\mathrm{in}}(\underline{k})\rangle = |\mathrm{in}\rangle\,, \qquad |\psi_{\mathrm{out}}(\underline{k})\rangle = |\mathrm{out}\rangle$$

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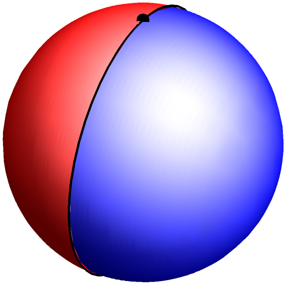

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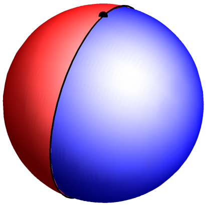

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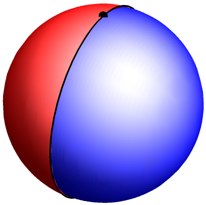

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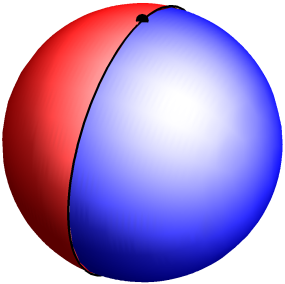
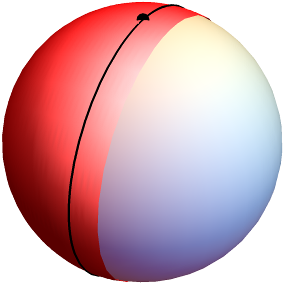
But there is a section covering more than the incoming hemisphere





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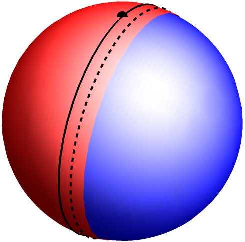

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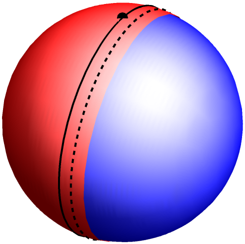

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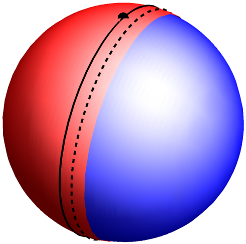

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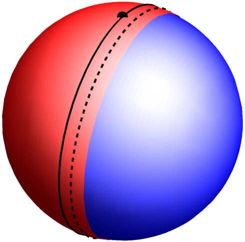

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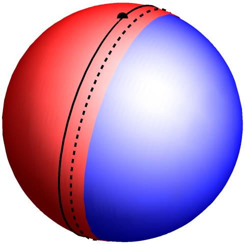

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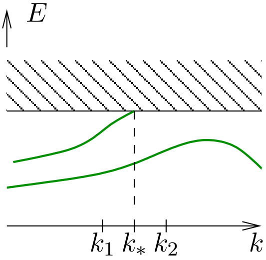
More precisely: Suppose $H(k)$ depends on some parameter $k \in \mathbb{R}$



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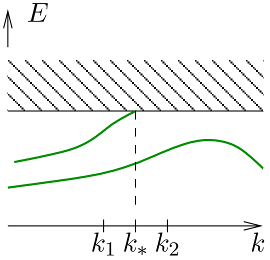

The scattering phase jumps when a bound state reaches threshold

$$\lim_{E \to 0} \arg S(k, E)\Big|_{k_1}^{k_2} = \mp 2\pi$$

The Levinson scenario

$$\lim_{E\to 0} \arg S(k_x, E)\Big|_{k_1}^{k_2} = \mp 2\pi$$

Structure of scattering phase

$$S(k_x, E) = -\frac{g(k_x, \tilde{k}_y)}{g(k_x, k_y)}$$

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The Levinson scenario

$$\lim_{E\to 0} \arg S(k_x, E)\Big|_{k_1}^{k_2} = \mp 2\pi$$

Structure of scattering phase

$$S(k_x, E) = -\frac{g(k_x, \tilde{k}_y)}{g(k_x, k_y)}$$

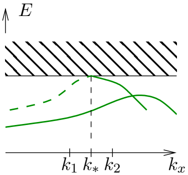
where

- $\tilde{k}_y$ and k_y are the incoming/outgoing momenta with $E(k_x, \tilde{k}_y) = E(k_x, k_y) = E$
- $\tilde{k}_y = -k_y$ if E is even
- g is analytic in k_y (Jost function)

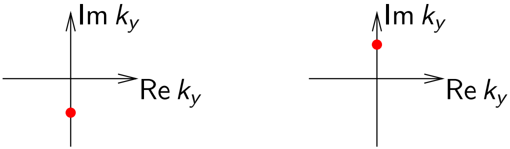
Bound states of $H(k_x)$ correspond to poles of $S(k_x, E)$ with $\operatorname{Im} k_y > 0$ ("bound out-state without in state");

Proof. From $S = \langle \mathrm{in} | \mathrm{out} \rangle$ get $|\psi\rangle = |\widetilde{\mathrm{in}}\rangle + |\mathrm{out}\rangle = |\widetilde{\mathrm{in}}\rangle + S|\mathrm{in}\rangle$; hence $|\psi\rangle \propto g|\widetilde{\mathrm{in}}\rangle - \tilde{g}|\mathrm{in}\rangle$ requires $g|\widetilde{\mathrm{in}}\rangle = 0$ (divergent part by $\operatorname{Im} \tilde{k} < 0$) to vanish)

The Levinson scenario



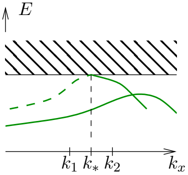
Bound states of $H(k_x)$ correspond to complex zeros k_y of $g(k_x, k_y)$



$(k_x < k_*)$ $(k_x > k_*)$

Fact 1: As k_x crosses k_*, a bound state appears.

The Levinson scenario



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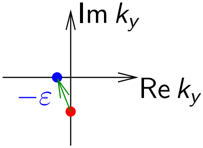

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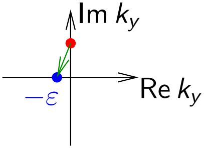


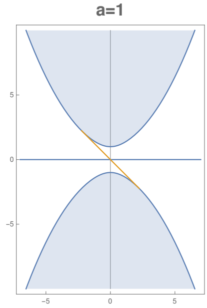
$(k_x > k_*)$

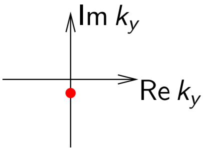
$$S(k_x, E) = -\frac{g(k_x, \tilde{k}_y)}{g(k_x, k_y)}$$

Fact 2: As k_x crosses k_*, $\arg g(k_x, \tilde{k}_y = -\varepsilon)$ changes by $-\pi$ (and $\arg g(k_x, \varepsilon)$ by π), hence S winds by -2π.

The Levinson scenario

As for waves, this is the relevant scenario for (almost) all critical, finite momenta k_x





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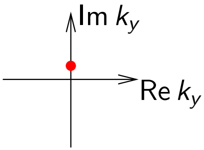

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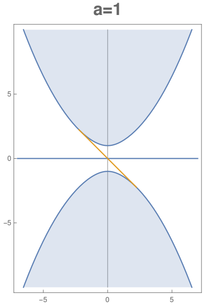

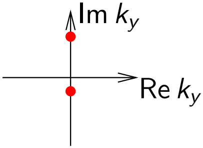

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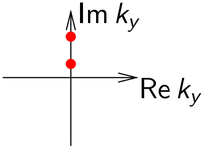

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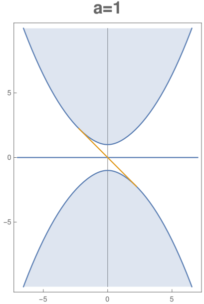

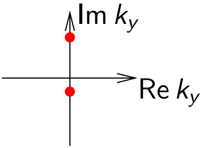

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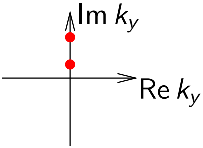

$(k_x > k_*)$

- Given k_x, the map $k_y \mapsto \omega(k_x, k_y)$ is 2 to 1, besides of $\pm k_y$
- For (k_x, ω) in the gap, all 4 preimages k_y are imaginary (2 decaying/diverging).

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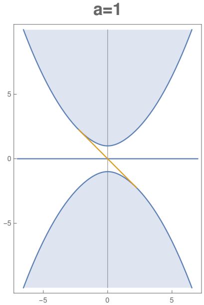

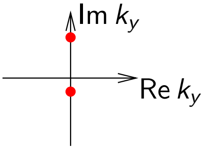

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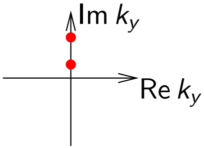

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- Given k_x, the map $k_y \mapsto \omega(k_x, k_y)$ is 2 to 1, besides of $\pm k_y$
- For (k_x, ω) in the gap, all 4 preimages k_y are imaginary (2 decaying/diverging).
- Bound states of $H(k_x)$ correspond to both momenta with $\operatorname{Im} k_y > 0$ having $g(k_x, k_y) = 0$
- At a critical point just one zero changes half-plane. Conclusions unchanged.

Waves at infinite momentum: Not the Levinson scenario

$k_x = \infty$ is always critical (regardless of whether an edge state merges there).

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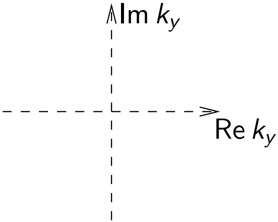
The complex plane



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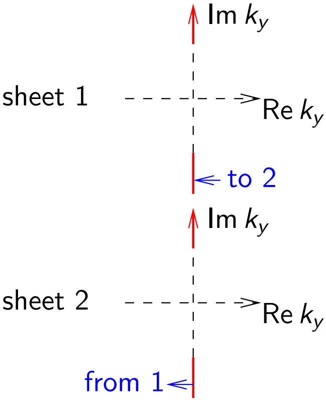
A Riemann surface in the variable k_y: Two sheets joined by slits extending to ∞



Waves at infinite momentum

The function $g = g(k_y)$

- is a function on the Riemann surface
- has one zero on each sheet
- depends parametrically on k_x as $g(k_x, k_y) = g(k_y/k_x)$ (scaling at $k_x \to \pm\infty$)

It takes $\text{Im}\, k_y > 0$ for both zeros to make a bound state (not necessarily realized)

Not the Levinson scenario: Alternative I

- The diagrams scale linearly with k_x
- It takes two zeros, both with $\text{Im}\, k_y > 0$, to make a bound state

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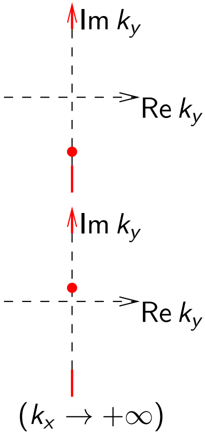

Fact 1: No bound state is created nor destroyed at transition.

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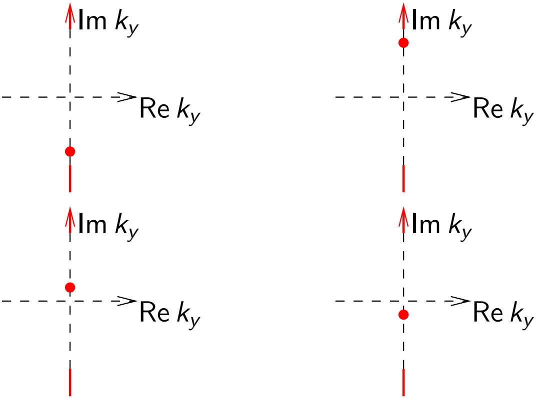


$(k_x \to +\infty)$ $(k_x \to -\infty)$

Fact 1: No bound state is created nor destroyed at transition.

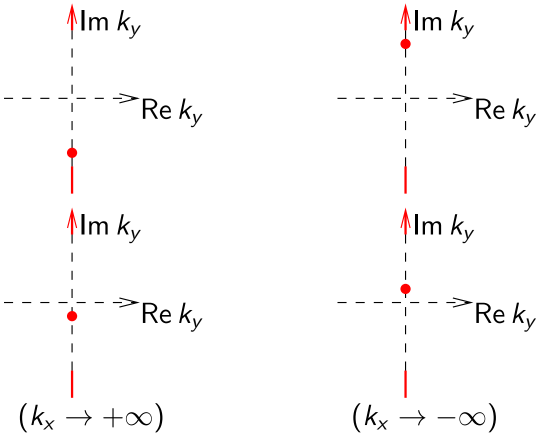
Fact 2: There is a jump of $\arg g$ by $\pm\pi$, hence S winds by $\pm 2\pi$

Not the Levinson scenario: Alternative II

It takes two zeros, both with $\operatorname{Im} k_y > 0$, to make a bound state.

Not the Levinson scenario: Alternative II

It takes two zeros, both with Im $k_y > 0$, to make a bound state.



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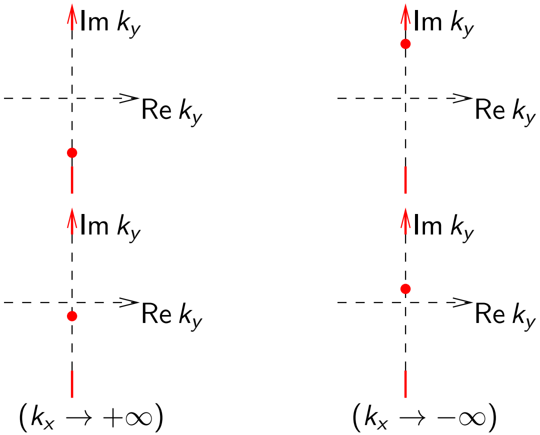


Fact 1: A bound state is created at transition

Fact 2: There is no jump of arg g and hence S does not wind.

Back to Theorem

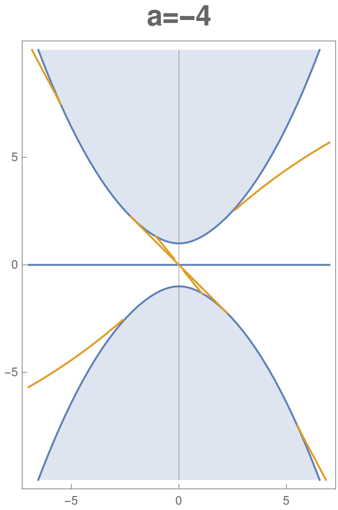
Edge:

$$\mathcal{N}_- = \begin{cases} 2 & (a < -\sqrt{2}) \\ 3 & (-\sqrt{2} < a < 0) \\ 1 & (0 < a < \sqrt{2}) \\ 2 & (a > \sqrt{2}) \end{cases}$$

Bulk:

$$\mathrm{ch}(P) = 2$$

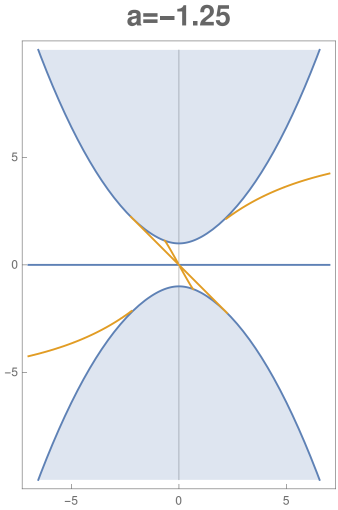
Back to Theorem, case by case



$$\mathcal{N}_- = 2\,, \qquad (a < -\sqrt{2})$$

Alternative II: Edge state merging at infinity; no winding of S there

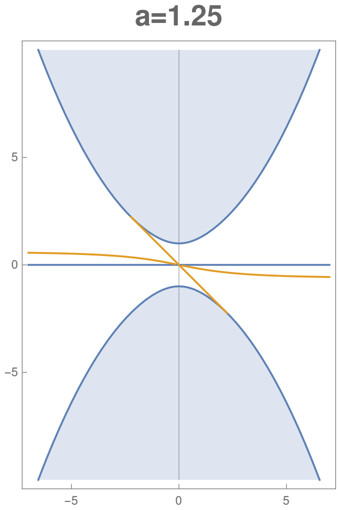
Back to Theorem, case by case



$$\mathcal{N}_- = 3\,, \qquad (-\sqrt{2} < a < 0)$$

Alternative I: No edge state merging at infinity; winding of S by -1

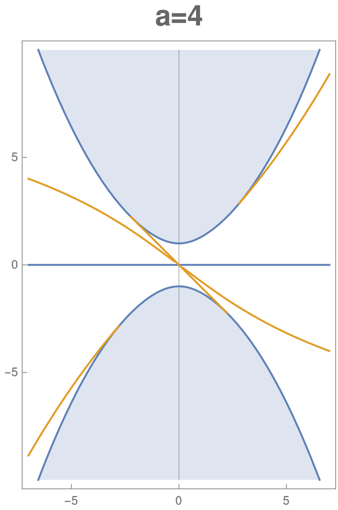
Back to Theorem, case by case



$$\mathcal{N}_- = 1\,, \qquad (0 < a < \sqrt{2})$$

Alternative I: No edge state merging at infinity; winding of S by $+1$

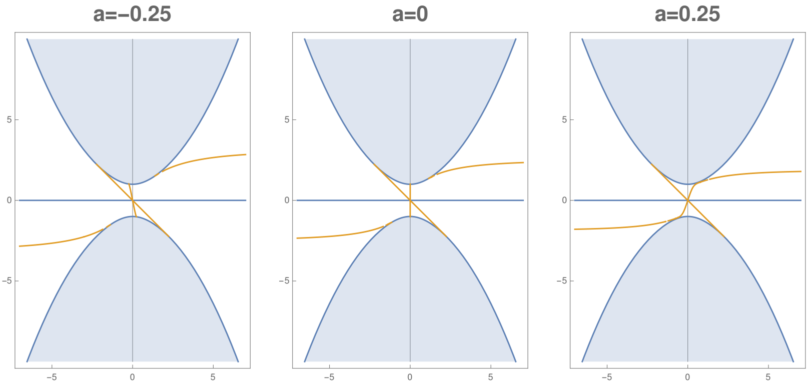
Back to Theorem, case by case



$$\mathcal{N}_- = 2\,, \qquad (a > \sqrt{2})$$

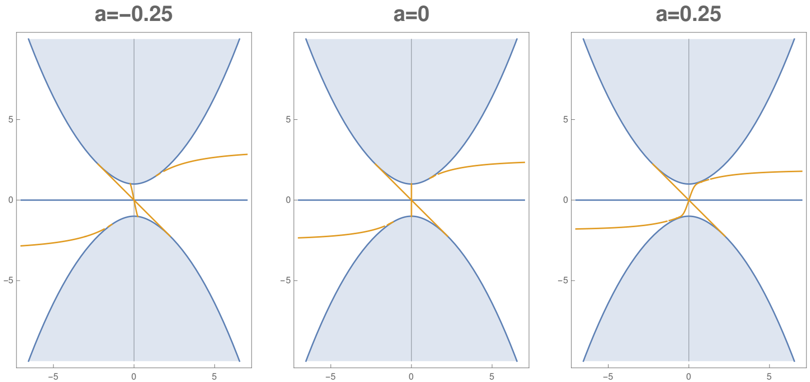
Alternative II: Edge state merging at infinity; no winding of S there

The transition at $a = 0$



- The transition occurs within Alternative 1.
- Winding of S at infinity changes from -1 to $+1$
- The fibers $H_a(k_x)$ of the edge Hamiltonian are self-adjoint for almost all k_x (as it must)

The transition at $a = 0$



- The transition occurs within Alternative 1.
- Winding of S at infinity changes from -1 to $+1$
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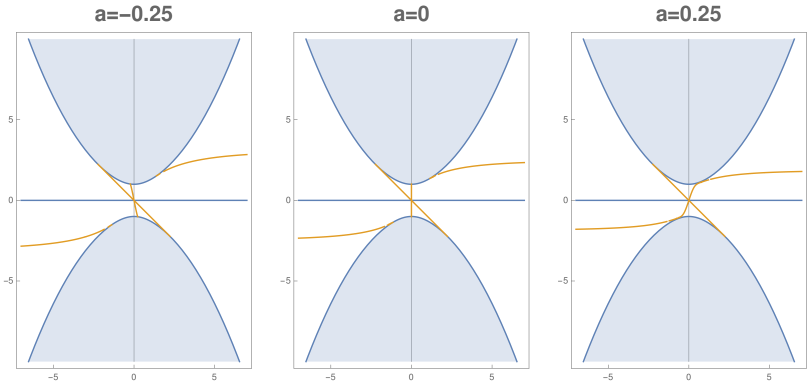

- The transition occurs within Alternative 1.
- Winding of S at infinity changes from -1 to $+1$
- The fibers $H_a(k_x)$ of the edge Hamiltonian are self-adjoint for almost all k_x (as it must), but not for $a = 0$, $k_x = 0$. In fact the boundary condition

$$\mathrm{i}k_x u + a\partial_y v = 0$$

becomes empty.

Summary

- The shallow water model has edge states in presence of Coriolis forces.
- The model is topological if compactified by odd viscosity
- The model violates bulk-boundary correspondence
- Scattering theory (of waves hitting shore) clarifies the cause
- Levinson's theorem does not apply in its usual form