Brain Storming in Mathematics and Physics

dedicated to Arthur Jaffe

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Mathematical Picture Language Seminar, Harvard University, Dec 22, 2020

Imagine that we are in Arthur's office, and I will give an informal lecture. This lecture has no main theme; instead I will talk about some insightful examples and fresh ideas inspired by math and physics as well as some potential connections between different areas.

Planar para algebras

2015 Fall, A. Jaffe $\rightarrow$ Z. Liu: parafermions and reflection positivity
Z. Liu $\rightarrow$ A. Jaffe: subfactor planar algebras

Jaffe-Liu: Planar para algebras, Reflection Positivity (CMP 2017)

Theorem [Jaffe-L 17]: Reflection positivity for a Hamiltonian H for any inverse temperature $\beta > 0$, if and only if

$$\mathfrak{F}(-H_0) \geq 0,$$

where H_0 is the restriction of Hamiltonian on the reflection mirror and $\mathfrak{F}$ is a *Fourier transform*.

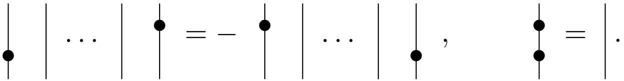
Key idea: (1D $\rightarrow$ 2D)

- Reflection positivity: *horizontal reflection*
- C^* Positivity: *vertical reflection*
- Fourier transformation: 90° *rotation*

Jaffe-Liu 2020: Reflection Positivity and Levin-Wen Models. (Expositiones Mathematicae, a special issue dedicated to Dick Kadison)

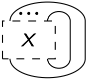
CAR Algebras and Pictorial Representations

Example: ($\mathbb{Z}_2$ case, Majorana fermions)
CAR algebras: $c_i c_j = -c_j c_i$, $i \neq j$; $c_i^2 = 1$; $\tau(1) = 1, \tau(c_{i_1} c_{i_2} \cdots c_{i_k}) = 0$.
The C^*-algebra $\mathscr{P}_n$ is generated by $c_1, c_2, \cdots c_n$
Pictorial representation:



Additional planar diagrams in $\mathscr{P}_n$:  and  .
Additional relations (not a full list):

Trace τ on $x \in \mathscr{P}_n$:

$\tau(x) = \delta^{-n}$ 

Remark: In general, τ is positive on $\mathscr{P}_n$ iff $\delta^2 = 2\times$ Jones index.

2-String Model for Quantum Information

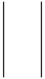
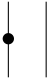
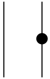
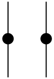
A. Wozniakowski $\rightarrow$ A. Jaffe, Z. Liu: quantum information
2-String Model: (Jaffe-L-Wozniakowski Sci. China Math. 2018)

Jordan-Wigner transformation: $\mathscr{P}_{2n} \cong \bigotimes_{k=1}^{n} M_2(\mathbb{C})$, n-qubit transformations.

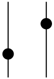
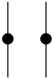
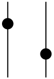
Qubits: $|0\rangle = \delta^{-1}$  and $|1\rangle = \delta^{-1}$ 

Measurements: $\langle 0| = \delta^{-1}$  and $\langle 1| = \delta^{-1}$ 

Pauli matrices:

$I =$  ; $X =$  ; $Y =$  ; $Z =$  ;

Interpolation:

 $= \sqrt{-1}$  $= -$  .

Topological Design of Protocols

We design communication protocols in a topological way.
(Jaffe-L-Wozniakowski NJP 2017)
Idea: Communication changes the location of information, but not the information itself. This is captured by topological isotopy.
Renato Renner suggested us to study multipartite communication.
What is the minimal cost of entangled states (+ LOCC) to implement n non-local (CNOT) gates?

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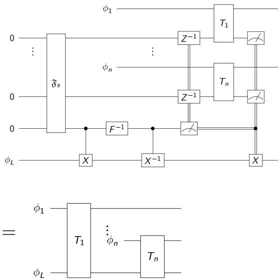
What is the minimal cost of entangled states (+ LOCC) to implement n non-local (CNOT) gates? Only one, the n-qubit Max state.



Max v.s. GHZ states

3-qubit case:
$|Max\rangle = 2^{-1}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$.
Greenberg-Horne-Zeilinger state: $|GHZ\rangle = 2^{-1/2}(|000\rangle + |111\rangle)$.

Hadamard gate:

$$H = 2^{-1/2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Fourier duality:

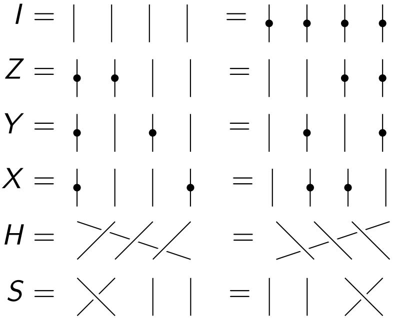
$$|Max\rangle = (H \otimes H \otimes H)|GHZ\rangle \; .$$

How to represent $|GHZ\rangle$, $|Max\rangle$ and the Hadamard gate pictorially at the same time?

Majorana zero modes (4-string model): See Sarma-Freedman-Nayak 2015 NJP QI for a recent survey.

$$|0\rangle_Z = 2^{-1/2}$$

1-qubit gates acting on 1-qubits: (Pictorial relations do NOT hold in $\mathscr{P}_4$.)



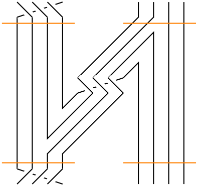
$$S = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{bmatrix}.$$

n-Qubit Gates

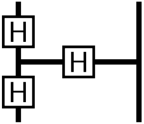
Grading operator:

CNOT: $|i,j\rangle \mapsto |i+j,j\rangle$.

CNOT in four-string model



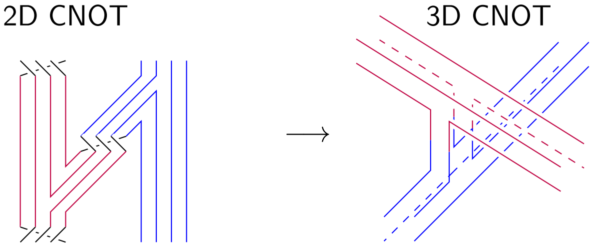
CNOT in tensor network



n-qubit gate: group the boundary points 4 by 4 for an element in $\mathscr{P}_{4n}$.

2D to 3D

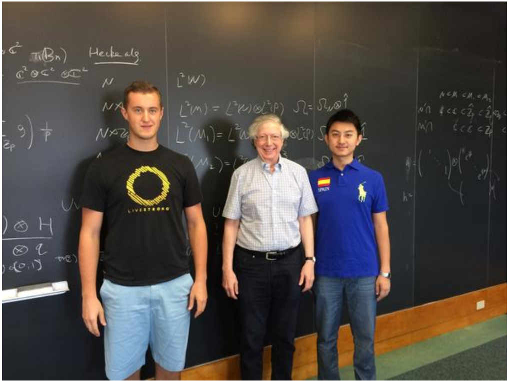
There were several examples indicating that we should extend the 2D theory to 3D.



Another Observation 2016: The *n*-qubit Max state also appeared in the study of *n*-interval Jones-Wassermann subfactors (for $\mathbb{Z}_2$) in chiral conformal field theory, with a quite different meaning. They appeared as 2D sub theories of a 3D theory, related by an *m-n duality*. (L-Xu Adv. Math. 2019)

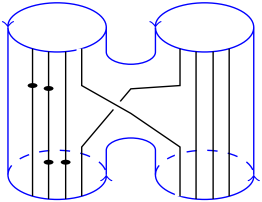
Quon Language

In the joint paper with Arthur Jaffe and Alex Wozniakowski (PNAS 2017), we introduce the **quon language** as a 3D picture language for quantum information.



Quon Language for Qubits

The quon language for qubits is an Ising TQFT with **charges**, the functor F is extended to a **projective** monoidal functor from the category of 1+1 cobordisms with braided strings and pairs of charges $\mathbf{Cob_{BCS}}$ to $\mathbf{Vec}$.



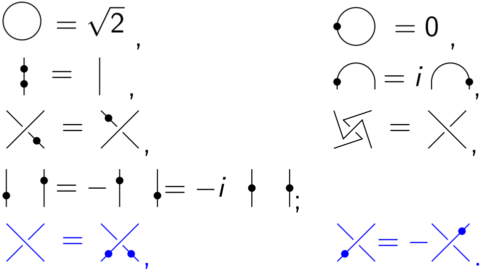
The quon language is projective in the bulk and linear on the boundary. This is good to simulate quantum theory, as the state space is linear in terms of the super position, and the transformations are defined projectively. (The functor F can be further extended to be super.)

Bulk Relations

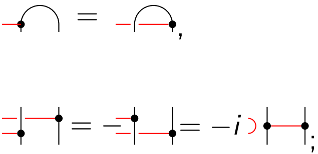
Bulk relations are relations of labelled tangles without linear sum, e.g. relations of projectively defined transformations.

The braid satisfies Reidemeister moves of type I, II, III.

The charge behaves like a Majorana fermion:



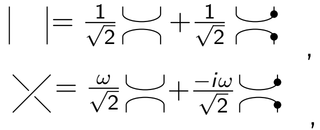
Topological Relations



Common properties: Topological relations also hold for qudits.

Difference between qubits and qudits: The last two relations are NOT topological. They only hold for qubits, and they are extremely useful.

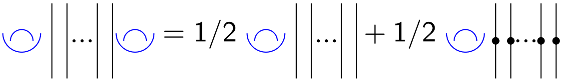
Boundary relations are linear relations of labelled tangles, e.g. relations of states.



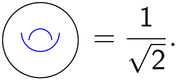
where $\omega^2 = \frac{1+i}{\sqrt{2}}$.

String-Genus Relation

Neutrality: (Generalizing the grading operator in Majorana zero modes.)



String-Genus Relation: If a hole of the surface is surrounded by a closed string, then they can be removed up to a scalar.



Significance: The string-genus relation is a bulk relation changing the shape of a surface.

Remark: The string-genus relation is important in the pictorial design of quantum error correcting codes on *quantized graphs.* (L, arXiv:1910.12065)

Topological Complexity Theory

What is the significance of bulk relations, particularly the String-Genus relation, in quantum information?

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Topological/Pictorial Complexity: Polynomial v.s. Exponential.

A fundamental problem is whether a closed diagram (tensor network or circuit,) can be evaluated in polynomial time, using bulk/boundary relations.

Bulk relation is combinatorial, the cost is usually polynomial.
Boundary relation is a linear sum, the cost is exponential.

For example, if we evaluate a genus-n surface only using the neutrality relation, then we will end up with a linear expansion of 2^n terms without genus, an exponential cost!

Remark: All known bulk relations changing the shape of the surface reduce to the string-genus relation.

Simulation of Clifford Gates

Clifford tensor network: X, Y, Z, H, S and the identity tensor $|000\rangle + |111\rangle$,.

Theorem (Gao-Jaffe-L-Ren-Wang)

Using the bulk relations above, we can efficiently simulate/evaluate a Clifford circuit/tensor network.

Actually these bulk relations implies all tensor network relations in ZX calculus. (See Bob and Duncan NJP 13 for ZX calculus.)

Universality

Clifford gates are fault tolerant in topological quantum computation of Kitaev's toric code.
However, Clifford gates are not universal for quantum computing. Clifford gates and phase gates $e^{i\theta X}$ are universal.

Question: Is there any pictorial feature for phase gates?

Universality

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However, Clifford gates are not universal for quantum computing. Clifford gates and phase gates $e^{i\theta X}$ are universal.

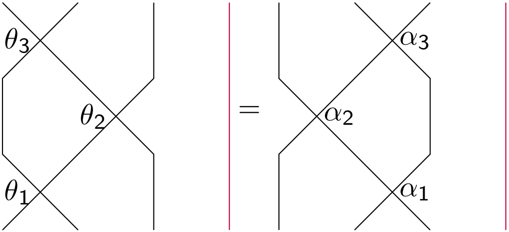
Question: Is there any pictorial feature for phase gates? Yes!

Euler Formula and Yang-Baxter equation

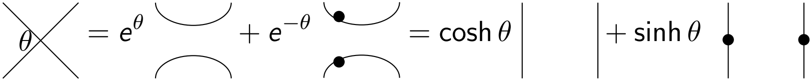
Euler formula: Any 1-qubit unitary

$$U = e^{i\theta_1 Z} e^{i\theta_2 X} e^{i\theta_3 Z} = e^{i\alpha_1 X} e^{i\alpha_2 Z} e^{i\alpha_3 X}.$$

Yang-Baxter equation (in $\mathscr{P}_3$):

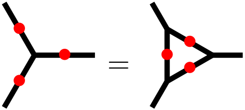


where



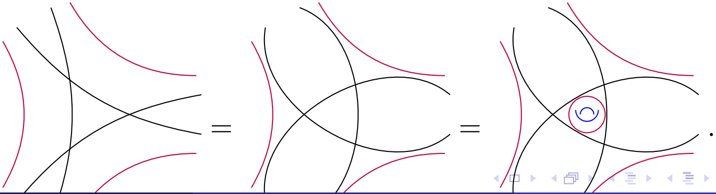
Star-Triangle Equation

Xun Gao's key observations

Star-triangle equation in tensor network: 

A red bullet represents a tensor $e^{i\theta X}$. (When $\theta = \pi/3$ for all three angles on the right, it is the W-state $|100\rangle + |010\rangle + |001\rangle$.)

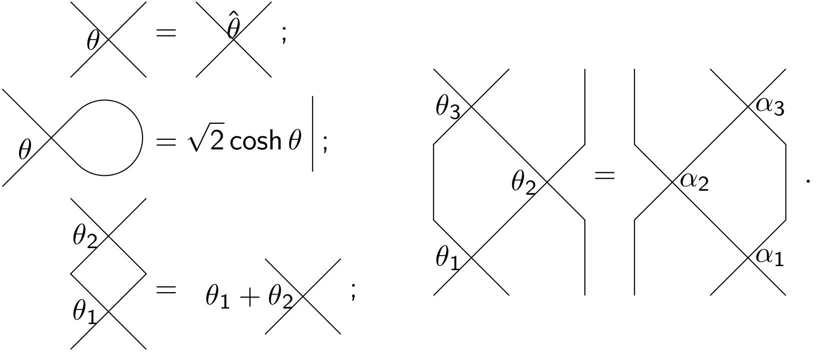
Yang-Baxter equation + string genus relations $\Rightarrow$ star-triangle equation:
(The crossings have parameters.)



Simulation of Matchgates

Matchgate tensor network: $e^{\theta X}$, $\forall \theta \in \mathbb{C}$, identity tensor $|000\rangle + |111\rangle$.
Matchgate tensor network reduces to a planar 4-valent graph without genus, like a 2D projection of a link with parameters at crossings.

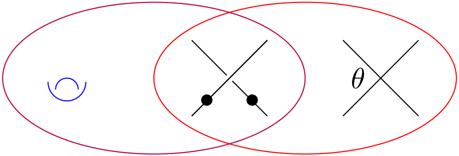
It can be evaluated in polynomial time $O(n^3)$ using the *Yang-Baxter relation* (L Thesis 15):



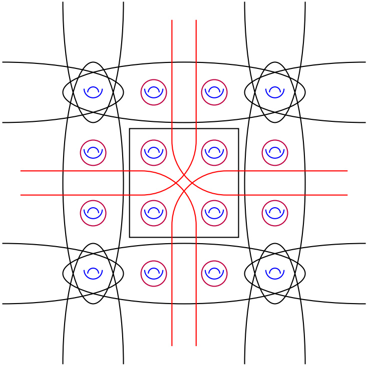
The last equation holds for $\theta_1, \theta_2, \theta_3 \in \mathbb{C}$ with probability 1.

(1) braided charged strings
(2) surface/genus
(3) parameterized crossings

Clifford gates: (1)+(2)
Matchgates: (1)+(3)
Universal gates: (1)+(2)+(3)



New Efficiently Simulable Tensor Networks



Red crossings have arbitrary parameters;
The other crossings are braids;
Black strings are above red strings.

This quon picture represents the local fractional data of a tensor network, neither Clifford nor matchgate, on a square lattice.
Interpretation: Ising model for interacting fermions.
Its partition function is the sum of two partition functions for free fermions.
Chain Reaction: Replacing one genus by a sum of two terms using neutrality relations, then the rest genus can be removed using bulk relations.

Outlook

Topological/Pictorial Design of

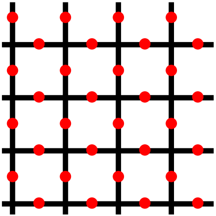
- Communication Protocols
- Quantum Error Correcting Codes
- Simulable tensor networks
- Exactly solvable models

Quantum circuit complier for simplifying circuits
Polynomial algorithms (for combinatorial/graphical problems)
Exactly solvable models with physics implementations
...

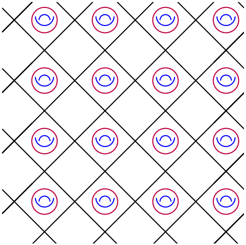
Ising model

The partition function of Ising model:

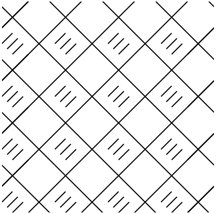
Tensor Network



Quon Language

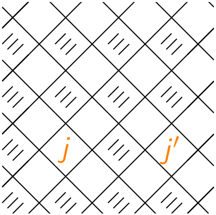


Spin Model
(Jones's planar algebras)

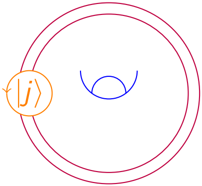


The red bullet in tensor network and the crossings in quon language have parameters corresponding to nearest interactions.
Kramer-Wannier duality: switch the position of the string-genus pairs.

Quantum Spin Lattice Modes (very recent work)



Topological Spin: $j \to d_j$



Using surface algebras (L CMP19), we can generalize the $\mathbb{Z}_2$ spin model for quantum symmetries, such as non-abelian anyons (unitary modular tensor category), Levin-Wen model (Unitary fusion category), and subfactors (unitary bi-module category). The quantum spin j is a simple object.

The String-Genus relation leads to a natural embedding functor Φ from the genus-0 theory (planar algebras) to a higher-genus theory (surface algebras), preserving the partition function.

Remark: The functor Φ captures the remarkable embedding theorem in subfactor theory of Jones-Penneys 2011 and of Morrison-Walker 2012.

Reconstruction Program

1+1 Conformal Field Theory $<->$ 2+1 Topological Quantum Field Theory
(Unitary Modular Tensor Category)

In the *quantum spin lattice mode*, we can study various concepts in statistical physics, such as quantum spins, interactions, partition function, correlation function, n-point function, large scale limit, continuum limit, renormalizations etc.

Long Term Project: We may obtain a quantum field theory in the limit, and conformal field theory at the critical temperature. Clearly tremendous deep work need to be established and proved.

A good sign: We know how to establish reflection positivity pictorially, as mentioned at the beginning.

Happy Birthday Arthur!

Thank you!