Fault-tolerant Coding for Quantum Communication

Matthias Christandl (Copenhagen)

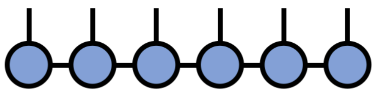
QMATH

joint work with Alexander Müller-Hermes (Lyon)

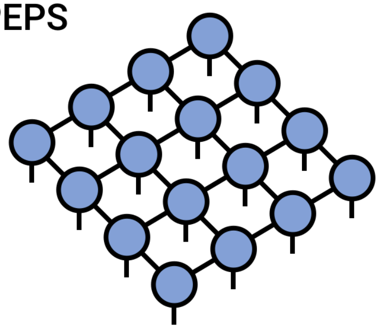
arXiv:2009.07161, QIP'21

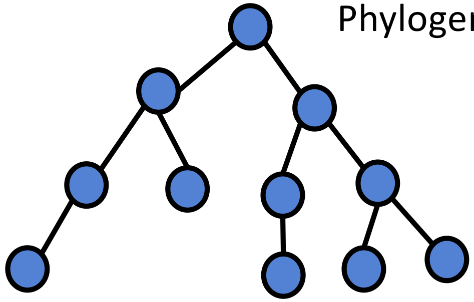
QMATH tracks coronavirus from genomic data

Matrix Product State / Tensor Train

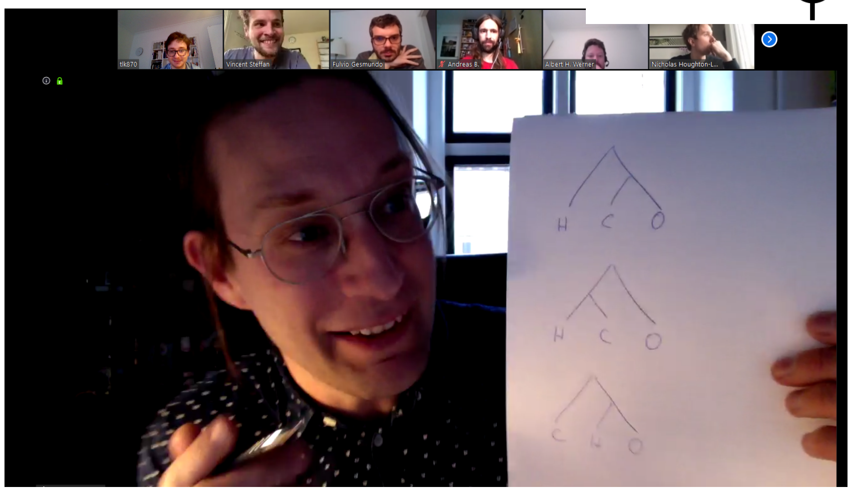


PEPS





Phylogeny/ancestry



PLOS ONE

RESEARCH ARTICLE

SARS-CoV-2 transmission routes from genetic data: A Danish case study

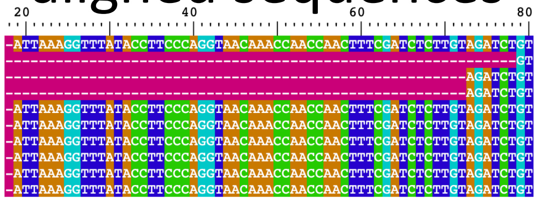
Andreas Bluhm, Matthias Christandl*, Fulvio Gesmundo, Frederik Ravn Klausen, Laura Mančinska, Vincent Steffan, Daniel Stilck França, Albert H. Werner

Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark

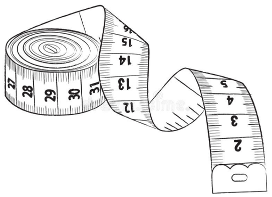
These authors contributed equally to this work.

From sequences to phylogenetic trees

aligned sequences

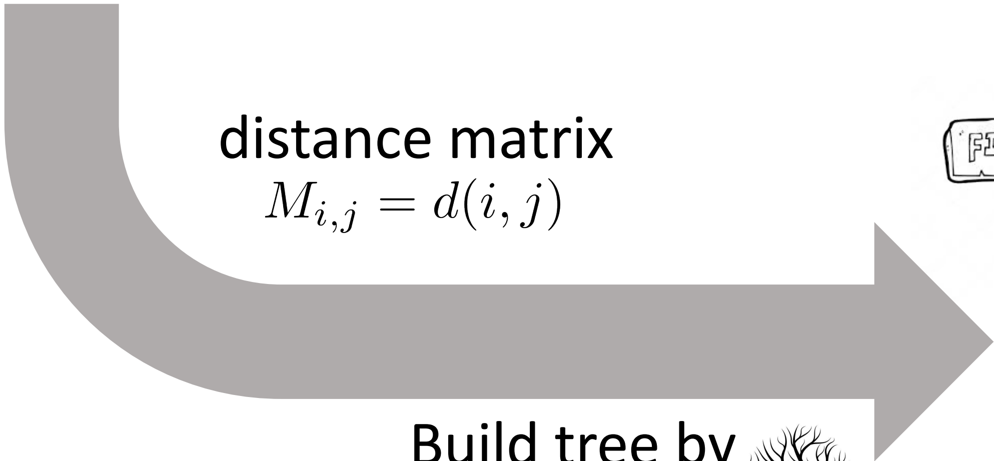


Idea: similar sequences are closely related



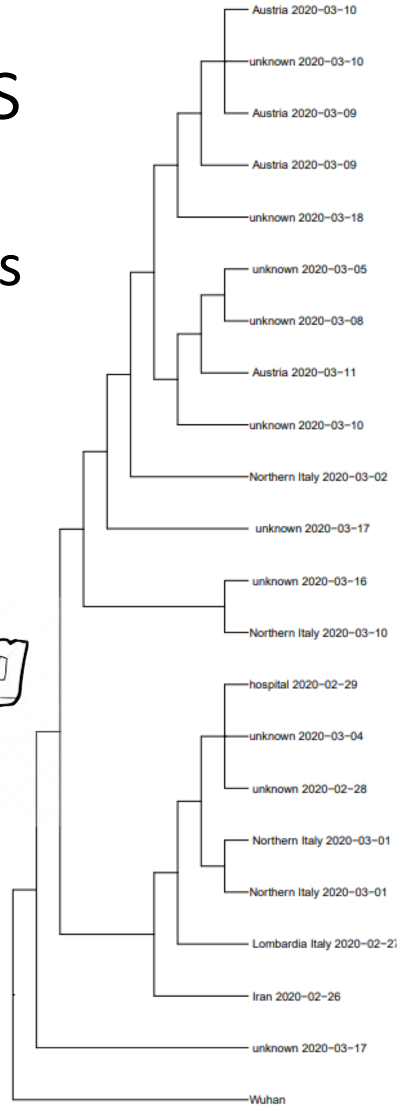
distance measure

e.g. Hamming distance
= # of differences



distance matrix
$M_{i,j} = d(i,j)$

Build tree by clustering



Fault-tolerant Coding for Quantum Communication

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QMATH

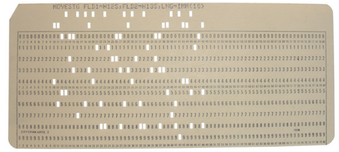
joint work with Alexander Müller-Hermes (Lyon)

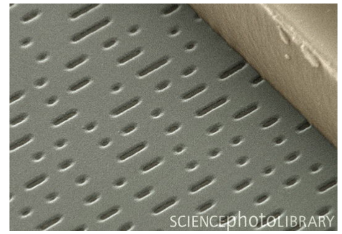
arXiv:2009.07161, QIP'21

Information Theory

- Abstract concept of information
- Independent of physical implementation

0100101
1001011

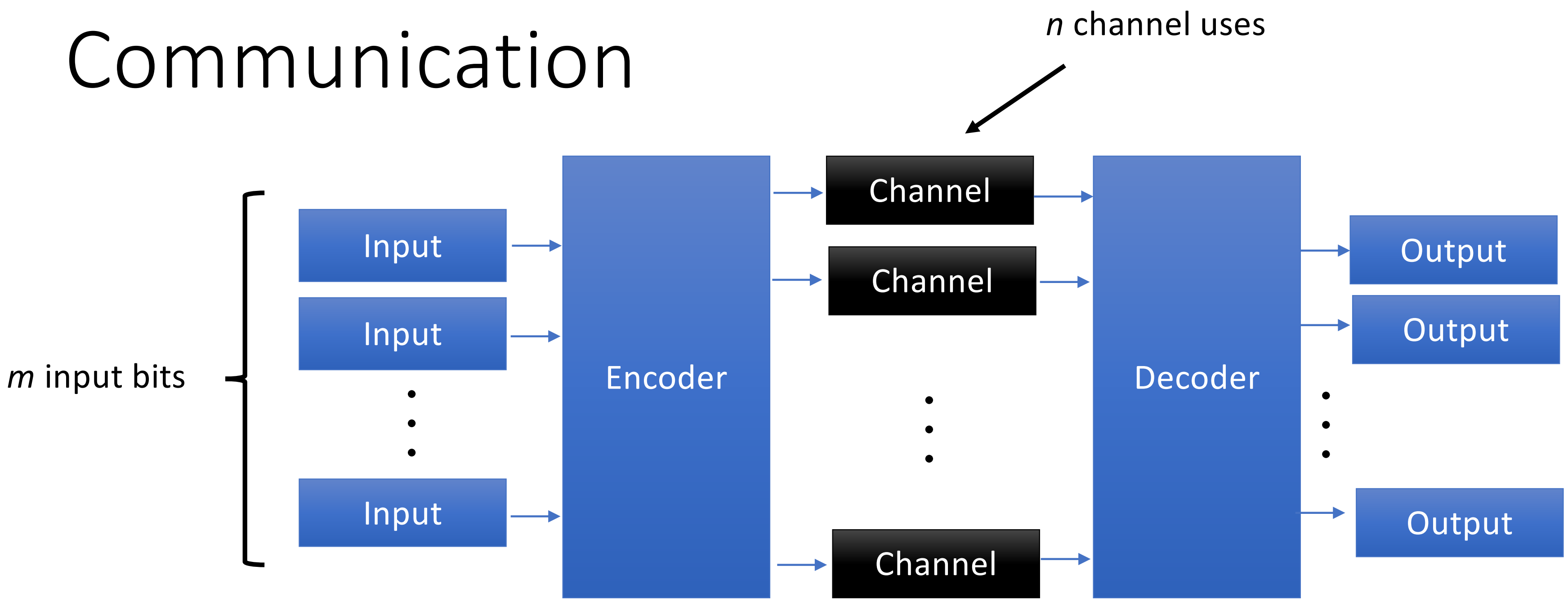




- Claim: all information can be abstracted this way

→ information theory

Communication



- Capacity $= \lim_{\epsilon\to 0}\lim_{n\to\infty}\sup\left\{\frac{m}{n} : Prob(input \neq output) \leq \epsilon\right\}$

$= $ I(X:Y) bits/channel

H(X)+H(Y)-H(XY) ← Shannon entropy

Quantum Information Theory

- Abstract concept of quantum information
- Independent of physical implementation

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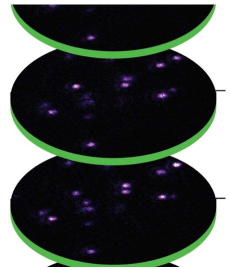
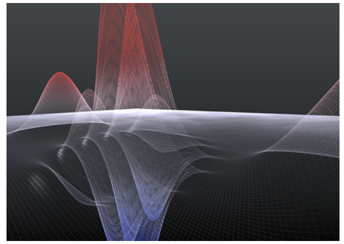




$$|0\rangle = e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

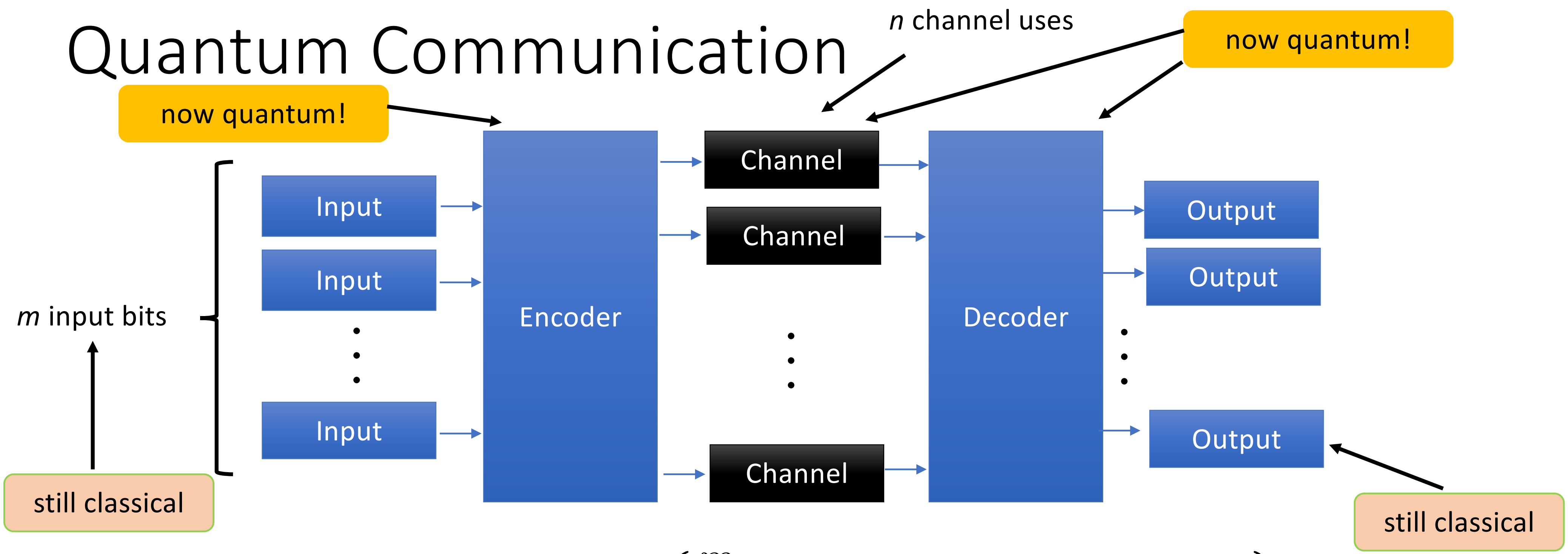
$$|1\rangle = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|000\rangle = e_0 \otimes e_0 \otimes e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cong \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- Claim: all (quantum) information can be abstracted this way

→ quantum information theory

Quantum Communication



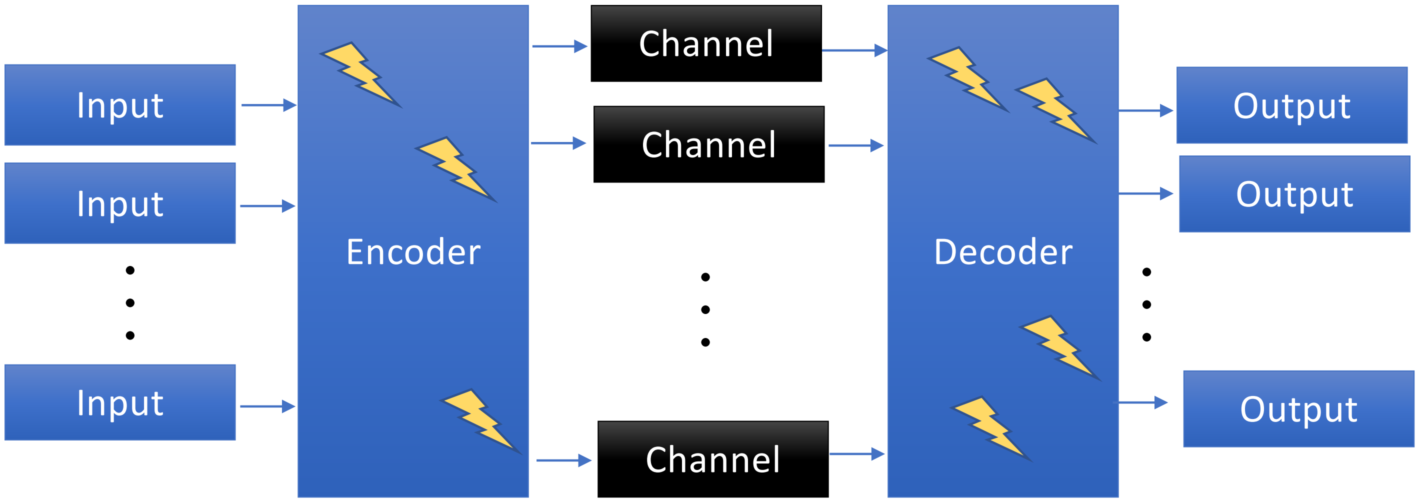
- **Capacity** $= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$

  = I(X:Y) bits/channel (+regularization)

H(X)+H(Y)-H(XY) ← von Neumann entropy

Observation

- Assumption of noiseless encoder/decoder unrealistic

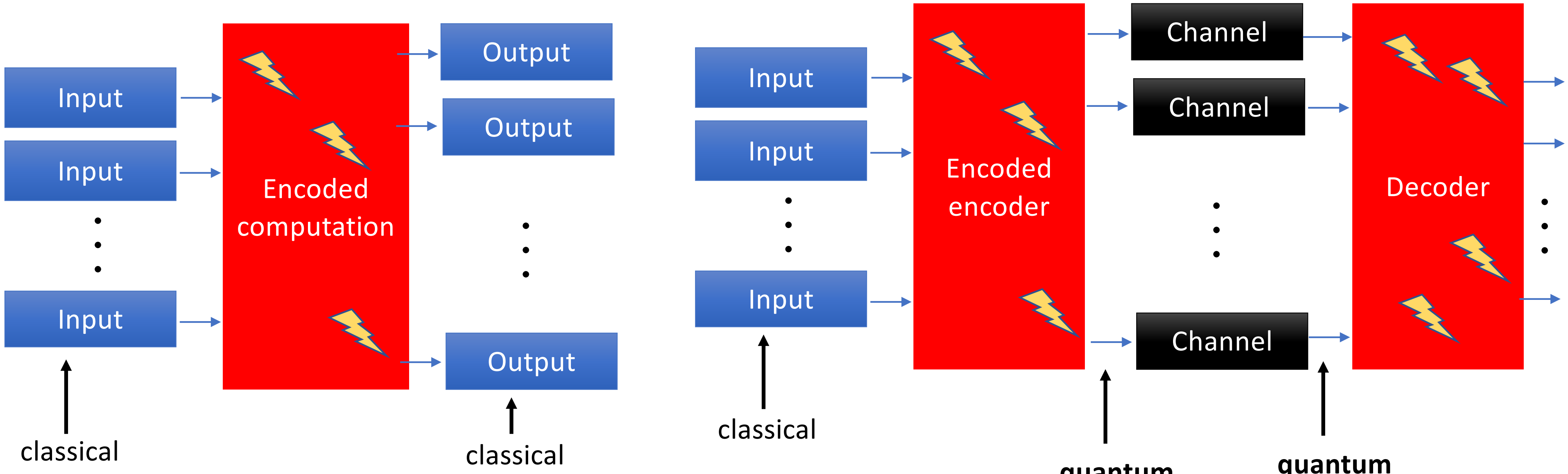


- Entirely new problems appear
  - Is quantum communication possible at all?
  - Is capacity formula continuous? $\delta \to 0$

gate error rate

Fault-tolerant quantum computing

- Has existed since the 90s, highly developed field
- Isn't the problem already solved?



- No, since output of encoder in communication case is not classical

The Plan

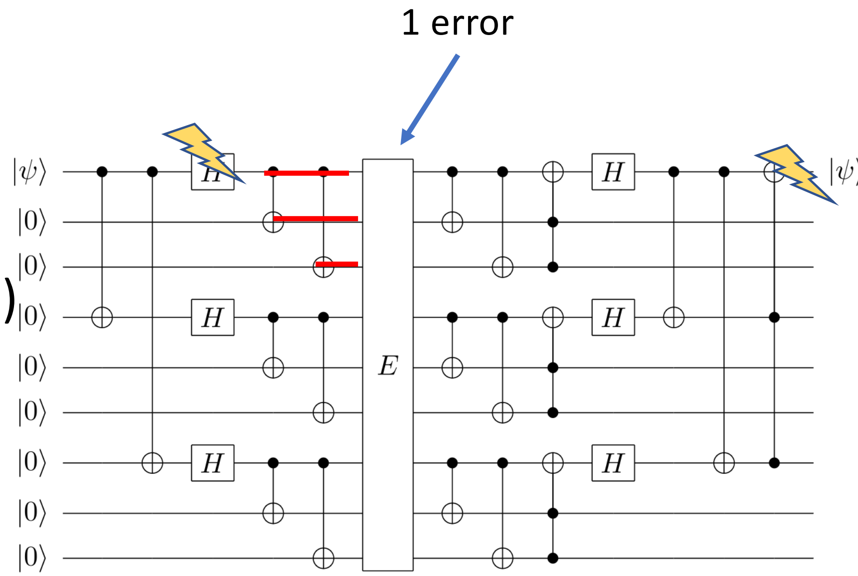
- Error Correction
- Fault-tolerant quantum computing
  - Setup
  - Threshold theorem
- Fault-tolerant quantum communication
  - Definition
  - Main result: threshold theorem
- Proof
  - Step 1: Reducing to effective channel
  - Step 2: Capacity of effective channel

Error correction

- Error correction (repetition code)
  - 0→(encoding) 000 → (1 error) 010 → (decoding) 0
  - 1→(encoding) 111 → (1 error) 101 → (decoding) 1
- Quantum error correction (Shor code, from Wiki)

$$|0_S\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

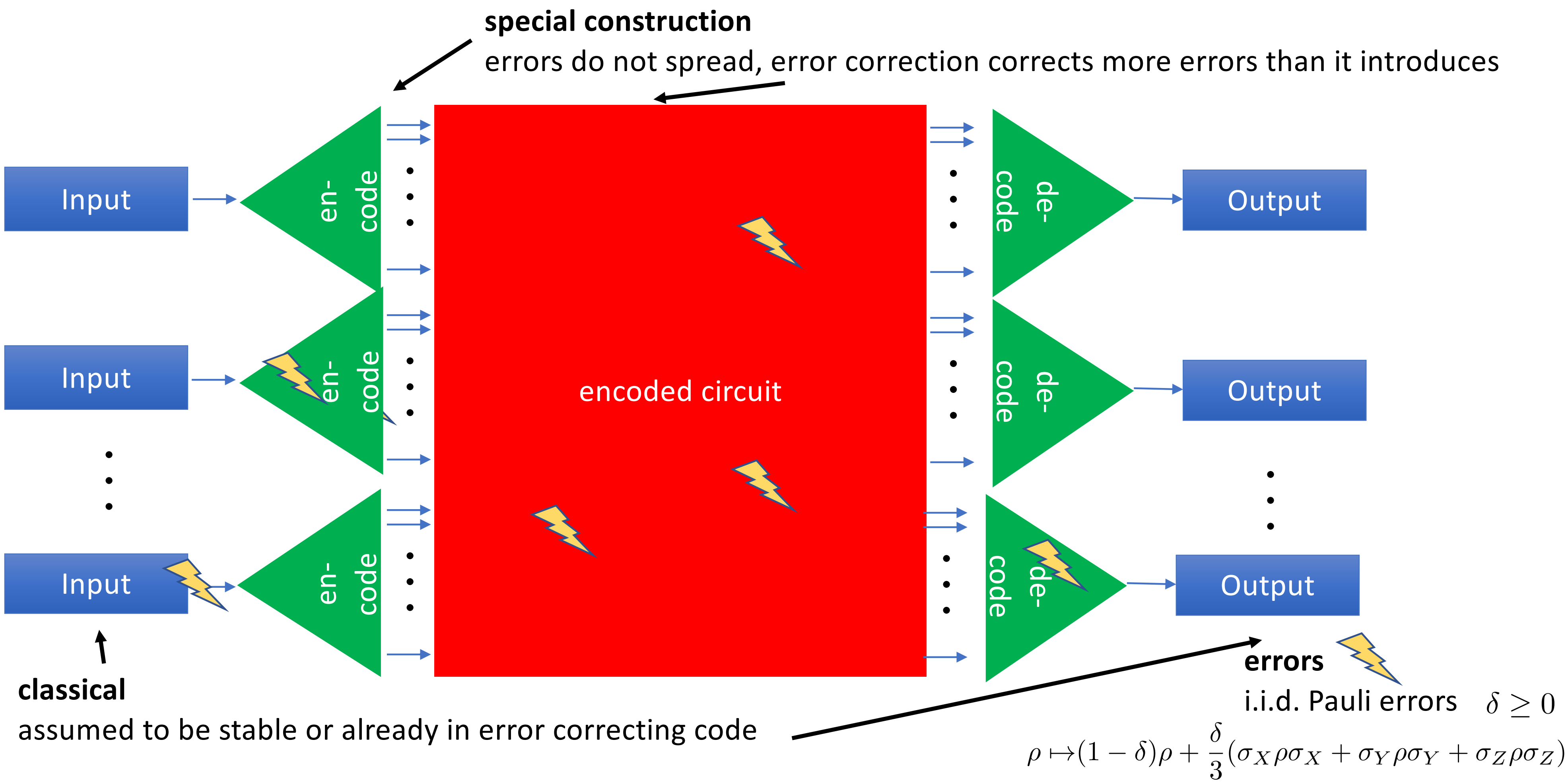
$$|1_S\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$



- What if gates in encoding or decoding have error?
  - Error could spread
  - Error correction could fail
- Cannot protect an unprotected qubit!

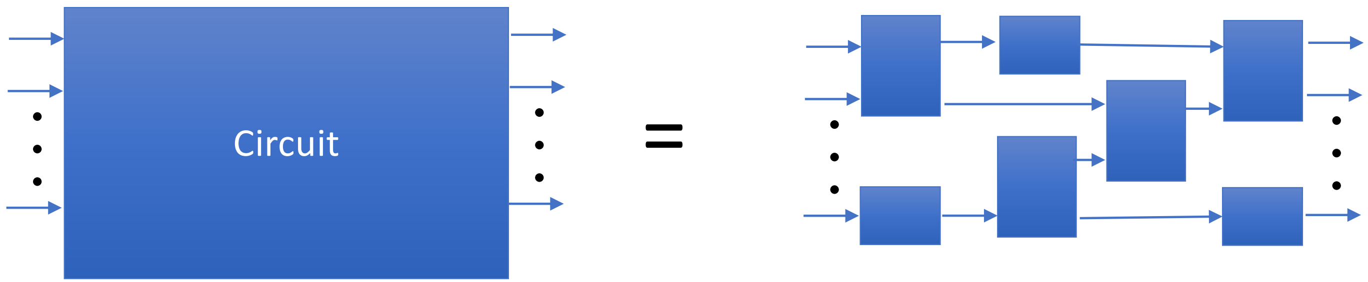
  At the last step, the noise might hit

Fault-tolerant quantum computing

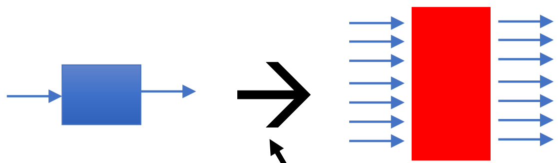


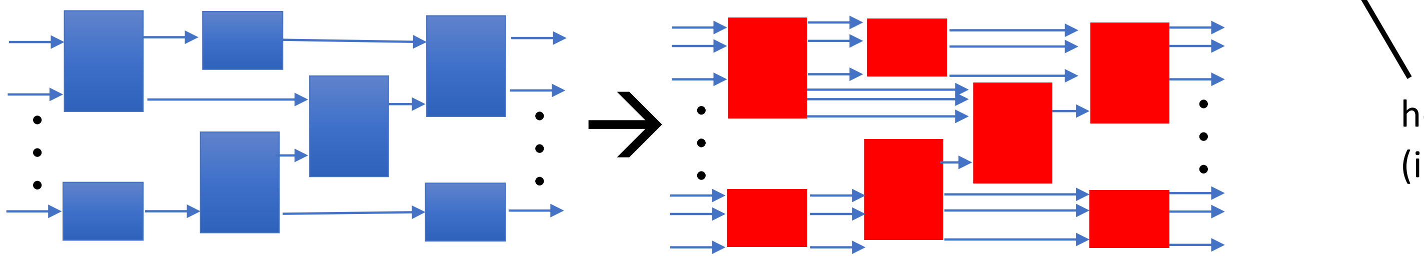
Gadgets

- pick a universal gate set (X, Y, Z, T, CNOT, id, prepare 0, 1, partial trace)



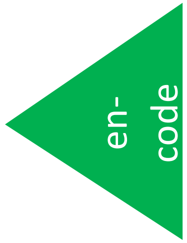
- every gate will be replaced by a gadget



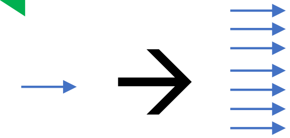


here for the 7-qubit Steane code (includes error correction)

Encoding

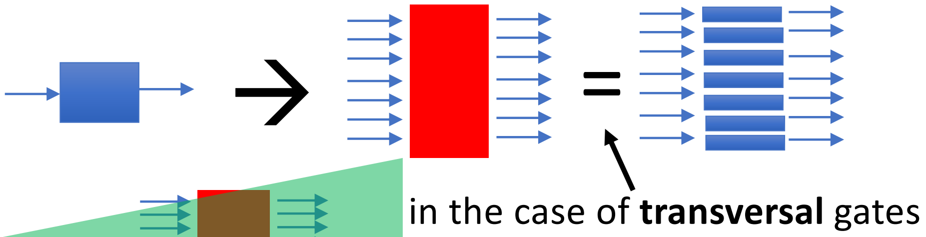


- 7 qubit Steane code

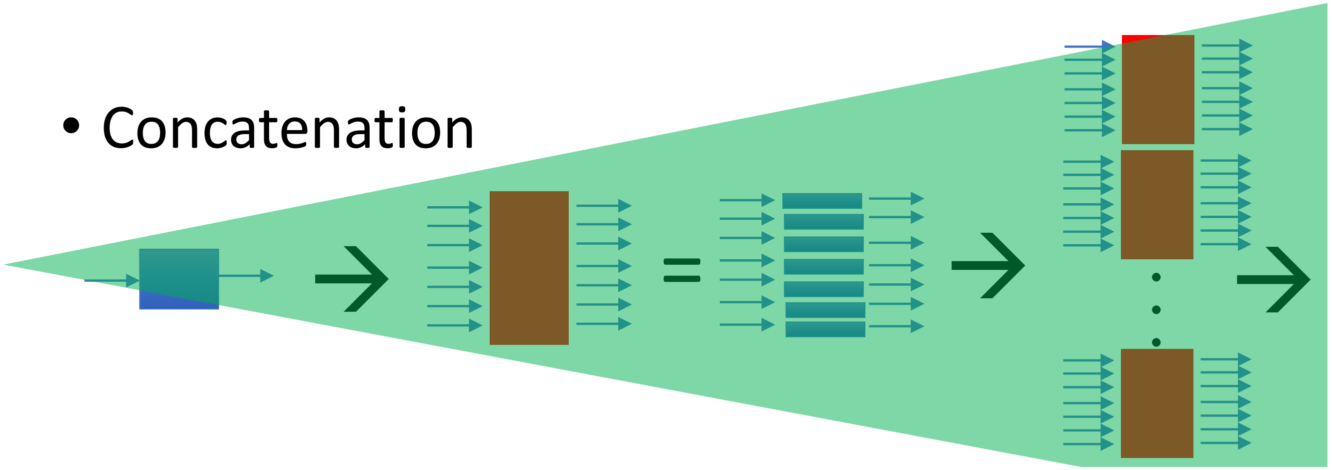


$$|0_L\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |101010101\rangle + |0110011\rangle + |1100110\rangle \\ + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$$

- Gates and gadgets
  Aliferis, Gottesman, Preskill

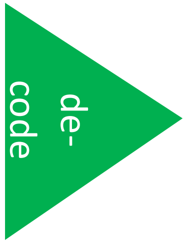


- Concatenation

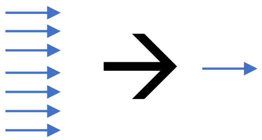


the encoding gets better
in every level
(doubly exponentially)

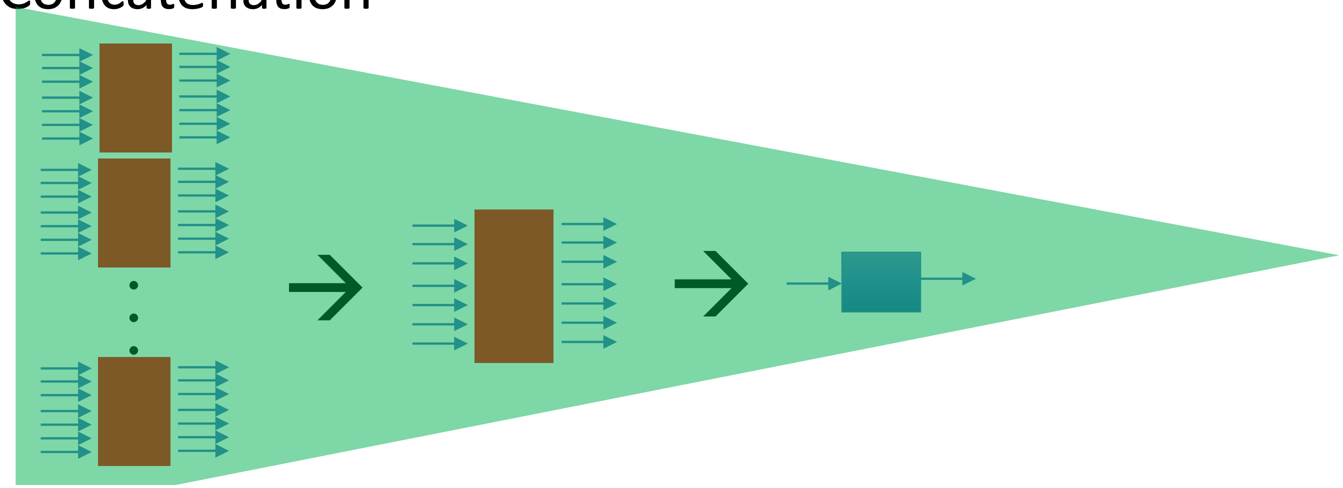
Decoding



- Error correction and decoding



- Concatenation



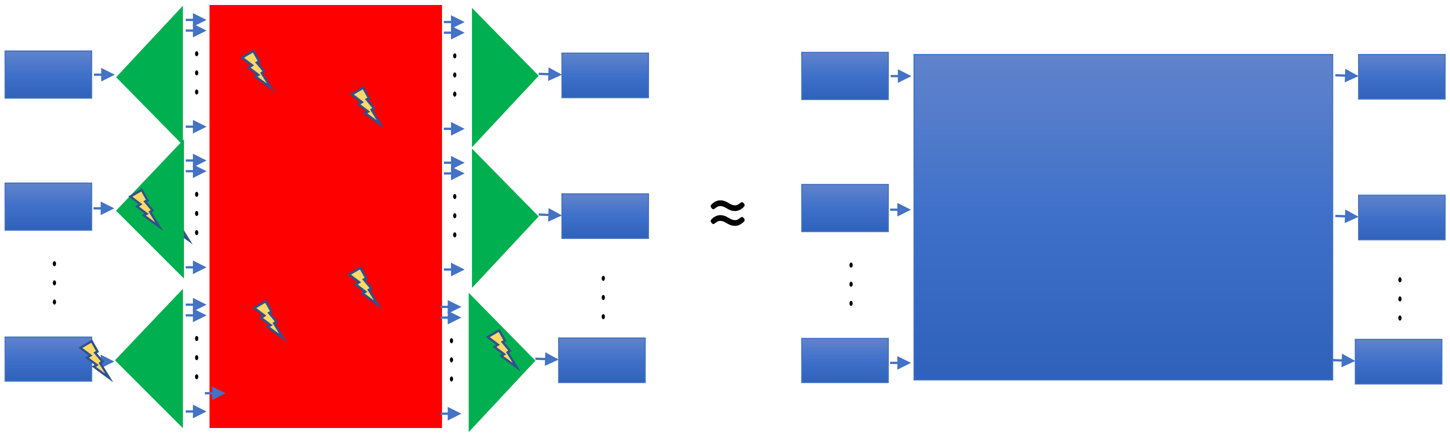
the total amount of error
= const* gate error
(Mazurek et al., 2014)

Threshold theorem for Quantum Computation

(Aliferis, Gottesman, Preskill, 2006)

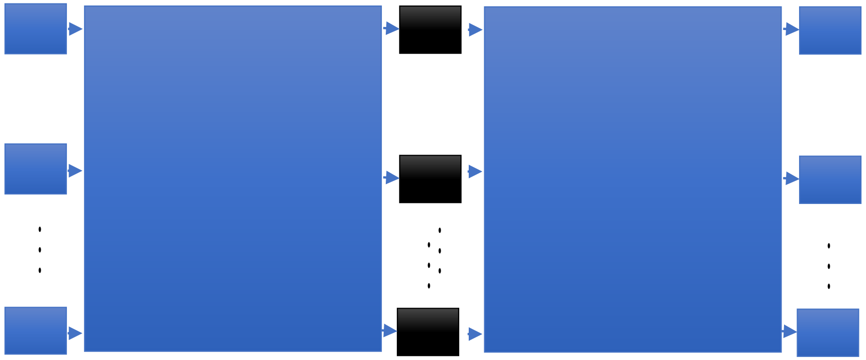
- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$ and all circuits and inputs

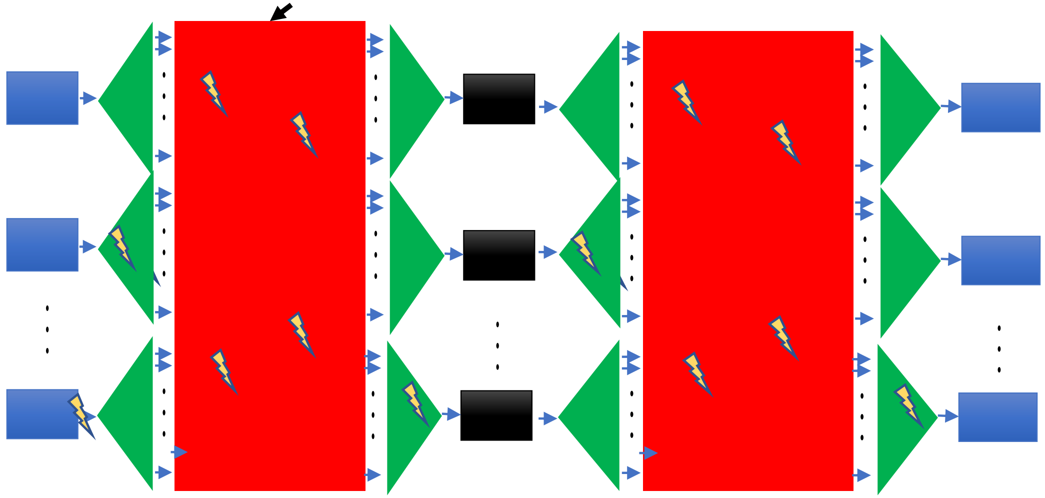
  Prob[Output(k-level, δ-noise)≠Output (0-level, 0-noise]<O(#gates) $\left(\frac{\delta}{\delta_0}\right)^{2^k}$



Fault-tolerant capacity (Christandl, Müller-Hermes, 2020)

noise is not allowed to decrease with growing *n*





Capacity(0-noise)

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$$

over encoders-decoder pairs with m input bits

number of channel uses

Capacity(δ-noise)

$$= \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \frac{m}{n} : Prob(input \neq output) \leq \epsilon \right\}$$

over fault-tolerant codes
& encoders-decoder pairs with m input bits

Threshold theorem for Quantum Communication

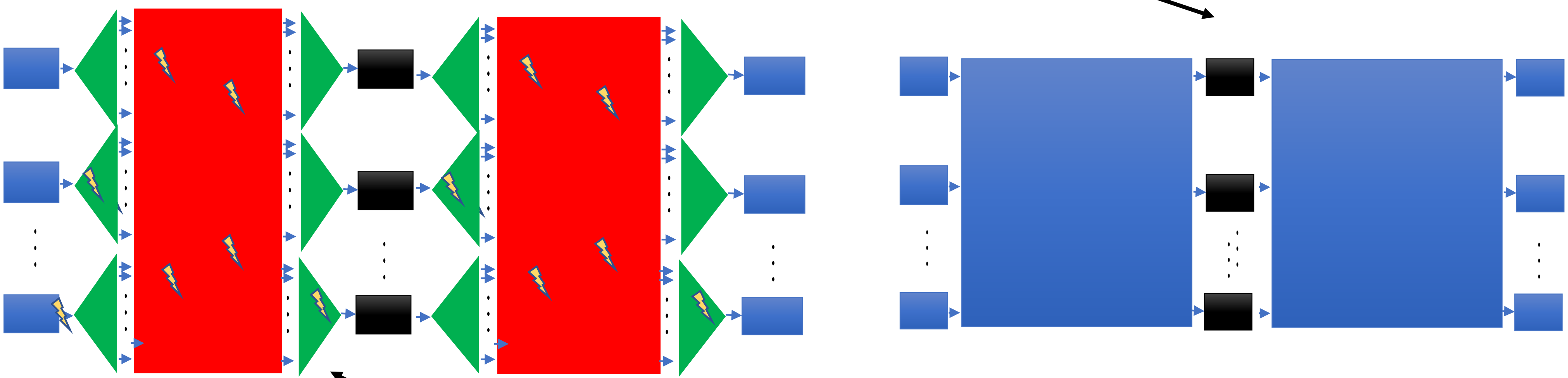
(Christandl, Müller-Hermes, 2020)

$f(\delta) \to 0, \text{ as } \delta \to 0$

can depend on channel

- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$

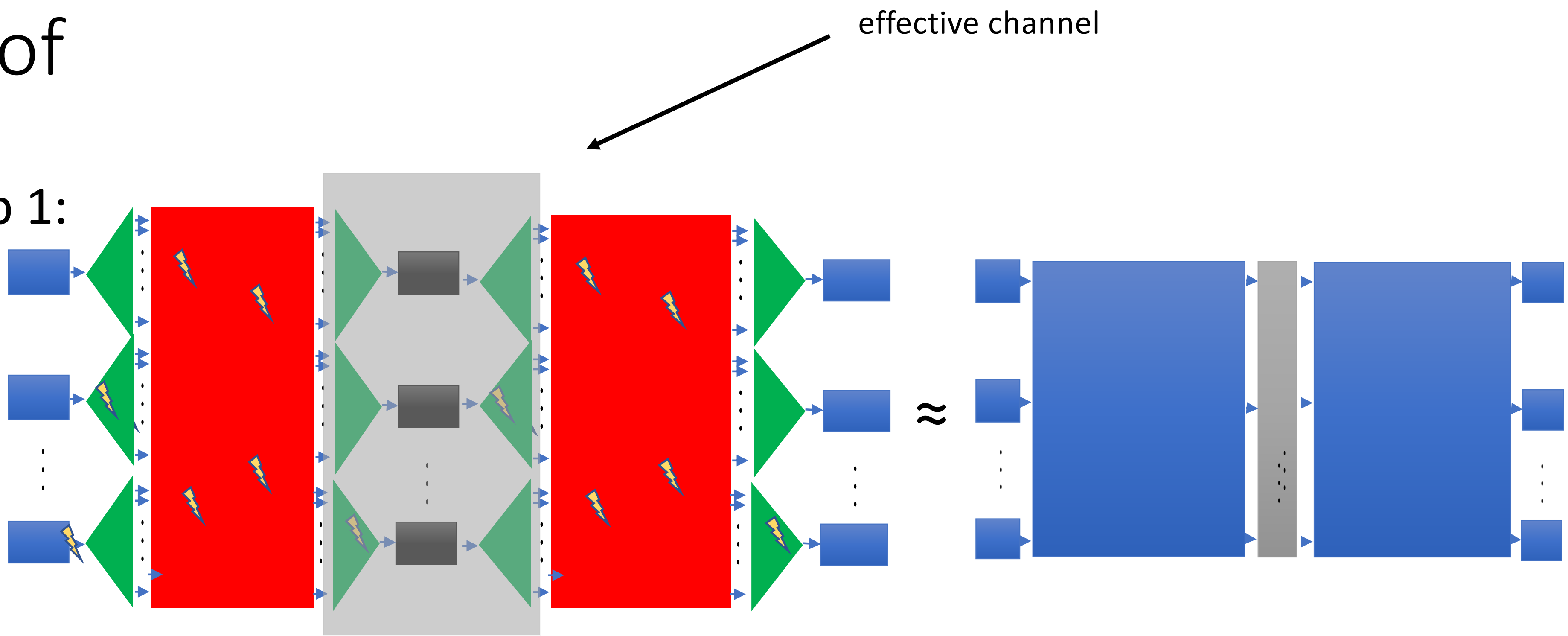
Capacity(δ-noise)>Capacity (0-noise]- $f(\delta)$



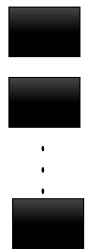
level can depend on *n*

Proof

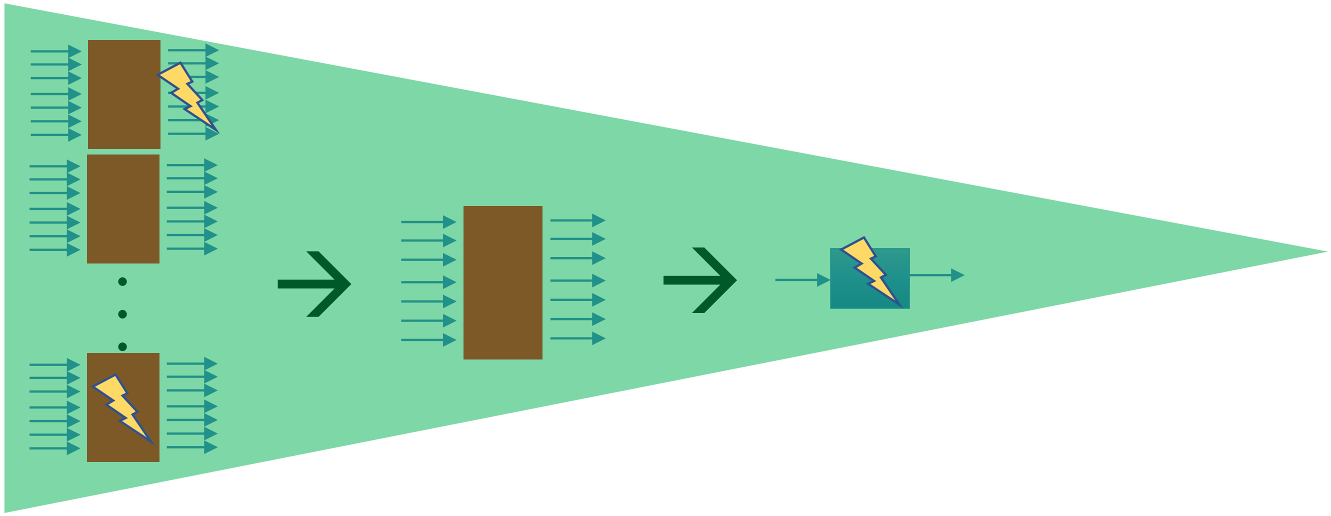
- Step 1:



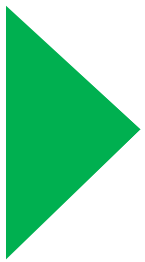
- Step 2:

  Effective channel  has similar capacity as 

Step 1: Single decoder

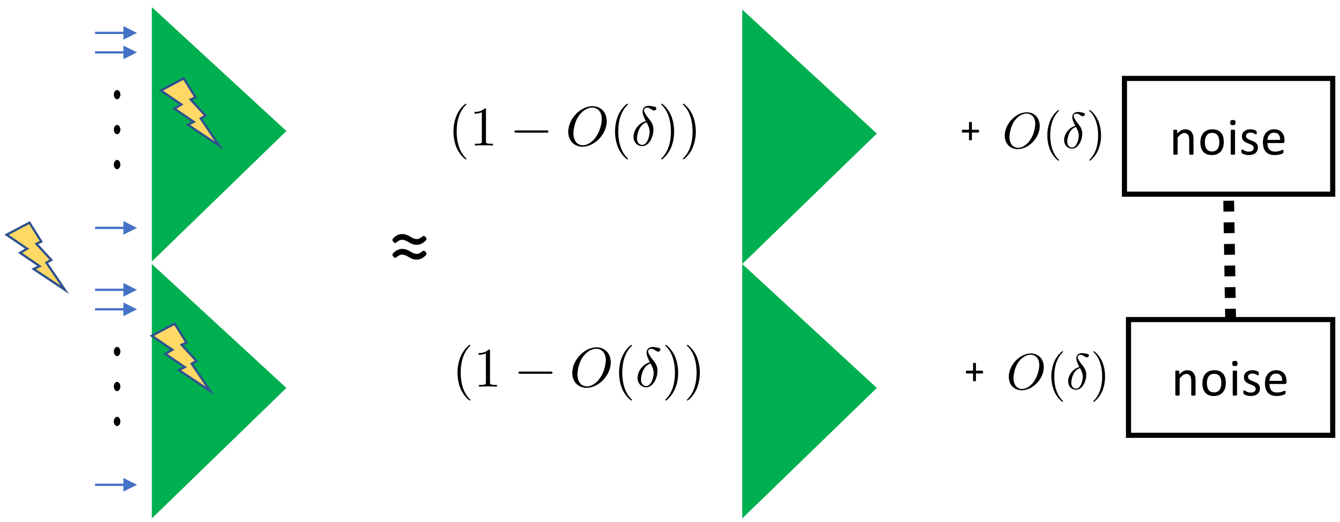


the total amount of error
= O(gate error)
(Mazurek et al., 2014)

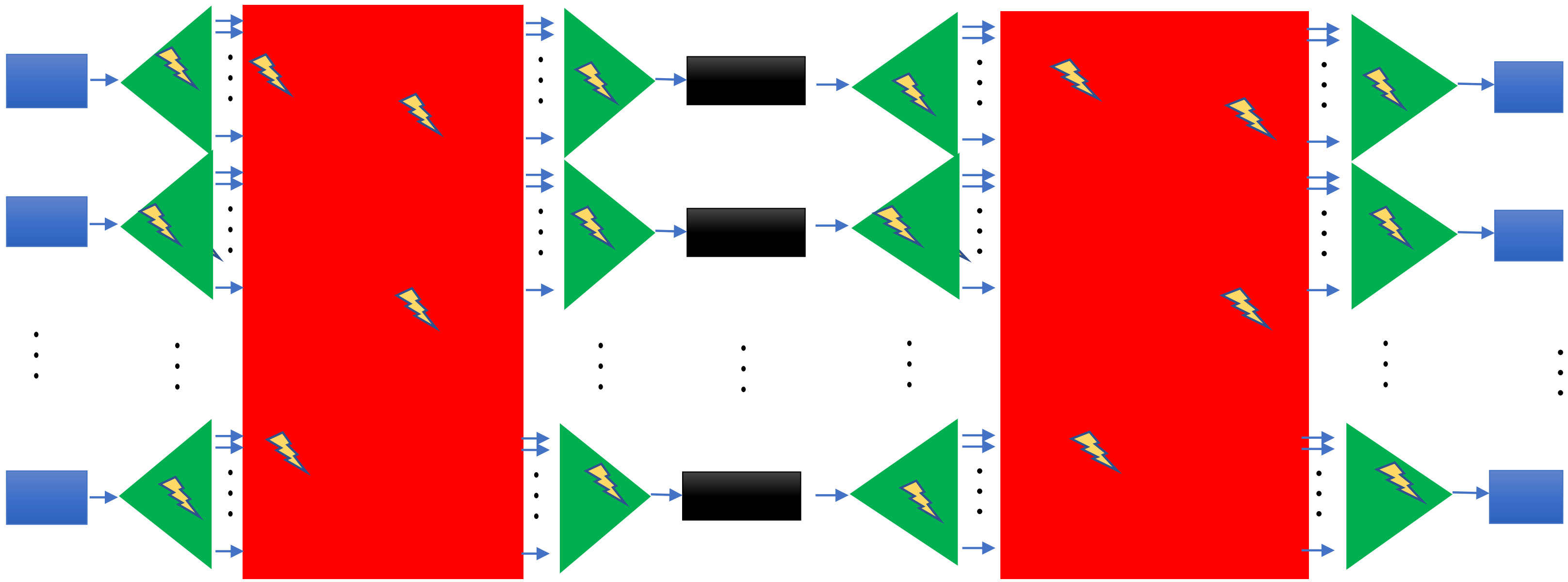
$\approx (1 - O(\delta))$  $+ O(\delta)$ noise

Step 1: Several decoders

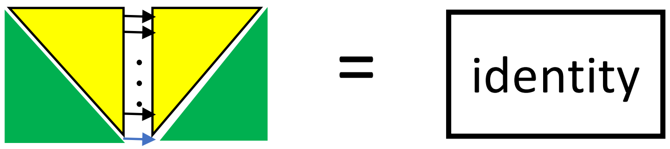
- Noise across channels can be correlated



Step 1: Construction of effective channel



- insert ideal decoder/encoder pair separating logical qubit from syndrome



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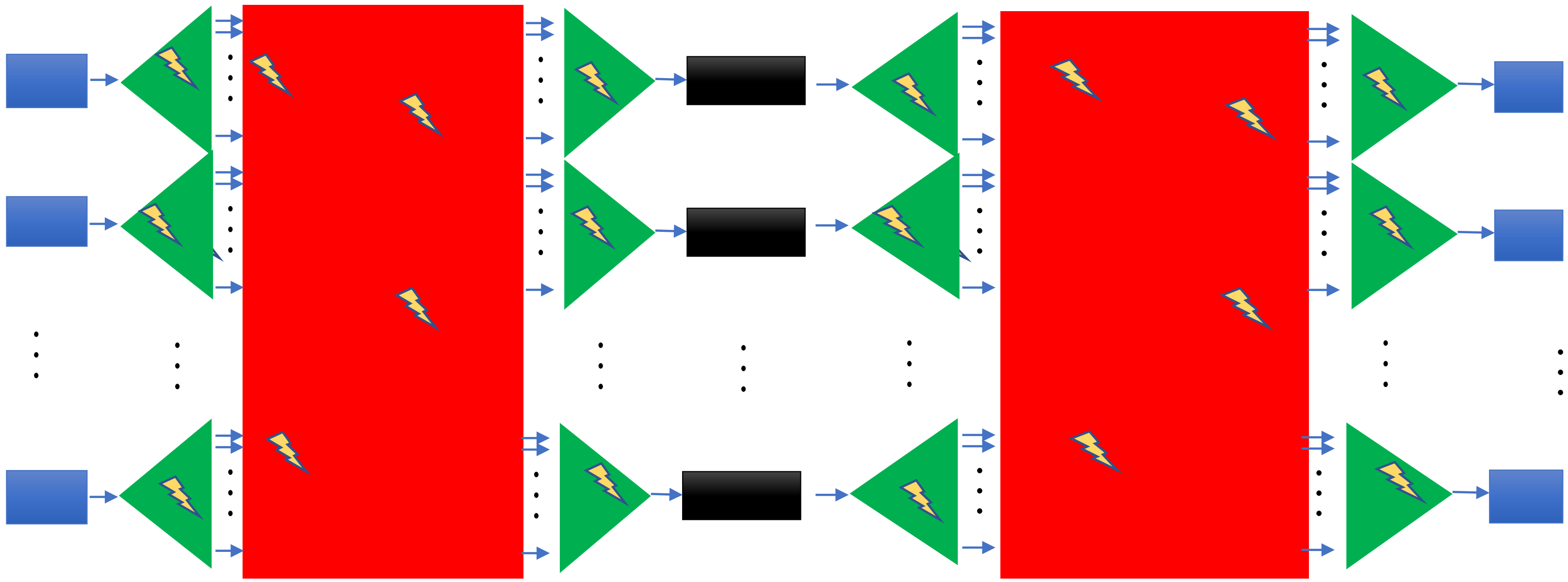
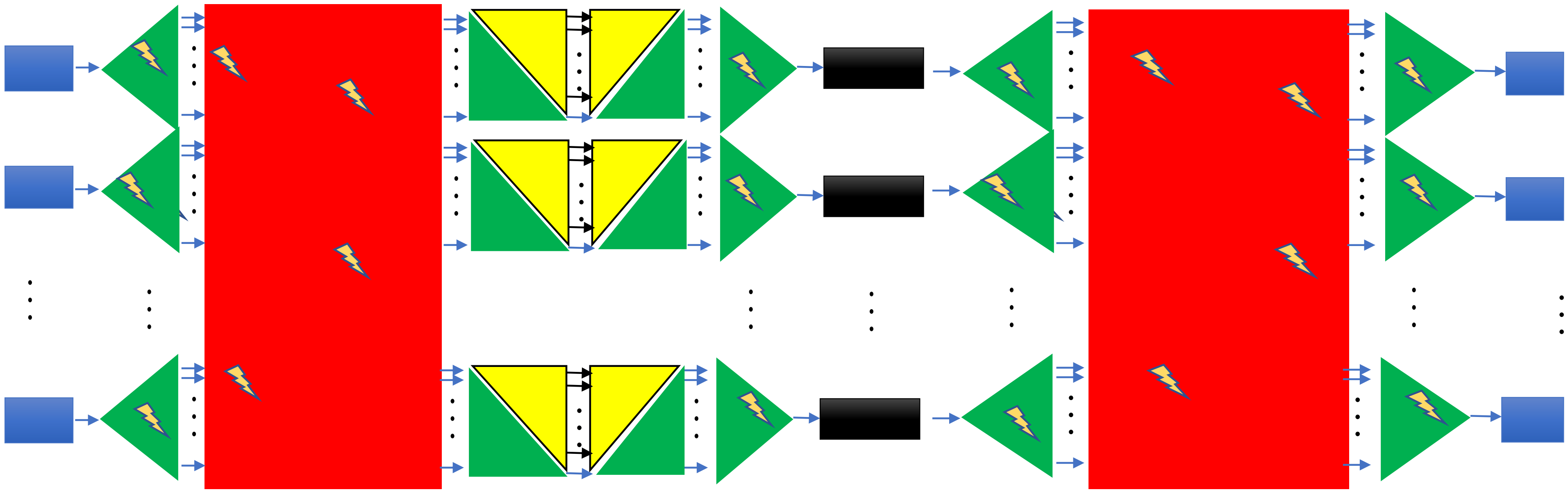
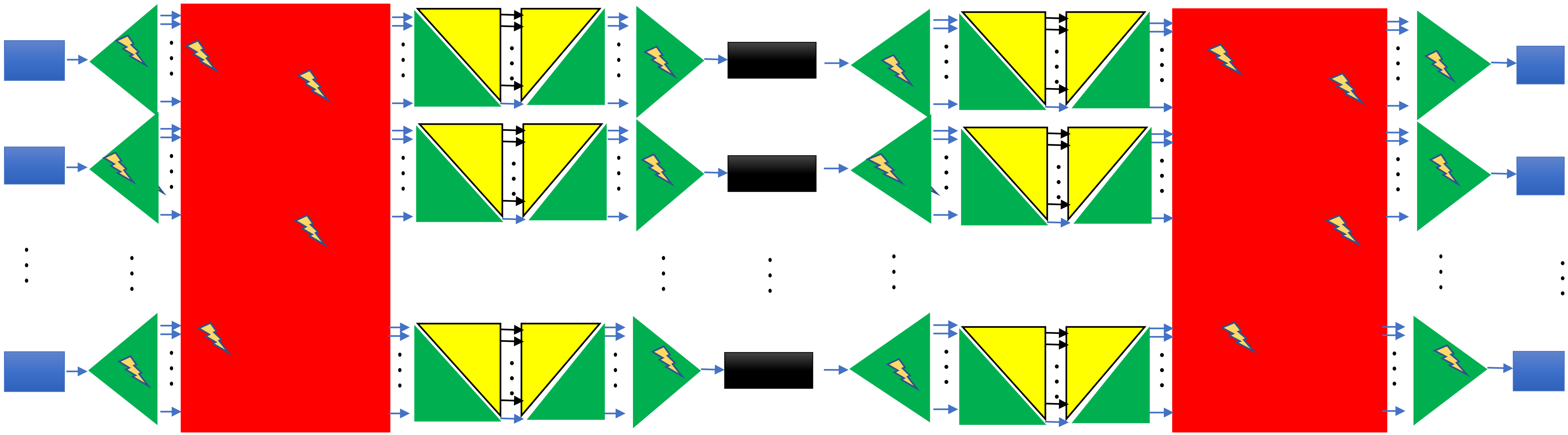


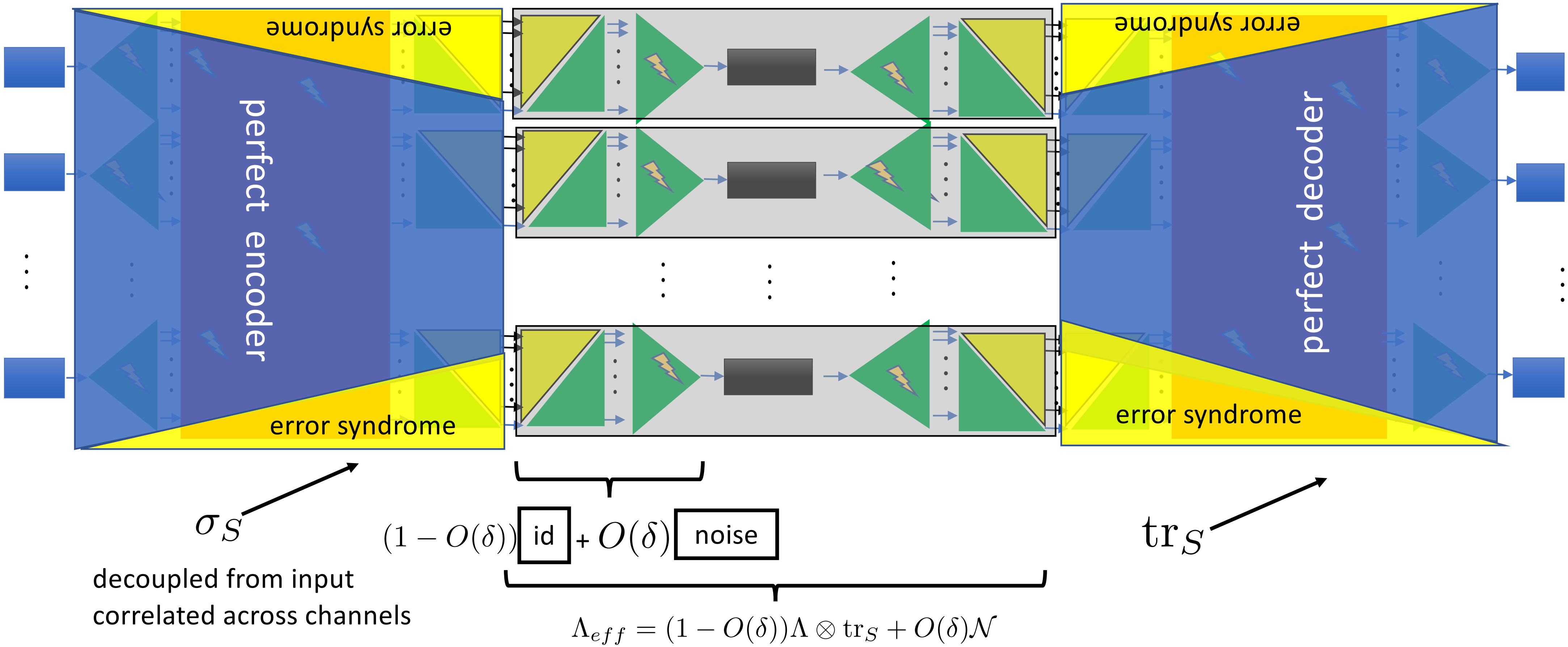
Step 1: Construction of effective channel



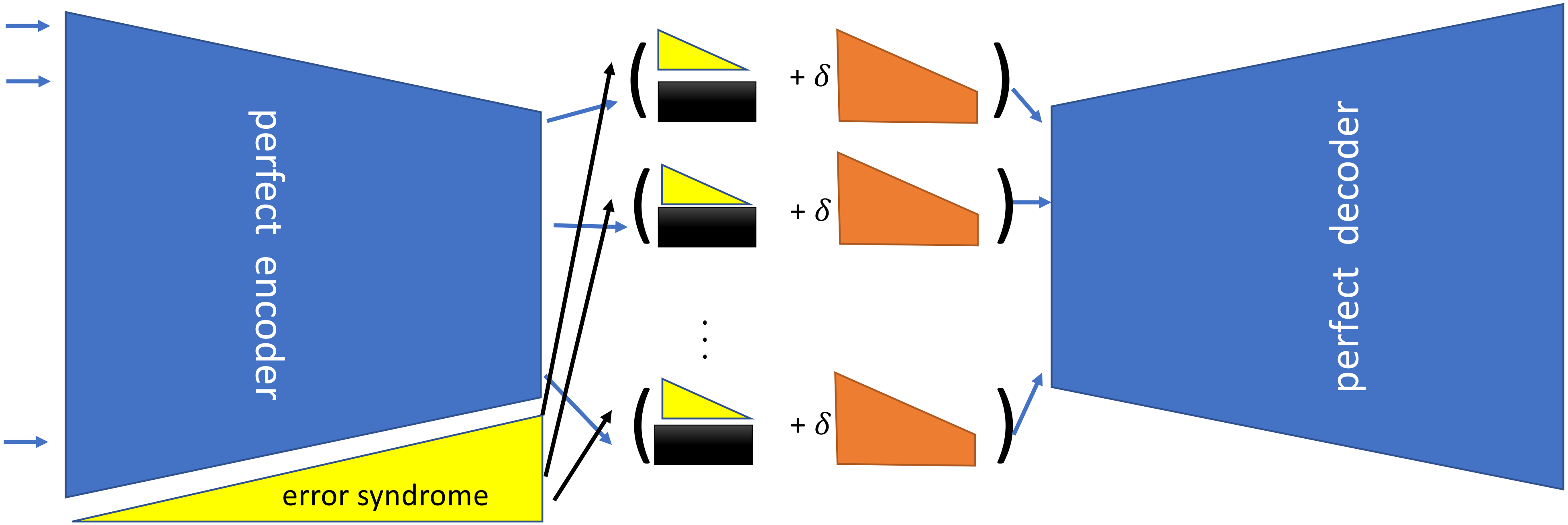
Step 1: Construction of effective channel



Step 1: Construction of effective channel



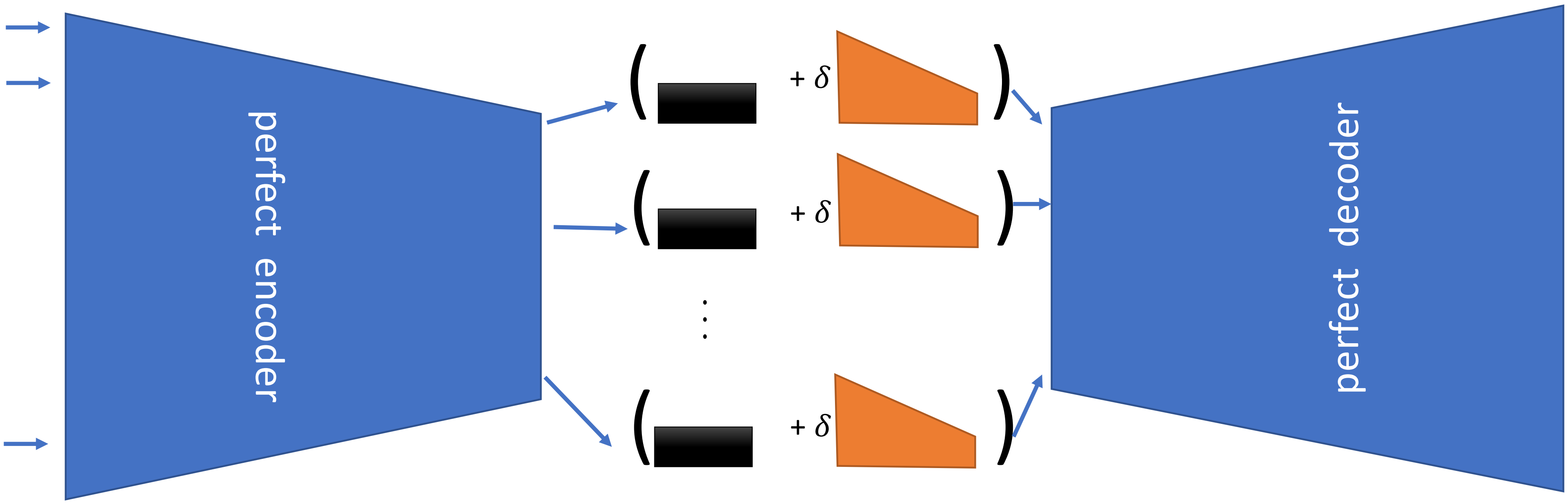
Step 1: Construction of effective channel



$$\Lambda_{eff} = (1 - O(\delta))\Lambda \otimes \mathrm{tr}_S + O(\delta)\mathcal{N}$$

Step 2: Capacity of effective channel

- Postselection technique→tensor product channels suffice



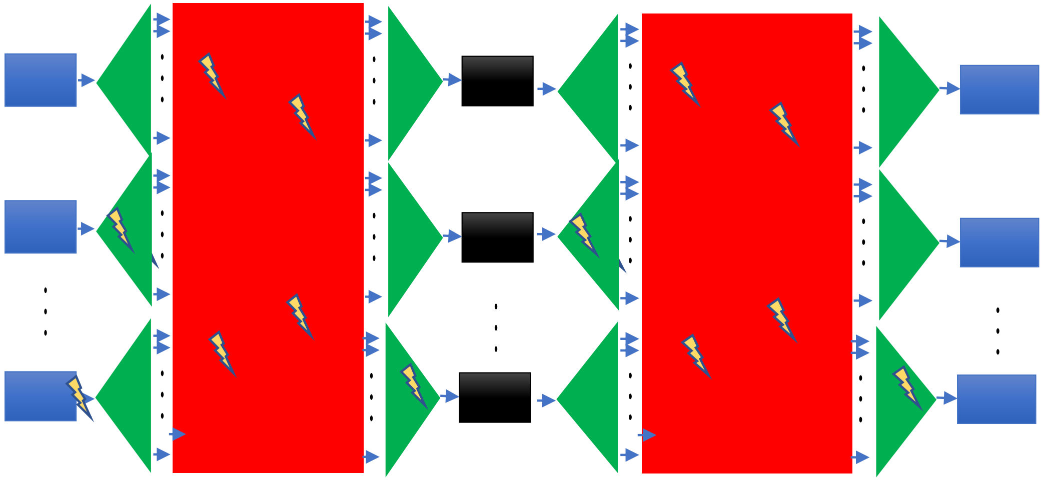
- Use standard random coding for capacity (explicit exp. bounds needed)
- Resulting Holevo capacity is continuous in δ, then regularize

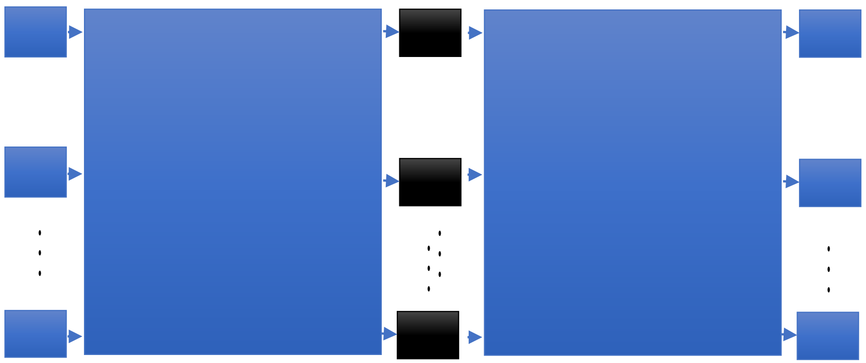
Threshold theorem for Quantum Communication

(Christandl, Müller-Hermes, 2020)

- There is a threshold $\delta_0 > 0$ s.th. for all $\delta \leq \delta_0$

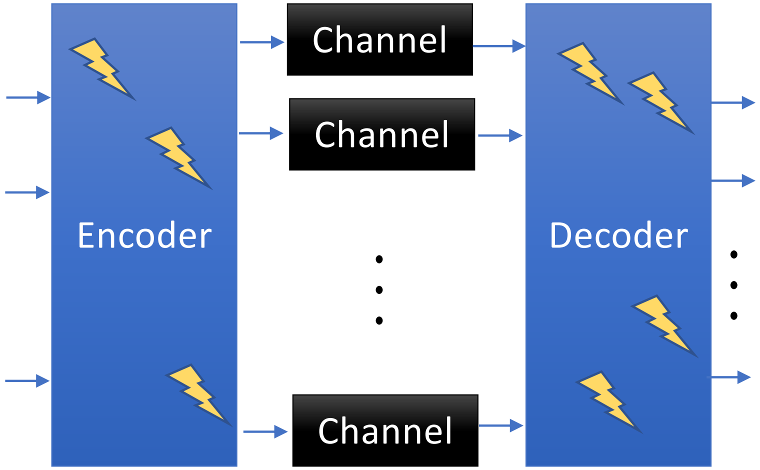
Capacity(δ -noise)>Capacity (0-noise]- $f(\delta)$





- Corollary: positivity and continuity

Summary



- New paradigm of fault-tolerant quantum communication
  - Fault-tolerant capacity defined
  - Positivity and continuity established
  - Also for quantum capacity
- Relevance for quantum communication
  - Theoretical: establishes that standard quantum Shannon theory is appropriate model
  - Experimental: gives explicit constructions
- Broader relevance in quantum computing
  - Several small quantum computers connected by noisy communication lines
  - e.g. superconducting qubits (since they are large)


Get in touch for PhD and postdoc positions!
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