Probabilistic Liouville Theory

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Liouville Theory

2d field theory with action functional

$$S(\phi) = \int (|\partial_z \phi(z)|^2 + \mu e^{\gamma \phi(z)}) d^2 z$$

- Uniformisation of Riemann surfaces (Picard, Poincaré1890's)
- Noncritical string theory (Polyakov 1981)
- 2d gravity Knizhnik, Polyakov, Zamolodchikov (1988)
- Quantum uniformisation (Taktajan, Zograf, Teschner)
- 4d SuSy Yang-Mills (Alday, Gaiotto, Tachikawa 2010)
- Random metric spaces (Le-Gall, Miermont, Duplantier, Sheffield, Miller)
- Probabilistic Liouville CFT (David, Guillarmou, K., Rhodes, Vargas)

2d Quantum Gravity

Polyakov '81: string theory in terms of **gravity on world sheet**

(Euclidean) gravity: Riemannian metric g on manifold Σ

(Euclidean) **quantum** gravity: g is **random**

$$\langle F(g,\Psi)\rangle = \int F(g,\Psi)e^{-S_{gravity}(g)-S_{matter}(g,\Psi)}DgD\Psi$$

Let Σ be **two dimensional**. Then modulo $Diff(\Sigma)$

$$g = e^{\sigma}\hat{g}$$

$\sigma:\Sigma\to\mathbb{R}$, $\hat{g}$ moduli.

What is the probability law of σ? Can we find $S_{effective}(\sigma)$:

$$\langle F(g)\rangle = \int F(e^{\sigma})e^{-S_{effective}(\sigma)}D\sigma.$$

Liouville Gravity

For **conformal** matter the answer is known (conjectured!).

Knizhnik, Polyakov, and Zamolodchikov ’88:

Let the matter be **conformal field theory** with central charge $c \leq 1$. Then

$$\sigma = \gamma\phi$$

and the probability distribution of ϕ is given by

$$\langle f(\phi) \rangle = \int f(\phi) e^{-S(\phi)} D\phi$$

with $S(\phi)$ the **Liouville action functional**:

$$S(\phi) = \int (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)}) d^2 z$$

KPZ correspondence

$$S(\phi) = \int (|\partial_z \phi(z)|^2 + \mu e^{\gamma \phi(z)}) d^2 z$$

The parameter γ depends on what the conformal field theory is:

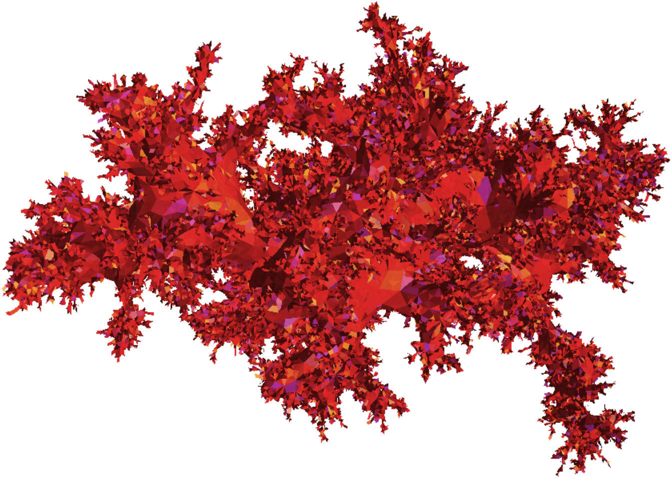
- $c = 25 - 6Q^2$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$
- $\gamma \in \mathbb{R} \Leftrightarrow Q^2 \geq 4 \Leftrightarrow c \leq 1$

The **cosmological constant** $\mu > 0$ but dependence on it is via simple scaling laws: it is **not** a perturbative parameter.

Liouville theory is the conjectured **scaling limit** of statistical mechanics models defined on **random lattices**. (Kazakov 1986)

$\gamma = \sqrt{2}$, Quantum Sphere

Random fractal surfaces



F. David

Gravitational Dressing

Example: Ising model $\gamma = \sqrt{3}$. Let

- σ be scaling limit of Ising spin
- $\tilde{\sigma}$ be scaling limit of Ising spin on a random triangulation

Then

$$\tilde{\sigma}(z) = e^{\alpha\phi(z)}\sigma(z)$$

with ϕ the $\gamma = \sqrt{3}$ Liouville field and $\alpha = \frac{5}{2\sqrt{3}}$.

Similar formuli for all $c \leq 1$ models (Potts, tricritical Ising, etc.)

Hence we need to understand correlation functions of **vertex operators** $e^{\alpha\phi(z)}$in Liouville theory:

$$\langle\prod_{i=1}^{n} e^{\alpha_i\phi(z_i)}\rangle = \int \prod_{i=1}^{n} e^{\alpha_i\phi(z_i)} e^{-S(\phi)} D\phi$$

Conformal Field Theory (CFT)

Euclidean QFT models **statistical physics**

At **critical temperature** such systems have **conformal symmetry** and the QFT is **conformal field theory**

This extra symmetry gives rise to strong constraints on correlation functions via **conformal bootstrap**

In 2 dimensions bootstrap was used by Belavin, Polyakov and Zamoldchicov (1984) to classify CFT's and find explicit expressions for the correlation functions in several cases

In more than 2 dimensions bootstrap has led to spectacular numerical predictions (e.g. 3d Ising model) by Rychkov and others.

Conformal invariance

Setup:

- **Scaling fields** $V_\alpha(x)$, $x \in \mathbb{R}^d$, e.g. Ising spin
- Expectation $\langle \cdot \rangle$

Correlation functions $\langle \prod_i V_{\alpha_i}(x_i) \rangle$ invariant under rotations and translations and under scaling

$$\langle \prod_i V_{\alpha_i}(\lambda x_i) \rangle = \prod_i \lambda^{-\Delta_{\alpha_i}} \langle \prod_i V_{\alpha_i}(x_i) \rangle$$

Δ_α scaling dimension or conformal weight

Conformal invariance: extends to conformal maps $x \to \Lambda(x)$,

E.g. in $d = 2$: $\mathbb{R}^2 \simeq \mathbb{C}$

$$\Lambda(z) = \frac{az + b}{cz + c} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

and $\lambda^{-\Delta_{\alpha_i}} \to |\Lambda'(z)|^{-\Delta_{\alpha_i}}$.

Structure Constants

3-point functions determined up to constants

$$\langle \prod_{k=1}^{3} V_{\alpha_k}(x_k) \rangle = |x_1 - x_2|^{2\Delta_{12}} |x_2 - x_3|^{2\Delta_{23}} |x_1 - x_3|^{2\Delta_{13}} C(\alpha_1, \alpha_2, \alpha_3)$$

with $\Delta_{12} = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$ etc.

$C(\alpha_1, \alpha_2, \alpha_3)$, the **structure constants** of the CFT.

Bootstrap

Operator Product Expansion Axiom:

$$V_{\alpha_1}(x_1)V_{\alpha_2}(x_2) = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(x_1, z_2, \partial_{x_2})V_{\alpha}(x_2)$$

a **convergent** sum assumed to hold when inserted to expectation:

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)\dots\rangle = \sum_{\alpha\in\mathcal{S}} C^{\alpha}_{\alpha_1\alpha_2}(x_1, x_2, \partial_{x_2})\langle V_{\alpha}(x_2)V_{\alpha_3}(x_3)\dots\rangle$$

- $C^{\alpha}_{\alpha_1\alpha_2}$ are **determined** by and **linear** in the structure constants
- $\mathcal{S}$ is called the **spectrum** of the CFT

Iterating OPE:

- All correlations are determined by $C(\alpha_1, \alpha_2, \alpha_3)$

Upshot: to “solve a CFT“ need to find its spectrum and structure constants.

Bootstrap equation for structure constants

Compute 4-point function in two ways:

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_2} \langle V_{\alpha} V_{\alpha_3} V_{\alpha_4} \rangle = \sum_{\alpha \in \mathcal{S}} C^{\alpha}_{\alpha_1 \alpha_3} \langle V_{\alpha} V_{\alpha_2} V_{\alpha_4} \rangle$$

This becomes a **quadratic equation** for structure constants.

It has proven to be a very constraining condition c.f. 3d Ising model.

In **two dimensions** many explicit solutions are known.

Solutions

Compare w. harmonic analysis on compact/noncompact groups:

1. **Compact CFT's**

(a) $\mathcal{S}$ is **finite**: minimal models (e.g. Ising model)
Belavin, Polyakov, Zamolodchicov (1983)

(b) $\mathcal{S}$ is **countable**: compact G WZW models, G/H coset theories

Explicit formuli for $C(\alpha_1, \alpha_2, \alpha_3)$ in terms of Coulomb gas integrals (Dotsenko,Fateev, ...)

2. **Non-compact CFT's**

$\mathcal{S}$ is **continuous**: **Liouville model**, Toda CFT's, WZW with noncompact group

Explicit formula for $C(\alpha_1, \alpha_2, \alpha_3)$ conjectured by Dorn, Otto, Zamolodchicov, Zamolodchicov (1995) (the **DOZZ formula**).

Constructive CFT

Try to find examples satisfying the Axioms from **functional integrals** over fields $\phi : \Sigma \to M$

$$\langle \prod_\alpha V_\alpha \rangle = \int \prod_\alpha V_\alpha(\phi) e^{-S(\phi)} D\phi$$

Minimal models $M = \mathbb{R}$ and S is (scaling limit of) $P(\phi)_2$ QFT:

$$S(\phi) = \int_{\mathbb{C}} (|\partial_z \phi(z)|^2 + P(\phi(z)))dz$$

with P, V_α polynomials in ϕ with unknown coefficients.

WZW models $M = G$ Lie Group, S explicit

Direct analysis from functional integral hard.

Liouville CFT

Scaling fields are formally the **vertex operators**

$$V_\alpha(z) = e^{\alpha\phi(z)}, \quad \alpha \in \mathbb{C}$$

and

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle = \int \prod_i e^{\alpha_i \phi(z_i)} e^{-\int (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)}) dz} D\phi$$

- $\mu > 0$ is **not** a perturbative parameter: $\phi \to \phi + a \Leftrightarrow \mu \to e^{\gamma a}\mu$
- γ only parameter
- $c = 1 + 6Q^2$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$
- $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$

Spectrum and structure constants

Curtright, Thorn (82) conjectured: **spectrum** of LCFT is **continuous** given by the vertex operators

$$V_{Q+ip}(z) = e^{(Q+ip)\phi(z)}, \quad p \in \mathbb{R}$$

What are the **structure constants**?

Polyakov: BPZ conformal field theory was an "unsuccessful attempt to solve Liouville theory"

In 1995 Zorn and Otto and Zamolodchicov and Zamolodchicov proposed a remarkable formula for the Liouville structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle$$

DOZZ formula

$$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3) = \hat{\mu}^{-s}\frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\alpha_1+\alpha_2+\alpha_3-2Q}{2})\Upsilon(\frac{\alpha_2+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_3}{2})\Upsilon(\frac{\alpha_1+\alpha_2}{2})}$$

- $\hat{\mu} = \frac{\pi\Gamma(\frac{\gamma^2}{4})(\frac{\gamma}{2})^{\frac{4-\gamma^2}{2}}}{\Gamma(1-\frac{\gamma^2}{4})}\mu$
- Υ is an entire function on $\mathbb{C}$ defined by

$$\Upsilon(\alpha)^{-1} = \Gamma_2(\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})\Gamma_2(2Q-\alpha|\tfrac{\gamma}{2},\tfrac{2}{\gamma})$$

$C_{DOZZ}(\alpha_1,\alpha_2,\alpha_3)$ has simple poles in α_i on

$$\{-\tfrac{\gamma}{2}\mathbb{N}-\tfrac{2}{\gamma}\mathbb{N}\}\cup\{Q+\tfrac{\gamma}{2}\mathbb{N}+\tfrac{2}{\gamma}\mathbb{N}\}$$

Liouville Bootstrap

C_{DOZZ} solves the quadratic bootstrap equations numerically

This and the above spectrum would imply the bootstrap formula

$$\langle e^{\alpha_1\phi(0)}e^{\alpha_2\phi(z)}e^{\alpha_3\phi(1)}e^{\alpha_4\phi(\infty)}\rangle = \int_{\mathbb{R}_+}|z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha,p,z)|^2 \times C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)C_{DOZZ}(\alpha_3,\alpha_4,Q-ip)dp$$

$\mathcal{F}(\alpha,p,z)$ purely representation theoretic **spherical conformal blocks** determined by c,α_i,p.

Constructive LCFT

1. Give a mathematical meaning to the functional integral

$$\langle \prod_i e^{\alpha_i \phi(z_i)} \rangle = \int \prod_i e^{\alpha_i \phi(z_i)} e^{-S_L(\phi)} D\phi$$

2. Prove

$$\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

3. Find the spectrum of LCFT and prove the bootstrap formula for the four point function and *n*-point functions as well as on Riemann surfaces of genus ≥ 1.

Probabilistic Liouville model

What is the mathematical meaning of the integral

$$\langle F \rangle = \int F(\phi) e^{-\int_\Sigma (|\partial_z \phi(z)|^2 + \mu e^{\gamma \phi(z)}) dz} D\phi$$

We define it in terms of the **Gaussian Free Field** (GFF) on (Σ, g):

$$X(z) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{\lambda_n}} e_n(z)$$

- e_n are eigenfunctions of Laplace-Beltrami Δ_g

$$-\Delta_g e_n = \lambda_n e_n, \quad n \geq 0$$

- x_n i.i.d. normal random variables variance 1

$$\phi(z) = c + X(z)$$

where $c \in \mathbb{R}$ is the constant mode of ϕ.

Renormalisation

The GFF X is not a function but a **distribution**:

$$\mathbb{E}X(z)X(z') = \log|z - z'|^{-1} + \text{regular}$$

i.e.

$$\mathbb{E}X(z)^2 = \infty.$$

To define $e^{\gamma X(z)}$ we need to first **regularize** $X \to X_\epsilon = \rho_\epsilon * X$ and **renormalize** by Wick ordering

$$e^{\gamma X(z)} \to \lim_{\epsilon \to 0} e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E}X_\epsilon(z)^2}.$$

In what sense this converges?

Gaussian Multiplicative Chaos

Convergence is in the sense of weak convergence of measures:

$$\lim_{\epsilon \to 0} \int f(z) e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E} X_\epsilon(z)^2} dv_g(z) = \int f(z) dM(z) \quad \text{in probability}$$

M is called **Gaussian Multiplicative Chaos** measure.

M is a **random multifractal measure**:

$$\mathbb{E}\Big(\int_{|z|<r} dM(z)\Big)^p \sim r^{\xi(p)}, \quad p < 4/\gamma^2$$

with $\xi(p) = \gamma Q p - \frac{1}{2}\gamma^2 p^2$.

Freezing transition (Kahane): $M = 0$ if $\gamma \geq 2$ "$c = 1$ barrier"

Probabilistic Liouville Theory

The functional integral is then defined by

$$\langle F(\phi)\rangle := \int_{\mathbb{R}} e^{-\chi(\Sigma)Qc}\mathbb{E}\left[F(c+X)e^{-\mu e^{\gamma c}M(\Sigma)}\right] dc$$

where $\chi(\Sigma)$ is the Euler characteristic $2-2\times\mathrm{genus}$ and

$$M_\gamma(\Sigma) = \int_\Sigma dM_\gamma(z)$$

is the total mass of the GMC.

Vertex operator correlation functions

$$\langle\prod_{i=1}^{n} V_{\alpha_i}(z_i)\rangle = \int_{\mathbb{R}} e^{-\chi(\Sigma)Qc}\mathbb{E}\left[\prod_{i=1}^{n} e^{\alpha_i(c+X(z_i))}e^{-\mu e^{\gamma c}M(\mathbb{C})}\right] dc$$

are defined by similar renormalisation (Wick ordering) as well.

Existence

Theorem (David, K, Rhodes, Vargas, 2015) *The Liouville correlation functions exist and are nontrivial if and only if the* ***Seiberg bounds*** *hold:*

$$(1) \quad \alpha_i < Q \ \ \forall i, \ \ \text{and} \quad (2) \ \sum_{i=1}^{n} \alpha_i + \chi(\Sigma) Q > 0$$

V_α *are primary fields with scaling dimension* $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

(2): convergence of c-integral

(1): regularity of GMC

0-mode

Integrate over the zero mode c:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \mathbb{E}\left[\prod_i e^{\alpha_i X(z_i)} \int_{\mathbb{R}} e^{(\sum_i \alpha_i + \chi(\Sigma)Q)c - \mu e^{\gamma c} M(\mathbb{C})} dc \right]$$

The c-integral converges **if** $\sum_i \alpha_i + \chi(\Sigma)Q$:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \frac{\Gamma(s)}{\mu^s \gamma} \mathbb{E}\left[\prod_i e^{\alpha_i X(z_i)} M(\mathbb{C})^{-s} \right]$$

where $s := \frac{\sum_i \alpha_i + \chi(\Sigma)Q}{\gamma}$.

Pinning transition

Shift in X-integral (Girsanov theorem):

$$X(z) \to X(z) - \sum_i \alpha_i \log|z - z_i|$$

gives

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \frac{\Gamma(s)}{\gamma \mu^s \prod |z_i - z_j|^{\alpha_i \alpha_j}} \mathbb{E} \left(\int \prod_i \frac{1}{|z - z_i|^{\gamma \alpha_i}} dM(z) \right)^{-s}$$

Now $\frac{1}{|z-z_i|^{\gamma\alpha_i}}$ is M-integrable (almost surely) **if and only if**

$$\alpha_i < Q \quad \text{i.e.} \quad \gamma \alpha_i < 2 + \tfrac{\gamma^2}{2}.$$

Particle in a random potential

$$-X(z) + \sum_i \alpha_i \log|z - z_i|$$

at inverse temperature γ

Structure constants

For the structure constants we take $\Sigma = S^2$

$$C(\alpha_1, \alpha_2, \alpha_3) := \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle =$$

$$= \frac{2}{\gamma} \mu^{-s} \Gamma(s) \lim_{u \to \infty} |u|^{4\Delta_{\alpha_3}} \mathbb{E} \left(\int \frac{|w \vee 1|^{\gamma(\alpha_1+\alpha_2+\alpha_3)}}{|w|^{\gamma\alpha_1} |w-1|^{\gamma\alpha_2} |w-u|^{\gamma\alpha_3}} M(dw) \right)^{-s}$$

in the region

$$s := \frac{\alpha_1 + \alpha_2 + \alpha_3 - 2Q}{\gamma} > 0, \quad \alpha_i < Q$$

Integrability

Does the probabilistic expression satisfy the DOZZ formula?

Theorem (K, Rhodes, Vargas, Annals of Mathematics **191**, 81) Let α_i satisfy the Seiberg bounds. Then

$$C(\alpha_1, \alpha_2, \alpha_3) = C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$$

Proof combines **probabilistic** analysis of GMC to derive **algebraic** identities for the structure constants that determine them uniquely (Teschner).

1. For $\alpha = -\frac{\gamma}{2}$ **or** $\alpha = -\frac{2}{\gamma}$

$$F(z) := \langle e^{\alpha\phi(z)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

satisfies a **hypergeometric equation**.

Proof: Gaussian integration by parts.

2. GMC analysis (OPE):

$$F(z) = C(\alpha_1 + \alpha, \alpha_2, \alpha_3) + |z|^\beta A(\alpha_1, \alpha) C(\alpha_1 - \alpha, \alpha_2, \alpha_3) + \dots$$

3. 1& 2 $\implies \alpha_1 \to C(\alpha_1, \alpha_2, \alpha_3)$ is **doubly periodic**

$$C(\alpha_1 - \alpha, \alpha_2, \alpha_3) = D(\alpha, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 + \alpha, \alpha_2, \alpha_3)$$

4. A'priori bounds $\implies$ unique solution, DOZZ.

4-point function

Möbius covariance: 4-point function on S^2 depends on $z \in \mathbb{C}$:

$$G_4(z) = \langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_2}(1) V_{\alpha_4}(\infty) \rangle$$

Probabilistic formula

$$G_4(z) = \frac{2\mu^{-s}}{\gamma}\Gamma(s) \lim_{u \to \infty} |u|^{4\Delta_{\alpha_3}} \mathbb{E}\left(\int \frac{|w \vee 1|^{\gamma(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}}{|w|^{\gamma\alpha_1}|w-z|^{\gamma\alpha_2}|w-1|^{\gamma\alpha_3}|w-u|^{\gamma\alpha_4}} M(dw) \right)^{-s}$$

Bootstrap : Can we express $G_4(z)$ in terms of 3-point functions?

4-point Bootstrap

Theorem. (GKRV 2020) Let α_i satisfy Seiberg bounds with $\alpha_1+\alpha_2>Q$ and $\alpha_3+\alpha_4>Q$. Then

$$\langle e^{\alpha_1\phi(0)}e^{\alpha_2\phi(z)}e^{\alpha_3\phi(1)}e^{\alpha_4\phi(\infty)}\rangle = \int_{\mathbb{R}_+}|z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha,p,z)|^2$$
$$\times\, C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)C_{DOZZ}(\alpha_3,\alpha_4,Q-ip)dp$$

$\mathcal{F}$ are purely representation theoretic **holomorphic conformal blocks**

Idea:

1. Express correlation functions as **scalar products**
2. $\mathcal{S}=$ **spectrum** of the **Hamiltonian** of the QFT
3. z-dependence from **conformal Ward identities**

General bootstrap

GKRV (in preparation)

$$\langle\prod_{i=1}^{n} V_{\alpha_i}(z_i)\rangle = \int_{\mathbb{R}_+^{3g+n-3}} |\mathcal{F}(\mathbf{q},\mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

where

- $\mathbf{q}$ are moduli parameters for a pants decomposition of Σ
- Conformal block $\mathcal{F}(\mathbf{q},\mathbf{P})$ is holomorphic in $\mathbf{q}$
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha,\alpha',\alpha'')$ with $\alpha,\alpha',\alpha'' \in \{\alpha_i, Q \pm iP_j\}$

Proof is based on the approach of G. Segal to CFT

- Build surfaces by gluing disks, annuli and pants
- Define amplitudes for surfaces with boundary
- Glue amplitudes with spectral resolution of H

Reflection positivity

Think about $\log|z|$ as **time**: $|z| = e^t$

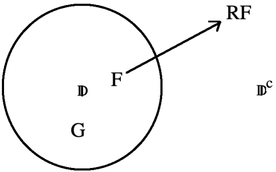
Hilbert space $\mathcal{F}_{\mathbb{D}}$ = functionals $F(\phi)$ that depend on $\phi|_{\mathbb{D}}$, $\mathbb{D}$ unit disc.

Reflection in unit circle $z \to \bar{z}^{-1}$ maps $\mathbb{D}$ to $\mathbb{D}^c$. It extends to

$$\mathcal{R} : \mathcal{F}_{\mathbb{D}} \to \mathcal{F}_{\mathbb{D}^c}$$

Scalar product $G, F \in \mathcal{F}_{\mathbb{D}} \to \langle G|F\rangle := \langle \bar{G}\mathcal{R}F\rangle$

Reflection positivity $\langle \bar{F}\mathcal{R}F\rangle \geq 0, \quad \forall F \in \mathcal{F}_{\mathbb{D}}$.



4-point function

The four-point function

$$G_4(z) := \left\langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(z)} e^{\alpha_3\phi(1)} e^{\alpha_4\phi(\infty)} \right\rangle \quad z \in \mathbb{D}$$

can be written as a scalar product

$$G_4(z) = \langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(z)} | e^{\alpha_4\phi(0)} e^{\alpha_3\phi(1)} \rangle \quad (*)$$

Bootstrap is obtained by factorising $(*)$ using the **spectral resolution** of the **Hamiltonian** of LCFT.

Hamiltonian of LCFT

Dilation $z \to e^{-t}z$ maps $\mathbb{D} \to \mathbb{D}$ and extends to a semigroup

$$e^{-tH} : \mathcal{F}_{\mathbb{D}} \to \mathcal{F}_{\mathbb{D}}$$

H is the **Hamiltonian** of the QFT

Proposition (GKRV 2020) H is a positive self adjoint operator on $\mathcal{H}$.
It has a continuous spectrum and a **complete set of generalized eigenfunctions** that can be constructed by **scattering theory**.
The states organise to **highest weight modules** of two commuting **Virasoro algebras** with weights Δ_{Q+iP}, $P \in \mathbb{R}$.

H as a Schrödinger operator

Write $\phi(e^{i\theta}) = c + \varphi(\theta) = c + \sum_{n\neq 0} \varphi_n e^{in\theta}$.

Hilbert space $\to$ wave functions $\psi(c,\varphi) \in L^2(dc \times \mathbb{P}(d\varphi))$.

Feynman-Kac formula gives

$$H = H_0 + \mu V$$

$$H_0 = \tfrac{1}{2}\left(-\tfrac{d^2}{dc^2} + Q^2 + \sum_{n=1}^{\infty}(a_n^* a_n + \bar{a}_n^* \bar{a}_n)\right)$$

$$V(c,\varphi) = e^{\gamma c} \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

where $a_n = i\frac{\partial}{\partial \varphi_n}$, $\bar{a}_n = i\frac{\partial}{\partial \bar{\varphi}_n}$etc.

We need to find a complete set of eigenfunctions $\psi(c,\varphi)$ of H:

$$(H_0 + \mu V)\psi = E\psi$$

Toy Liouville

Keep only c variable:

$$H = \tfrac{1}{2}(-\frac{d^2}{dc^2} + Q^2) + \mu e^{\gamma c}$$

Schrödinger operator on $L^2(\mathbb{R}, dc)$ with a wall potential

$$V(c) = e^{\gamma c} \to \begin{cases} 0 & \text{if } c \to -\infty \\ \infty & \text{if } c \to \infty \end{cases}$$

Scattering theory: Generalized eigenfunctions

$$\psi_p(c) \sim \begin{cases} e^{ipc} + R(p)e^{-ipc} & c \to -\infty \\ 0 & c \to \infty \end{cases}$$

with $p \in \mathbb{R}_+$ and eigenvalue $\frac{1}{2}(Q^2 + p^2) = 2\Delta_{Q+ip}$.

Spectrum of H_0

$H_0 = \frac{1}{2}(-\frac{d^2}{dc^2} + Q^2 + \sum_{n=1}^{\infty}(a_n^* a_n + \bar{a}_n^* \bar{a}_n))$

- Free particle moving on $\mathbb{R}$
- ∞ number of harmonic oscillators.

Orthonormal basis of generalized eigenstates

$$\psi^0_{p,k}(c,\varphi) = e^{ipc} h_k(\varphi) \quad p \in \mathbb{R}, \quad k \in \mathbb{N}$$

h_k Hermite polynomials

$$H_0 \psi^0_{p,k} = E_{p,k} \psi^0_{p,k}$$

Spectrum of LCFT

Theorem (GKRV 2020). H has an orthonormal basis of generalized eigenstates with the **same** eigenvalues

$$H\psi_{p,k} = E_{p,k}\psi_{p,k}$$

$$\psi_{p,k}(c,\varphi) \sim \psi^0_{p,k}(c,\varphi) + \text{reflected waves} \quad \text{as } c \to -\infty$$

Corollary. Plancharel identity holds

$$G_4(z) = \sum_k \int_{\mathbb{R}_+} \langle V_{\alpha_1}(0)V_{\alpha_2}(z)|\psi_{p,k}\rangle\langle\psi_{p,k}, V_{\alpha_3}(1)V_{\alpha_4}(0)\rangle dp$$

Remains to connect $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)|\psi_{p,k}\rangle$ to structure constants.

Bootstrap

Theorem (GKRV2020)

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)|\psi_{p,k}\rangle = C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)\times \text{ explicit factor}$$

Heuristic explanation Eigenstates $\{\psi_{p,k}\}_{k=0}^{\infty}$ can be organised to a representation of the **Virasoro algabra** with **highest weight state**

$$\psi_{p,0} = V_{Q+ip}(0)$$

so that

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)|\psi_{p,0}\rangle = \langle V_{\alpha_1}(0)V_{\alpha_2}(z)|V_{Q+ip}(0)\rangle = C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)$$

Corollary. Bootstrap formula holds:

$$\langle e^{\alpha_1\phi(0)}e^{\alpha_2\phi(z)}e^{\alpha_3\phi(1)}e^{\alpha_4\phi(\infty)}\rangle = \int_{\mathbb{R}_+}|z|^{2(\Delta_{Q+iP}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}|\mathcal{F}(\alpha,p,z)|^2$$
$$\times C_{DOZZ}(\alpha_1,\alpha_2,Q+ip)C_{DOZZ}(\alpha_3,\alpha_4,Q-ip)dp$$

where $\mathcal{F}$ comes from representation theory of the Virasoro algabra.

Remarks

1. There is **no local field** $V_{Q+ip}(z)$. The spectral state $\psi_{p,0}$ is an **analytic continuation** of the state $V_\alpha(0)$. It is a **macroscopic state**.

The **microscopic state** $V_\alpha(0)$ is **not** in the Hilbert space. This has been emphasized before by Seiberg and Teschner.

2. The Liouville potential

$$V(c,\varphi) = e^{\gamma c} \int_0^{2\pi} e^{\gamma\varphi(\theta) - \frac{\gamma^2}{2}\mathbb{E}\varphi(\theta)^2} d\theta$$

is a well defined multiplication operator if $\gamma < \sqrt{2}$ but it **vanishes** identically if $\gamma \geq \sqrt{2}$!. It has to be defined as a measure in the Hilbert space if $\gamma \in [\sqrt{2}, 2)$. Then the Hamiltonian exists as a Friedrichs extension.

Riemann surfaces

Riemann surface Σ of genus g, n punctures can be obtained by gluing pairs of **pants** T, **annuli** A_q of modulus $q \in \mathbb{C}$ and **discs** $\mathbb{D}$.

Probabilistic **amplitudes**

$$\mathcal{A}(T) \in \mathcal{H}^{\otimes 3}, \;\; \mathcal{A}(A_q) \in \mathcal{H}^{\otimes 2}, \;\; \mathcal{A}(D, \alpha) \in \mathcal{H}$$

Gluing $\implies$ **scalar products** of amplitudes

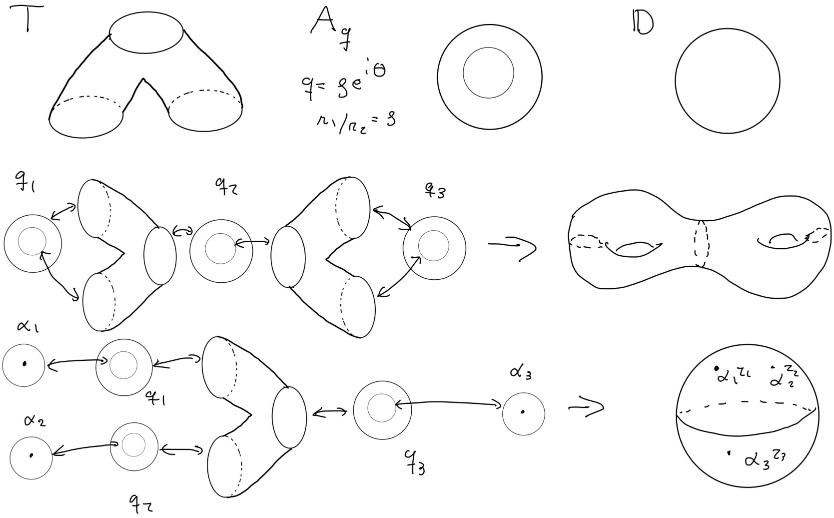
Bootstrap: insert a complete set of states to factorise

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle = \int_{\mathbb{R}_+^{3g+n-3}} |\mathcal{F}(\mathbf{q}, \mathbf{P})|^2 \rho(\mathbf{P}) d\mathbf{P}$$

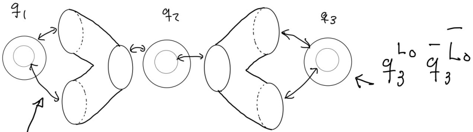
where

- Conformal block $\mathcal{F}(\mathbf{q}, \mathbf{P})$ is holomorphic in $\mathbf{q}$
- $\rho(\mathbf{P})$ is a product of structure constants $C(\alpha, \alpha', \alpha'')$ with $\alpha, \alpha', \alpha'' \in \{\alpha_i, Q \pm iP_j\}$

Gluing of surface

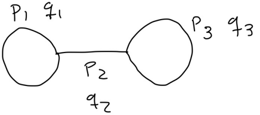


Factorisation



$$\int dp_1 \sum_{\substack{\nu,\bar\nu \\ \nu',\bar\nu'}} |\psi_{p_1,\nu,\bar\nu}\rangle\langle\psi_{p_1,\nu',\bar\nu'}|\, G_{\nu\nu'}(p)\, G_{\bar\nu\bar\nu'}(p)$$

$\Rightarrow$



$$\Rightarrow \int |\mathcal{F}(\tilde{p},\tilde{q})|^2 \rho(\tilde{p}) \prod_{i=1}^{3} dp_i$$

$$\rho(\tilde{p}) \sim C(Q+ip_1, Q-ip_1, Q+ip_2)\, C(Q+ip_2, Q+ip_3, Q-ip_3)$$

Open questions

Analytic continuation in γ?

Connection of LCFT to scaling limits

Connection of LCFT to Liouville gravity metrics

Toda CFT's

Other noncompact CFT: $G^{\mathbb{C}}/G$ WZW model, 2d black hole?

Thank you!