The wondrous world of hyperfinite subfactors

Mathematical Picture Language Seminar

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Subfactors: $N \subset M$ inclusion of II_1 factors

- $M \subseteq B(H)$ **von Neumann algebra** (*-subalgebra, $1 \in M$, closed in top. of ptwise conv. on vectors)
- M is a **factor** if $Z(M) = \mathbb{C}1$.
- A factor M is of **type II_1** if $\dim M = \infty$ and if M has a tracial state $tr: M \to \mathbb{C}$.
- M is **hyperfinite** if there are fin. dim. *-subalg. $A_n \subseteq A_{n+1}$ with $\overline{\bigcup_{n=1}^{\infty} A_n}^{w} = M$. Construction:

$$A_n = M_{2^n}(\mathbb{C}) \ni x \hookrightarrow \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in M_{2^{n+1}}(\mathbb{C}) = A_{n+1}$$

$A = \overline{\bigcup_{n\geq 1} M_{2^n}(\mathbb{C})}^{\|\cdot\|}$ C^*-algebra with trace tr (normalized matrix trace)

$(x|y) = \mathrm{tr}(y^*x)$, $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ $x, y \in A$

$H = \overline{A}^{\|\cdot\|_2}$, $A \curvearrowright H$ left multiplication

$R = \overline{A}^{w} \leq B(H)$ hyperfinite II_1 factor

- tr extends to R, $Z(R) = \mathbb{C}$, tr(all projections) $= [0,1]$
  continuous dimension
- Murray-VN (1943): There is a unique hyperfinite II_1 factor.
- M a II_1 factor, then $R \hookrightarrow M$ (R is the smallest II_1).

Connes (1976): $N \subseteq R$ subfactor ($\dim N = \infty$), then $N \cong R$.
G amenable ICC group, $\neq \{e\}$, then $L(G) \cong R$.

Jones (1983): Position of N in R, i.e. $N \subseteq R$ up to isom.

Invariants? $H = \overline{R}^{\|\cdot\|_2} =: L^2(R)$, N-R bimodule:

$n \in N$, $m \in R$, $n \cdot \hat{r} \cdot m = \widehat{nrm}$, $\hat{R} \subseteq L^2(R)$ dense

$\rho = {}_N L^2(R)_R$ standard rep. of $N \subseteq R$

ρ is irreducible iff $N' \cap R = \mathbb{C}$ (we say $N \subseteq R$ is irreducible)

Coupling constant measures size of $L^2(R)$ as left N-mod:

$$L^2(R) \underset{N\text{-module}}{\overset{\text{left}}{\cong}} \bigoplus_{i=1}^{k} L^2(N) \oplus L^2(N)p \qquad p \in N \text{ a proj.}$$

Jones index: $[R:N] = k + \mathrm{tr}(p) \in [1,\infty]$

($[R:N] \in \{4\cos^2\frac{\pi}{n} \mid n \geq 3\} \cup [4,\infty]$, Jones' rigidity theorem)
1983

We assume $[R:N] < \infty$.

$$\rho := {}_N L^2(R)_R \qquad \text{standard rep.}$$

$$\bar{\rho} \overset{R-N}{\cong} {}_R L^2(R)_N \qquad \text{contragredient rep.}$$

Proceed à la H. Weyl:

$(\mathfrak{s} \otimes_R \bar{\mathfrak{s}})^{\otimes k}$ $\forall k$ $\xrightarrow{\text{decompose}}$ irred. N-N bimodules

$(\bar{\mathfrak{s}} \otimes_N \mathfrak{s})^{\otimes k}$ $\forall k$ $\xrightarrow{\text{decompose}}$ irred. R-R bimodules (*)

irred. N-R, R-N bim.

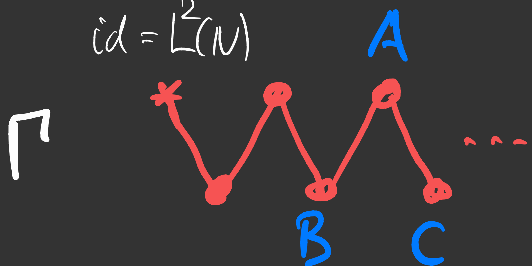
Standard invariant:

$$\mathcal{G}_{N \subset R} := \left\{ \begin{array}{ccc} \mathrm{Hom}_{N\text{-}N}((\mathfrak{s}\otimes\bar{\mathfrak{s}})^{\ell}) & \overset{\cdot \otimes id}{\subseteq} & \mathrm{Hom}_{N\text{-}R}((\mathfrak{s}\otimes\bar{\mathfrak{s}})^{\ell}\otimes \mathfrak{s}) \\ id\otimes \cdot \, \cup\!| & & id\otimes \cdot \, \cup\!| \\ \mathrm{Hom}_{R\text{-}N}((\bar{\mathfrak{s}}\otimes\mathfrak{s})^{\ell-1}\otimes\bar{\mathfrak{s}}) & \overset{\cdot \otimes id}{\subseteq} & \mathrm{Hom}_{R\text{-}R}((\bar{\mathfrak{s}}\otimes\mathfrak{s})^{\ell}) \end{array} \right\}_{\ell}$$

(planar algebra, λ-lattice, rigid C^*-tensor cat. / fusion cat., ...)

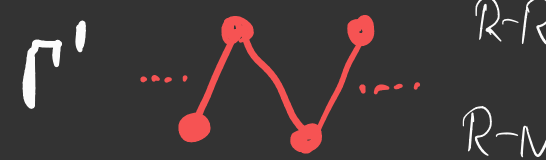
$\mathcal{G}_{NCR}$ consists of a double-sequence of multi-matrix algebras + orthogonality (commuting squares).

Fusion graphs



irred. N-N bim.

irred. N-R bim.



$$A \otimes \mathfrak{s} \overset{N-R}{\cong} B \oplus C$$

$$\cdot \otimes \bar{\mathfrak{s}}$$

(Γ, Γ') bipartite, connected, possibly infinite graphs + standard eigenvectors (e.g. PF-e.v.)

Example: $\Gamma = \Gamma' = A_\infty$  $\|A_\infty\|^2 = 4.$

Standard invariant consists of Temperley-Lieb-Jones algebras with parameter $\delta = [R:N]^{1/2}$ (minimal or "trivial")

Popa (1990's): Amenability of $N \subset R$.

Γ is amenable if $\|\Gamma\|^2 = [R:N]$ $(= \|\Gamma'\|^2)$.

Theorem (Popa, 1994): $\mathcal{G}_{N\subset R}$ is a complete invariant for $N \subset R$ with Γ amenable.

Thus, $N \subseteq R$ with $[R:N] > 4$ and TLJ $(= A_\infty)$ standard invariant is non-amenable.

Finite depth subfactors ($\Leftrightarrow \Gamma$ finite graph) are amenable. Classification led to many "exotic" fusion categories (Haagerup $\frac{5+\sqrt{13}}{2}$, EH, ...). Up to index 5.25, countably many finite depth subfactors, and the only stand. inv. with infinite graphs are A_∞, D_∞ ($A_{-\infty,\infty}$) and:

Thm (B., Jones): There are non-amenable, irreducible subfactors with index $3+\sqrt{5}$ ($\approx 5.23606\ldots$) and fusion graphs

($\mathfrak{g}_{A_3} * \mathfrak{g}_{A_4} \overset{\text{Popa}}{\rightsquigarrow}$ exist as hyperfinite subf.)

Some open problems

1) • Which stand. inv./planar alg. arise from hyperf. subf.?
   • What is $I(R) = \{ [R:N] \mid N \subseteq R$ irred. subf.$\}$ (Jones)

2) Given $N \subseteq R$ with A_∞-stand. inv. and $[R:N] > 4$, is there $Q \subseteq R$ with same stand. inv., but $(N \subseteq R) \not\cong (Q \subseteq R)$?
   (compare Ocneanu, Jones result on group actions)

3) How many hyperfinite subf. are there with $\mathcal{G}_{A_3} * \mathcal{G}_{A_4}$ stand. inv. (index $3+\sqrt{5}$)? Many?

We need to look for invariants beyond $\mathcal{G}_{N \subseteq R}$.

We need to construct (∞ depth) hyperfinite subfactors.

Constructions (of irred. hyperfinite subfactors)

- Use group actions, group reps, quantum groups etc.

Bisch–Haagerup subfactors: $R^H \leq R \rtimes K$

(H, K finite groups with outer actions on R)

Thm (B–Haagerup, 1996): Let $G = \langle H, K \rangle \leq \mathrm{Out}\, R = {}^{\mathrm{Aut}\, R}\!/_{\mathrm{Inn}\, R}$.

1) $R^H \leq R \rtimes K$ is irreducible $\iff$ $H \cap K = \{e\}$ in $\mathrm{Out}\, R$

2) $R^H \leq R \rtimes K$ has infinite depth $\iff$ $|G| = \infty$.

3) $R^H \leq R \rtimes K$ is amenable $\iff$ G is amenable.

• Commuting squares

$$\begin{array}{ccc} B_0 & \overset{H}{\subset} & B_1 \\ K\ \cup & G & \cup\ L \\ A_0 & \subset & A_1 \end{array}$$

tr normal faithful trace (weight vector)

A_i, B_i multi-matrix algebras

It is a **commuting square** if

$A_0^\perp \cap A_1 \perp A_0^\perp \cap B_0$ in B_1 w.r.to $(x|y) = tr(y^* x)$

$(\Leftrightarrow E_{B_0} E_{A_1} = E_{A_0} \Leftrightarrow E_{A_1} E_{B_0} = E_{A_0} \Leftrightarrow E_{A_1}(B_0) \subseteq A_0$ etc.$)$

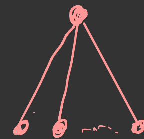
<u>Natural structure:</u> $A \subset B \subset C \subset D$ fin. VN alg., tr n.f. trace on D,

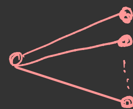
then

$$\begin{array}{ccc} A' \cap C & \subseteq & A' \cap D \\ \cup\!\mid & & \cup\!\mid \\ B' \cap C & \subseteq & B' \cap D \end{array}$$

is a commuting square.

E.g.:
$$\begin{array}{ccc} u\Delta u^* & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & \Delta \end{array}$$





tr = matrix trace, $u \in \mathcal{U}(n)$

It is a commuting square iff $|u_{ij}| = \frac{1}{\sqrt{n}}$, $1 \le i,j \le n$.

complex Hadamard matrix

$$\begin{array}{ccccccccccccc} B_0 & \overset{L}{\subset} & B_1 & \overset{e_1}{\subset} & B_2 & \overset{e_2}{\subset} & B_3 & \subset & \cdots & \subset & \overline{\bigcup_{n\ge 1} B_n}^{w} & =: & B \\ {\scriptstyle K}\cup & & \cup {\scriptstyle H} & & \cup & & \cup & & & & & & \cup \\ A_0 & \overset{G}{\subset} & A_1 & \overset{e_1}{\subset} & A_2 & \overset{e_2}{\subset} & A_3 & \subset & \cdots & \subset & \overline{\bigcup_{n\ge 1} A_n}^{w} & =: & A \end{array}$$

$GH = KL$
$HL^t = G^tK$
+ easy condit.

+ tr a Markov trace (PF...)

$\rightsquigarrow$ $A \subset B$ irred. hyperfinite subf. with $[B:A] = \|K\|^2 = \|H\|^2$.

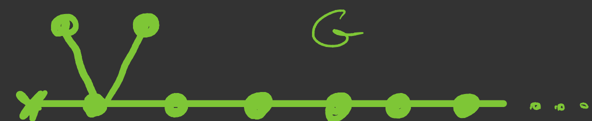
Note: K, H finite graphs $\rightsquigarrow$ index is an algebraic integer.

Smallest known (Oc, H-S): $[B:A] = \|E_{10}\|^2 = 4.026\ldots$

Bisch (1994): There is an irred. hyperfinite subfactor with index 4.5.

Built from a c.s.

$$\begin{array}{ccccc} & B_0 & \overset{2G^t}{\subset} & B_1 & \\ G & \cup & & \cup & G^t \\ & A_0 & \overset{2G}{\subset} & A_1 & \end{array}$$



By classification, it has A_∞-standard invariant.

Rem: Interesting (infinite) c.s. have been constructed.

"Bare hand constructions"

How noncommutative can a subfactor be?

$N \subset M$ $(\cong R)$, commutativity of M relative to N

<u>Def:</u> $N \subset M$ has property (Γ) if there are nontrivial central sequences for M in N, i.e. $\exists\, (x_n) \in \ell^\infty(\mathbb{N}, N)$ with

$$\| x_n y - y x_n \|_2 \to 0,\ n \to \infty,\ \forall\, y \in M \quad \text{and} \quad \inf_n \| x_n - tr(x_n) \cdot 1 \| > 0.$$

<u>Central sequence algebras:</u> ω a free ultrafilter on $\mathbb{N}$ $(\omega \in \beta\mathbb{N} \setminus \mathbb{N})$, let

$$I_\omega = \{ (x_n) \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n \to \omega} \|x_n\|_2 = 0 \}, \quad M^\omega = \ell^\infty(\mathbb{N}, M) / I_\omega.$$

M^ω is a vN algebra with trace $tr_\omega((x_n)) = \lim_{n \to \omega} tr(x_n)$.

Prop.: $N \subset M$ has $(\Gamma) \iff M' \cap N^\omega \neq \mathbb{C}$. In this case, $M' \cap N^\omega$ contains a diffuse, abelian subalg.

Def.: $N \subset M$ is McDuff (or stable) if $(N \subset M) \cong (N \subset M) \otimes R$.

Thm (B., 1990): $N \subset M$ is McDuff $\iff M' \cap N^\omega$ is non-abel.

Degrees of non-commut. for $N \subset M$:

- $M' \cap N^\omega = \mathbb{C}$
- $M' \cap N^\omega$ abelian, diffuse
- $M' \cap N^\omega$ non-abel., hence II_1 VN alg.

factor non-factor

$N \subset M$ more commutative

Examples: • $N \subset M$ fin. depth or (strongly amen.), then $M' \cap N^\omega$ is a II_1 factor.

• $N \subset R$ constructed from commuting squares are McDuff (the sequences $(e_i)_{i \geq 1}$ and $(e_{i+1})_{i \geq 1}$ are nontrivial, central sequ. for M, contained in N).

$\rightsquigarrow$ "very commutative" (irred.) hyperfinite subfactors

Non-commutative examples?

Note: If $N \subset M$ satisfies $M' \cap N^\omega = \mathbb{C}$, then $N \otimes R \subset M \otimes R$ has the same stand. inv., but big relative central sequ. algebra (hence $\not\cong (N \subset M)$) by my thm.

Thm (B.): Let $G = \langle H, K \rangle$, H, K fin. groups, $G \overset{\sigma}{\curvearrowright} R$ outer action. Then

1) $R^H \subset R \rtimes K$ has prop. (Γ) iff $R \subset R \rtimes_\sigma G$ has prop. (Γ).

2) $R^H \subset R \rtimes K$ has McDuff iff $R \subset R \rtimes_\sigma G$ has McDuff.

This allows us to construct many "non-commutative" irred., hyperf. subfactors.

- By 1), it suffices to look for $G \overset{\sigma}{\curvearrowright} R$ s.th. $R \rtimes_\sigma G$ does not have property (Γ).
- Lots of examples: Strongly ergodic actions of non-inner amenable groups on R
- <u>Concrete examples (index 6):</u> $G = \langle \mathbb{Z}_2, \mathbb{Z}_3 \rangle$, G property (T) (e.g. $G = SL(n, \mathbb{Z})$, $n \geq 29$), σ = Bernoulli shift on R. Then $R \rtimes_\sigma G$ does not have property (Γ) (Choda), hence $R^{\mathbb{Z}_2} \subset R \rtimes \mathbb{Z}_3$ are very non-commutative (do not have prop. (Γ)).

They cannot be constructed from a commuting square!