Conjugation of words, self-intersection of planar curves, and non-commutative divergence

based on joint works with

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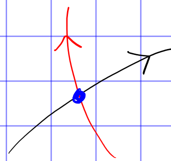
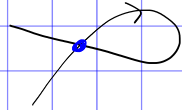
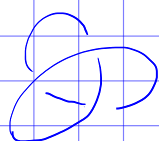
	Easy	Hard
2D topology		
non-commutative geometry	$\mathrm{Tr}(xyz)$	Div
Lie theory	$\log(e^x e^y)$	$\det^{1/2}\left(\frac{e^{ad_x}-1}{ad_x}\right)$
Geometry	Poisson $[\pi,\pi]=0$	BV $\Delta^2=0$
3D topology		$\Phi\Phi = \Phi\Phi\Phi$
Number theory		$\zeta(k_1,\dots,k_r)$

...

Non-commutative geometry

à la Kontsevich:

- Define geometric structures on free non-commutative algebras

- $k\langle x_1, \dots, x_n\rangle \xrightarrow{\rho} \mathrm{Mat}_N(k)^n$

  $\rho(x_i) = X_i$ matrices of any SIZE

Conjugation of words

- $k\langle\langle x_1, \dots, x_n \rangle\rangle$ formal power series in non-commutative variables $x_1, \dots, x_n$

Example: $x_1 x_2 \neq x_2 x_1$

Grading: # of x_i's

- $\mathbb{L}(x_1, \dots, x_n)$ Lie series

$\cup\!\!|$

$x_1, \quad x_2, \quad [x_1, x_2] = x_1 x_2 - x_2 x_1, \dots$

Baker - Campbell - Hausdorff series:

$$\log(e^{x_1} e^{x_2}) = x_1 + x_2 + \frac{1}{2}[x_1, x_2] + \dots$$

- $\exp(\mathbb{L}) \subset k\langle\langle x_1, \dots, x_n \rangle\rangle$

a group

Examples: e^{x_1}

$$e^{x_1} e^{x_2} = e^{\log(e^{x_1} e^{x_2})}$$

Def : $a, b \in \mathbb{L}$ are conjugate

if $\exists\, g \in \exp(\mathbb{L})$ s.t. $b = g\, a\, g^{-1}$

Example :

$$\log(e^{X_2} e^{X_1}) = e^{X_2} \log(e^{X_1} e^{X_2})\, e^{-X_2}$$

Question : how to decide

whether a and b are conjugate ?

Cyclic words

- $A = k\langle\langle x_1, \dots, x_n \rangle\rangle$

- $\mathrm{Cycl} = A/[A,A] \xleftarrow{\mathrm{Tr}} A$

$$\Downarrow$$

$$\mathrm{Tr}(x_1 x_2) = \mathrm{Tr}(x_2 x_1)$$

- $b = g a g^{-1}$

$\Rightarrow \quad \mathrm{Tr}\, b^n = \mathrm{Tr}\, a^n \quad \forall n$

$\Leftarrow$? not for matrices (because of Jordan blocks)

Thm (AKKN)

If

- $a_{1,n} \neq 0$

- $a_{1,n} = 0, \; a = \sum_{i=1}^{m} [X_{2i-1}, X_{2i}]$

$n = 2m$

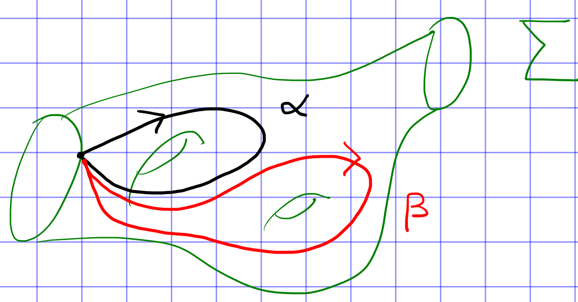
$\Rightarrow \quad \mathrm{Tr}\, b^n = \mathrm{Tr}\, a^n \quad \forall n$

implies $\quad b = g a g^{-1}$

Applications to 2D topology

k = field of characteristic 0

Σ = compact oriented 2-manifold



$$\mathfrak{g}(\Sigma) = k\langle \text{homotopy classes of closed oriented curves} \rangle$$

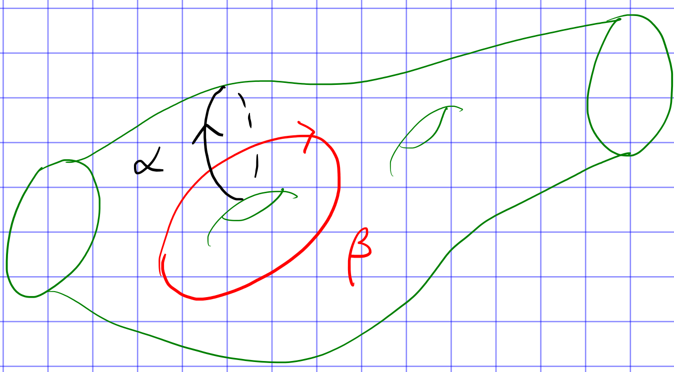
$$\cong k\langle \text{conjugacy classes in } \pi_1(\Sigma) \rangle$$

$$\cong k\pi_1 / [k\pi_1, k\pi_1]$$

$$\mathrm{Tr} \uparrow \qquad \rotatebox{90}{$\in$}$$

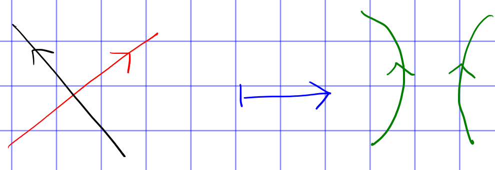
$$k\pi_1 \qquad \mathrm{Tr}\,\alpha\beta = \mathrm{Tr}\,\beta\alpha$$

Goldman bracket



$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p \, \mathrm{Tr}\, \alpha \#_p \beta$$

$$\varepsilon_p = \frac{d\alpha \wedge d\beta}{\mathrm{vol}_\Sigma}$$



Thm [Goldman]

$[\cdot, \cdot]$
- is well defined
- is a Lie bracket

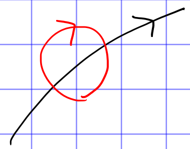
Thm [Etingof]

$$\mathfrak{z}\left(\mathfrak{g}(\Sigma), [\cdot, \cdot] \right) =$$

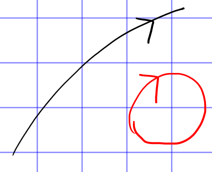
$$= \Bbbk \left\langle \operatorname{Tr} \gamma_j^n \; ; \quad n \in \mathbb{Z} \right.$$

$$\left. \gamma_j = \text{boundary loops} \right\rangle$$

Remark: $1 = \operatorname{Tr} \gamma_j^0$ trivial loop



$\longmapsto$



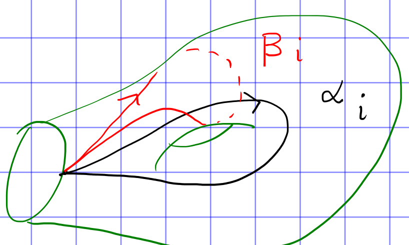
Filtrations

- $\pi_1 \supset [\pi_1, \pi_1] \supset [\pi_1, [\pi_1, \pi_1]] \supset \dots$

$\Rightarrow$ filtration on $k\pi_1$

$\Rightarrow$ filtration on $\mathfrak{g}(\Sigma)$

- Associated graded



$\mathrm{Cycl}(H_1(\Sigma))$

$\Downarrow$

$\mathrm{Tr}(x_i\, y_j)$

$[\alpha_i] = x_i, \; [\beta_i] = y_i$

- Fact: $[\cdot, \cdot]_{gr}$ has degree (-2)

<u>Question</u> (Goldman formality):

$$\left(\mathfrak{g}(\Sigma), [\cdot, \cdot]\right) \overset{?}{\cong} \left(\mathrm{gr}\,\mathfrak{g}(\Sigma), [\cdot, \cdot]_{\mathrm{gr}}\right)$$

<u>Thm</u> [Kawazumi-Kuno, Massuyeau-Turaev, Naef]

- $\theta : k\pi_1 \longrightarrow TH = \bigoplus_n H_1^{\otimes n}$

  map of coalgebras

- normalization: • $\theta(\alpha_i) = 1 + x_i + \dots$

  • $\theta(\beta_i) = 1 + y_i + \dots$

$\Rightarrow$ $\theta\left(\prod_i [\alpha_i, \beta_i]\right) =$

$$= g \exp \sum_i [x_i, y_i] \; g^{-1}$$

implies Goldman formality

Thm [AKKN] $\Leftarrow$ holds

(Idea of) proof:

- $\mathfrak{z}(\mathcal{G}(\Sigma)) = \Bbbk\langle \mathrm{Tr}\,\gamma_0^n \,;\, n \in \mathbb{Z}\rangle$

$$\gamma_0 = \prod_i [\alpha_i, \beta_i]$$

- $\mathfrak{z}(\mathrm{gr}\,\mathcal{G}(\Sigma)) = \Bbbk\langle \mathrm{Tr}\,\omega^n \,;\, n = 0, 1, \dots\rangle$

$$\omega = \sum_i [x_i, y_i]$$

- $\theta(\gamma_0) = 1 + \omega + \dots$

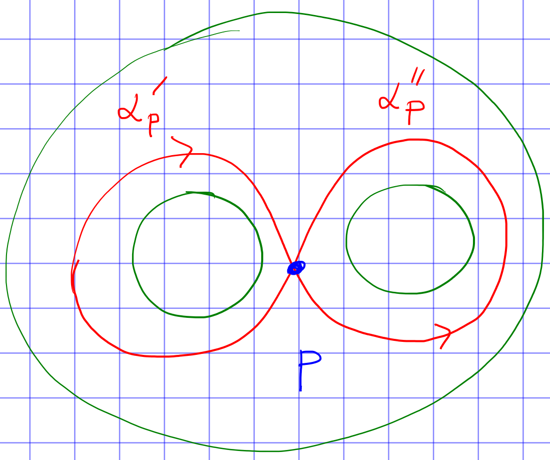
- $\theta(\mathrm{Tr}\,\gamma_0^n) \in \mathfrak{z}(\mathrm{gr}\,\mathcal{G}(\Sigma))$

$\Rightarrow$ $\theta(\mathrm{Tr}(\log \gamma_0)^n) = \mathrm{Tr}\,\omega^n \quad \forall n$

$\Rightarrow$ $\theta(\log \gamma_0) = g\,\omega\,g^{-1}$

Turaev cobracket

$$\delta(\operatorname{Tr}\alpha) = \sum_{p \in \alpha \cap \alpha} \varepsilon_p \operatorname{Tr}\alpha'_p \wedge \operatorname{Tr}\alpha''_p$$



Thm [Turaev, Chas]

- δ is well defined
- $(\mathfrak{g}(\Sigma), [\cdot,\cdot], \delta)$ is an involutive Lie bialgebra

Involutive Lie bialgebras

- $[\cdot,\cdot] \circ \delta = 0$

- $\delta [u, v] = [\delta u, v] + [u, \delta v]$

  and moore ...

- δ_{gr} has degree (-2)

Question (Goldman-Turaev formality)

$$(\mathfrak{g}(\Sigma), [\cdot,\cdot], \delta) \cong (\mathrm{gr}\,\mathfrak{g}(\Sigma), [\cdot,\cdot]_{gr}, \delta_{gr})$$

?

genus 0

Massuyeau $\Leftarrow$ Kontsevich knot invariant

AKKN $\Leftarrow$ Kashiwara-Vergne problem

Naef & AA $\Leftarrow$ Knizhnik-Zamolodchikov connection

genus ≥ 1

AKKN $\Leftarrow$ elliptic associators

Hain $\Leftarrow$ Hodge theory

non-commutative divergence

- partial derivatives

$$\partial_i : A \longrightarrow A \otimes A$$

Example: $\partial_2 (x_1 x_2 x_3) = x_1 \otimes x_3$

Notation: $\partial_i a = \partial_i' a \otimes \partial_i'' a$

- derivations

$u \in \mathrm{Der}(A) \Rightarrow u(ab) = u(a)b + a\,u(b)$

Notation: $u(x_i) = u_i$

- divergence

$$\mathrm{Div}(u) = \sum_i \mathrm{Tr}(\partial_i' u_i) \otimes \mathrm{Tr}(\partial_i'' u_i)$$

<u>Thm</u> Divergence is a Lie algebra 1-cocycle on $\mathrm{Der}(A)$ with values in $\mathrm{Cycl} \otimes \mathrm{Cycl}$:

$$\mathrm{Div}([u,v]) = u(\mathrm{Div}(v)) - v(\mathrm{Div}(u))$$

Goldman-Turaev formality

(Idea of) AKKN proof:

- use Goldman bracket:

$$\mathrm{Tr}\,\alpha \longmapsto h_{\mathrm{Tr}\,\alpha} : a \longmapsto [\mathrm{Tr}\,\alpha, a]$$

$$\rotatebox{90}{$\in$}$$

$$\mathrm{Der}(A)$$

- $$\delta(\mathrm{Tr}\,\alpha) = \mathrm{Div}(h_{\mathrm{Tr}\,\alpha})$$

  + graded version

- $$\theta : \gamma_0 \longmapsto g \exp(\omega) g^{-1}$$

$$\mathrm{Div} \longmapsto \mathrm{Div}$$

the KV problem $\Leftarrow$ Lie theory !

Rigidity of Goldman-Turaev:

Conjecture:

$$\mathrm{Der}\left(\mathrm{gr}\, \mathfrak{g}(\Sigma),\ [\cdot,\cdot]_{\mathrm{gr}},\ \delta_{\mathrm{gr}} \right)$$

AKKN $\cup$ $\cong$

$$\mathfrak{grt}_1 \supset \mathbb{L}(\sigma_3, \sigma_5, \ldots)$$

[Brown]

Thank

you !