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Aspects of M Theory

$$p_{i\alpha} p_{i\alpha} + \frac{1}{M!} g_{\alpha\alpha_1 \ldots \alpha_M} g_{\alpha\beta_1 \ldots \beta_M} x_{i_1\alpha_1} x_{i_1\beta_1} \cdots x_{i_M\alpha_M} x_{i_M\beta_M}$$

spectrum of $-\Delta + V(x)$, solutions of $\ddot{x}_{i\alpha} = -\frac{\partial V}{\partial x_{i\alpha}}$

$i = 1 \ldots d$, $\alpha = 1, 2, \ldots$

M-dimensional surface, $\int Y_\alpha(\varphi) Y_\beta(\varphi) \rho \, d^M\varphi = \delta_{\alpha\beta}$

$$x_i(t, \varphi^1 \ldots \varphi^M) = \sum_{\alpha=1}^{\infty} x_{i\alpha}(t) Y_\alpha(\varphi \ldots \varphi^M)$$

$$g_{\alpha\alpha_1 \ldots \alpha_M} = \int Y_\alpha \epsilon^{r_1 \ldots r_M} \partial_{r_1} Y_{\alpha_1} \cdots \partial_{r_M} Y_{\alpha_M} \, d^M\varphi$$

$\varphi^1 \ldots \varphi^M \to \tilde{\varphi}^1 \ldots \tilde{\varphi}^M$ $\qquad \rho(\varphi) \, d^M\varphi = \rho(\tilde{\varphi}) \, d^M\tilde{\varphi}$

$\tilde{\varphi}^a = \varphi^a + \epsilon f^a(\varphi^1 \ldots \varphi^M) + \ldots$ $\qquad \nabla_a f^a = \frac{1}{\rho} \partial_a (\rho f^a) = 0$

M-brane Theory

$$\mathrm{Vol}(\mathcal{M}) = \int \sqrt{G}\, d^{M+1}\varphi$$

M+1 dimensional manifold $\subset \mathbb{R}^{1,D-1}$, described by $X^{\mu=0,1,\dots,D-1}(\varphi^0\varphi^1\dots\varphi^M)$

$$|\det(G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})|$$

$$\begin{pmatrix} G_{00} & 0 \cdots 0 \\ 0 & \\ \vdots & -g_{ab} \\ 0 & \end{pmatrix}$$

ONLCG: choose $\varphi^0 = \frac{X^0 + X^{D-1}}{2} = \tau$

$\zeta = X^0 - X^{D-1}$

$X^{i=1\dots D-2=d}$

$$\partial_a \zeta - \dot{\vec{x}}\,\partial_a \vec{x} = 0$$

$$\Delta X^\mu := \frac{1}{\sqrt{G}}\partial_\alpha \sqrt{G}\, G^{\alpha\beta}\partial_\beta X^\mu = 0$$

$$\Delta \tau = 0 \Leftrightarrow \partial_+ \Big(\underbrace{\sqrt{g/(2\dot{\zeta} - \dot{\vec{x}}^2)}}_{=}\Big) = 0$$

$$\Delta X^i = 0 \Leftrightarrow \eta^2 \ddot{x}^i = \frac{1}{\rho}\partial_a\left(g\, g^{ab}/\rho\; \partial_b x^i\right) =: \Delta x^i$$

$$\rho_+(\varphi) = \eta\rho(\varphi^1 \dots \varphi^M) = P_+\rho$$

$$P_- = \frac{1}{2\eta}\int\left(\frac{\vec{p}^{\,2}}{\rho^2} + g/\rho^2\right)\rho\, d^M\varphi \qquad \int \rho\, d^M\varphi = 1$$

$$\int f^a \vec{p}\,\partial_a \vec{x}\, d^M\varphi = 0 \text{ whenever } \nabla_a f^a = \frac{1}{\rho}\partial_a(\rho f^a) = 0$$

$$\mathbb{M}^2 = 2P_+P_- - \vec{P}^2$$

(see e.g. H 13, J. Phys. A 46, 023001)

Dynamical Symmetry ($\overset{H}{2011}$ + 2103.16540)

$$M_{i-} = \frac{1}{2\eta}\int x_i\left(\frac{\vec{p}^{\,2}}{\rho^2} + g/\rho^2\right) - \int \zeta\, p_i\, d^M\varphi$$

$$\partial_a \zeta = \frac{\vec{p}}{\eta\rho}\partial_a \vec{x}, \quad \dot{\zeta} = \frac{1}{2}\frac{\vec{p}^{\,2} + g}{\eta^2\rho^2}\quad \left(\eta\rho = \sqrt{g/(2\dot{\zeta} - \dot{\vec{x}}^2)}\right)$$

$$\zeta(\varphi) = \zeta_0 + \frac{1}{\eta}\int G(\varphi,\tilde{\varphi})\,\tilde{\nabla}^a\left(\widetilde{\frac{\vec{p}}{\rho}\partial_a\vec{x}}\right)\tilde{\rho}\, d^M\tilde{\varphi}$$

$$\Delta G(\varphi,\tilde{\varphi}) = \frac{\delta(\varphi,\tilde{\varphi})}{\rho} - 1, \quad \int G(\varphi,\tilde{\varphi})\rho\, d^M\varphi = 0$$

$$M_{i-} = (X_i P_- - \zeta_0 P_i) + \mathbb{M}_{i-} + \mathbb{M}_{ik}P_k \;\Big|\Rightarrow \{M_{i-}, M_{j-}\} = 0 \quad (G85)$$

$$\Rightarrow \{\mathbb{M}_{i-}, \mathbb{M}_{j-}\} = \mathbb{M}^2\mathbb{M}_{ij}$$

$$\{\mathbb{M}_{ij}, \mathbb{M}_{kl}\} = -\delta_{jk}\mathbb{M}_{il} \pm 3 \text{ more}$$

$SO(d+1)$

$$\{\mathbb{M}_{ij}, \mathbb{M}_{k-}\} = -\delta_{jk}\mathbb{M}_{i-} + \delta_{ik}\mathbb{M}_{j-}$$

finite-dimensional blocks and multiplicities $\rightarrow$ dimension + topology of the extended object

$$\mathbb{M}^2 = p_{i\alpha}p_{i\alpha} + \frac{1}{M!}g_{\alpha\alpha_1\dots\alpha_M}\,g_{\alpha\beta_1\dots\beta_M}\,x_{i_1\alpha_1}x_{i_1\beta_1}\cdots x_{i_M\alpha_M}x_{i_M\beta_M} \quad \text{(internal)}$$

$$= \int z \int\left(z(\nabla p)^2 + \frac{1}{z}\right) - \left(\int z\nabla p\right)^2$$

$$\mathbb{M}_{ij} = L_{ij} - \frac{1}{P_+}(L_{i+}P_j - L_{j+}P_i)$$

$$\mathbb{M}_{i-} = P_+L_{i-} - (L_{i+}P_- + L_{+-}P_i) - \mathbb{M}_{ik}P_k$$

Hydrodynamics $(d=M)$

$t\varphi^1\ldots\varphi^M \to t x^1\ldots x^M$, $\int \eta\rho\, d^M\varphi = \int q(t,\vec{x})\, d^M x$

$$P_- = H = \frac{1}{2}\int\left((\vec{\nabla}p)^2 + \frac{1}{q^2}\right) q\, d^M x \qquad \left(\vec{p}/\eta\rho(\varphi^\alpha) = \vec{\nabla}p(x^\alpha)\right)$$

$P_+ = \int q$, $\vec{P} = \int q\vec{\nabla}p$, $L_{+-} = \tau P_- - \int q p$, $L_{a+} = \int (x_a q) - \tau P_a$

$L_{a-} = \frac{1}{2}\int\left(x_a\left(q(\nabla p)^2 + \frac{1}{q}\right) - q\,\partial_a p^2\right) d^M x$, $L_{ab} = \int q(x_a\partial_b p - x_b\partial_a p)$

$M+2$-dimensional Poincaré-invariance (BH93, J98)

of the Chaplygin-Kármán-Tsien gas (C 1902?)

$$\left(\dot{q} + \vec{\nabla}(q\vec{\nabla}p) = 0,\quad \dot{p} = \frac{1}{2}\left(\frac{1}{q^2} - (\nabla p)^2\right)\right)$$

isentropic inviscid irrotational flow

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Membrane Matrix Model (GH82)

$$H = \frac{1}{2}\int\left(\vec{p}'^2 + \sum_{i<j}\{x_i, x_j\}^2\right)\rho\, d^2\varphi$$

$x_i(t,\varphi^1\varphi^2) \to X_i(t)$ $\downarrow$ $\ddot{x}_i = \{\{x_i, x_j\}, x_j\}$, $\{x_i, \dot{x}_i\} = 0$

$H_N = \frac{1}{2}\mathrm{Tr}\left(\vec{P}^2 - \sum_{i<j}[X_i, X_j]^2\right)$, $\ddot{X}_i = -[[X_i, X_j], X_j]$, $[X_i, \dot{X}_i] = 0$

$(\Sigma_2, \rho\, d^2\varphi)$ $\exists\, T_N : f \to T_N(f) = F^{(N\times N)}$

BMS94

$\| T_N(f)T_N(g) - T_N(f\cdot g)\| \to 0$

$\| \frac{1}{i\hbar_N}[T_N(f), T_N(g)] - T_N(\{f,g\})\| \to 0$

$2\pi\hbar_N \mathrm{Tr}\, T_N(f) \to \int f\rho\, d^2\varphi$

$\{f,g\} := \frac{\epsilon^{rs}}{\rho}\partial_r f\,\partial_s g$

Fuzzy Sphere (GH 82!)

$$Y_\alpha = Y_{lm}(\theta,\varphi) = \sum C^{(m)}_{a_1\ldots a_l} x_{a_1}\cdots x_{a_l}\Big|_{\vec{x}^2=1}$$

$$\downarrow$$

$$T^{(N)}_{lm} = \sum C^{(m)}_{a_1\ldots a_l} X^{(N)}_{a_1}\cdots X^{(N)}_{a_l} \qquad [X^{(N)}_{a}, X^{(N)}_{b}] = \frac{2i}{\sqrt{N^2-1}}\epsilon_{abc}X^{(N)}_c$$

$l = 1,\ldots,N-1$
$m = -l,\ldots,+l$ $\quad N^2-1$ l.i. traceless $N\times N$ matrices $\quad \vec{X}^2 = \mathbb{1}$

$\Rightarrow$ basis of $sl(N)$

$$\frac{1}{N}\mathrm{Tr}\,\tilde{T}_{lm}\tilde{T}^\dagger_{l'm'} = \delta_{ll'}\delta_{mm'}, \qquad \tilde{T}_{lm} = \sqrt{4\pi N}\sqrt{\frac{(N^2-1)^l(N-l-1)!}{(N+l)!}}\,T^{(N)}_{lm}$$

$$\tilde{f}^{(N)}_{lm\,l'm'\,l''m''} := 2\pi\,\mathrm{Tr}\left(\tilde{T}_{l''m''}[\tilde{T}_{lm},\tilde{T}_{l'm'}]\right) \xrightarrow[N\to\infty]{} \tilde{g}_{lm\,l'm'\,l''m''} := i\int \tilde{Y}_{l''m''}\{\tilde{Y}_{lm},\tilde{Y}_{l'm'}\}\rho\, d^2\varphi$$

$$\rho = \frac{\sin\theta}{4\pi} \qquad \tilde{Y}_{lm} = \sqrt{4\pi}\,Y_{lm}$$

$$f = \sum f_{lm}Y_{lm}(\theta,\varphi)$$

$$\to \tilde{T}_N(f) = \sum_{lm} f_{lm}\tilde{T}^{(N)}_{lm}$$

note: $\tilde{T}^{(N)}_{l\geq N} = 0$ (automatically)

Supersymmetrizable Systems (2101: 01803, 04495, 11510)

$$\mathbb{M}^2_N = \mathrm{Tr}\left(\vec{P}^2 - \sum_{i<j}[X_i,X_j]^2\right) \to \mathbb{M}^2_{susy} = \mathbb{M}^2_N\cdot 1 + i f_{abc}\theta_{\alpha a}\theta_{\beta b}x_{kc}\gamma^k_{\alpha\beta}$$

$$Q^{(N)}_\beta = \left(p_{ia}\gamma^i_{\beta\alpha} + \frac{1}{2}f^{(N)}_{abc}x_{ib}x_{jc}\gamma^{ij}_{\beta\alpha}\right)\theta_{\alpha a}, \quad \{Q_\beta, Q_{\beta'}\} = \delta_{\beta\beta'}\mathbb{M}^2_{susy} + 2\gamma^i_{\beta\beta'}x_{ia}J_a$$

$$J_a = f_{abc}\left(x_{ib}p_{ic} - \frac{i}{2}\theta_{\alpha b}\theta_{\alpha c}\right); \quad \begin{array}{l} abc = 1\ldots N^2-1 \\ ij = 1\ldots d\end{array} \quad d = 2,3,5,9 \qquad \alpha\beta = 1\ldots \tilde{d} = 2,4,8,16$$

$$Q_\beta = \int\left(p_{i}/\rho\,\gamma^i_{\beta\alpha} + \frac{1}{2}\{x_i,x_j\}\gamma^{ij}_{\beta\alpha}\right)\theta_\alpha\, d^2\varphi$$

$\to$ Lax-pairs for (classical) bosonic systems:

$$H = \frac{1}{2}\left(p_{ia}p_{ia} + \frac{1}{2}f_{abc}f_{ade}x_{ib}x_{jc}x_{id}x_{je}\right)$$

$$\dot{x}_{ia} = \frac{\partial H}{\partial p_{ia}} = p_{ia}, \quad \dot{p}_{ia} = -\frac{\partial H}{\partial x_{ia}} = \ldots \overset{?}{\Leftrightarrow} \dot{L}_\beta = [L_\beta, M]$$

$$L_\beta = Q_\beta\ (\dot\theta = 0) \qquad M = \frac{1}{2}f_{abc}\theta_{\alpha a}\theta_{\beta b}\gamma^i_{\alpha\beta}x_{ic}$$

$$\dot{L}(\lambda) = [L(\lambda), M] \quad \left(L(\lambda) := \sum_1^{\tilde{d}}\lambda_\beta L_\beta\right)$$

any d! $\quad \Rightarrow \frac{d}{dt}\left(\mathrm{Tr}\,L^k(\lambda)\right) = 0$

Lax-pairs and r-matrix (a simple example)

$$H = \frac{1}{2}\left(\vec{p}^{\,2} + (\nabla W)^2\right)_{W=W(x_1 \ldots x_N)} \qquad \dot{\vec{x}} = \vec{p} \qquad \dot{p}_a = -\vec{\nabla} W \, \partial_a \vec{\nabla} W$$

$$L_1 = \sum_1^N \left(\gamma_a p_a - \gamma_{a+N}\partial_a W\right), \quad L_2 = \sum \left(\gamma_a \partial_a W + \gamma_{a+N} p_a\right) \qquad \left(H_{susy} = H \cdot \mathbb{1} + i \gamma_a \gamma_{b+N} \partial^2_{ab} W\right)$$

$$\dot{L}_\beta = \left[L_\beta, M = -\frac{1}{2}\gamma_a \gamma_{b+N}\partial^2_{ab} W\right] \qquad \dot{L}(\lambda) = [L(\lambda), M]$$

Using fundamental rep. of $SO(2N+1)$ (instead of spinor rep.)

$$\tilde{L} = i\begin{pmatrix} 0 & V^T \\ -V & 0_{2N \times 2N}\end{pmatrix}, \quad V = \begin{pmatrix} \lambda_1 \vec{p} + \lambda_2 \nabla W \\ \lambda_2 \vec{p} - \lambda_1 \nabla W \end{pmatrix} \qquad \dot{\tilde{L}} = [\tilde{L}, \tilde{M}]$$

$$V^2 = (\lambda_1^2 + \lambda_2^2)\, 2H \qquad \tilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & -A \end{pmatrix} \Updownarrow A = \begin{pmatrix} 0 & W_{ab} \\ -W_{ab} & 0 \end{pmatrix}$$

$$J(\lambda) := \frac{\tilde{L}}{\sqrt{2}\sqrt{\lambda_1^2 + \lambda_2^2}} = U(\lambda) \begin{pmatrix} H & & \\ & -H & \\ & & 0 \ldots 0 \end{pmatrix} U^\dagger(\lambda) \qquad \dot{V} = AV \quad \text{resp. } \dot{e} = Ae$$

$$\{J \times 1, 1 \times J\} = \{J_1, J_2\} = [r_{12}, J_1] - [r_{21}, J_2]$$

$$r_{12} = \frac{1}{2}\left([U_{12}, J_2] - \frac{M_1 J_2}{H}\right) \qquad U_{12} = \{U_1, U_2\} U_1^{-1} U_2^{-1}$$

Quantum Minimal Surface Algebras

(work in progress, with R. Köhl and R. Lautenbacher)

$$\sum_\nu [[X^\mu, X^\nu], X_\nu] = 0$$ (96 H, IKKT / 99 CW / 02 N, CD / 07 S / 08 HS / 12 ACH / 19 with J. Arnlind and M. Kontsevich)

$$\sum_j [[M_i, M_j], M_j] = \mu_i M_i$$ (82 H / 85 CWW / 97 W / 03 AH+T / 05 BD / 09) DMSA (0903.5237)

Berman (89) (Com. Alg. 17), for simply laced KM (also for GIM, cp. Slodowy 84)

$$[[Y_i, Y_j], Y_j] = Y_i \text{ (no sum) if } A_{ij} = -1, \quad [Y_i, Y_j] = 0 \text{ if } A_{ij} = 0$$

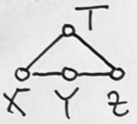
$$(\Rightarrow) \sum_j [[Y_i, Y_j], Y_j] = m_i Y_i; \quad m_i := -\sum_{k \neq i} A_{ik},$$

resp. $\sum_j [[X_i, X_j], X_j] = -m_i X_i$ if $[[X_i, X_j], X_j] = -X_i$)

$X_i \to e_i \Rightarrow \ldots = 0$ (KL, D; in particular $\mathbb{R}^{1,9} \to E_{10}$)

QMSA representations (HKL)

$$[[X^\Gamma, X^\nu], X_\nu] = [[X^\Gamma, T], T] \underset{(+)}{-} \sum_{i=1}^{3} [[X^\Gamma, X^i], X^i] = 0$$



Example $D=4$, $\eta_{\mu\nu} = \mathrm{diag}\underset{(+\,+\,+\,+)}{(1,-1,-1,-1)}$; $\text{"}\mathcal{L}(so(1,3))\text{"} = S(\tilde{A}_3)$ (Subalgebra invariant under a specific involutive automorphism)

$\underset{\mathrm{Fix}(\mathcal{L}(sl(4)))}{}$

$X^1 = X = E_{12} - E_{21}$, $X^2 = E_{23} - E_{32}$, $X^3 = E_{34} + E_{43}$, $X^4 = T = sE_{41} \underset{(+)}{-} \frac{1}{s} E_{14}$

$\left([E_{IJ}, E_{KL}] = \delta_{JK} E_{IL} - \delta_{LI} E_{KJ}\right.$; $E_{JK}^\dagger = -E_{JK}$ unitarizable, infinite dimensional $\rightarrow \left. X^{\mu\dagger} = -X^\mu \right)$

Classically: $\int G^\nu d^{M+1}\varphi \rightarrow \partial_\alpha (G^\nu G^{\alpha\beta} \partial_\beta X^\Gamma) = 0 \underset{(\nu \neq 1/2)}{\Rightarrow} G = \mathrm{const}.$

In the example: $[X^\Gamma, X^\nu][X_\mu, X_\nu] = (E_{13} - E_{31})^2 + (E_{24} + E_{42})^2 \underset{(+)}{-} (sE_{31} \underset{(-)}{+} \frac{1}{s} E_{13})^2 \underset{(+)}{-} (sE_{42} \underset{(+)}{-} \frac{1}{s} E_{24})^2$

Note: Just as spaces of solutions to relativistically invariant wave-equations provide unitary representations of the Poincaré group ($(\Box + m^2)\phi = 0 \leftrightarrow$ scalar), Solutions of the (infinite component) 'composite' equations* (#9602020 p.14) should be related to representations of ∞-dim. extensions of $so(1, D-1)$

* $\{\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_{M+1}}\}, X_{\mu_1}; \ldots; X_{\mu_M}\} = 0$ (classical and quantum)

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