Fractal properties of the Hofstadter butterfly, eigenvalues, and topological phase transitions

Svetlana Jitomirskaya

Picture Language Seminar, June 1, 2021

Discrete 2D Laplacian

$$H : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$$

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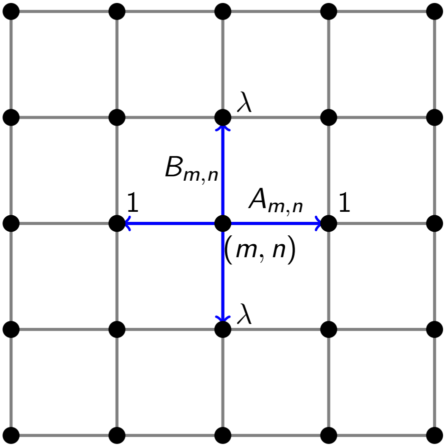
is unitarily equivalent (Fourier transform) to $H_{2\cos x + 2\cos y}$ on $L^2(\mathbb{T}^2)$, so $\sigma(H) = [-4, 4]$.

Adding a magnetic field



Choice of gauge

$$A_{m,n} + B_{m+1,n} - A_{m,n+1} - B_{m,n} = 2\pi\alpha,$$

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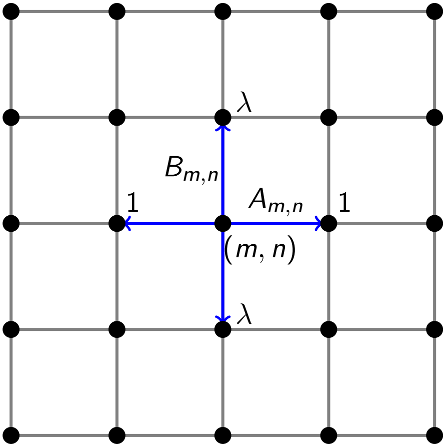

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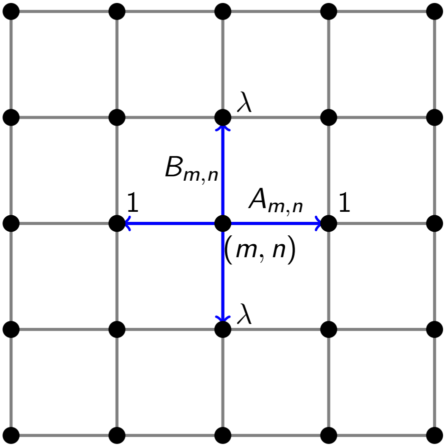

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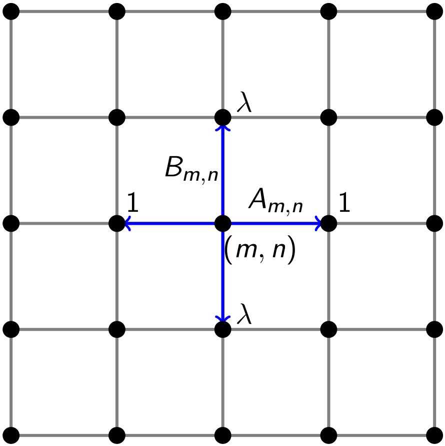

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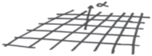
The Harper's model

The tight-binding model of 2D Bloch electrons in magnetic fields



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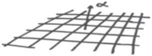

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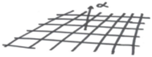

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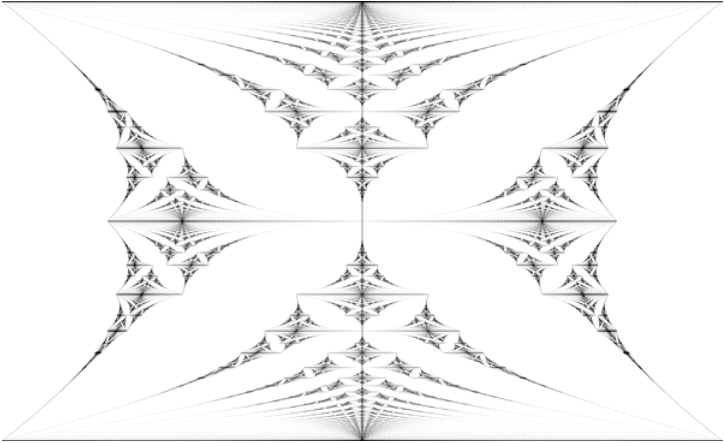
$$\alpha > 0?$$



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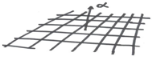

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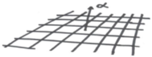

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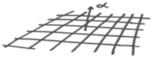

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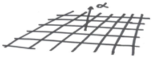

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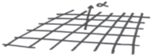

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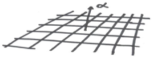

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- Anisotropic lattice:

$$(H(\alpha,\lambda)\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \lambda e^{-i\alpha m}\psi_{m,n-1} + \lambda e^{i\alpha m}\psi_{m,n+1}$$

- α is a dimensionless parameter equal to the ratio of flux through a lattice cell to one flux quantum.

Prizes

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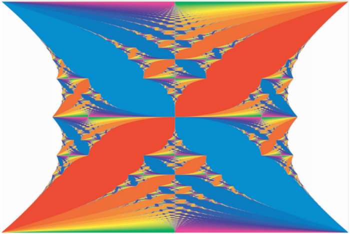
- Fields medal 2014
- Topological insulators (Nobel, 2016)
- Heineman prize 2020

Thouless theory of IQHE illustrated



Avron-Osadchy

Aubry-Andre model aka the almost Mathieu operator

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With a choice of Landau gauge effectively reduces to

$$(H_{\alpha,\theta}\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)\Psi_n$$

Metal-insulator transition

For E in the spectrum

- Coupling $\lambda > 1$ (supercritical)
  a.e. pure point spectrum (SJ 99)
- Coupling $\lambda < 1$ (subcritical) no point spectrum for all (α, θ) (Delyon 87) (in fact, pure ac spectrum, SJ 99,Avila-SJ 09, Avila 10)
- Coupling $\lambda = 1$ (critical) - boundary of both regions.

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Theorem (J, 2021)

No!

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On point spectrum of critical almost Mathieu operators

S. Jitomirskaya[1]

Abstract. There isn't any.

1. Introduction

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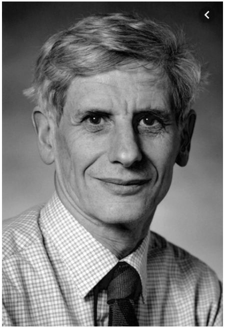
Popularized by Simon as one of the XXI century problems
Avila-Krikorian 2006: solved for a.e. (all remaining!) α.

Dimension of the spectrum for the critical case: Thouless' conjecture



David Thouless (1934-2019)

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Not true:

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2. J-Shiwen Zhang: box counting equal to one for α with $\beta(\alpha) > 0$ where $\beta(\alpha) := \limsup_{n\to\infty} -\frac{\ln q_{n+1}}{q_n}$.
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Barry Simon's new list of hard unsolved problems (March 2019)

$1/2$ of the Thouless' conjecture

Theorem (SJ-Igor Krasovsky, preprint)

For any irrational α, $dim_H(\sigma(H_\alpha)) \leq 1/2$.

Aubry duality

$$(H_{\alpha,x}\Psi)_n = \Psi_{n+1} + \Psi_{n-1} + 2\lambda\cos 2\pi(x+n\alpha)\Psi_n$$

For $u \in \ell^2(\mathbb{Z})$, set $\hat{u}(x) = \sum e^{2\pi inx} u_n$, the Fourier transform of u. If u solves $H_{\alpha,\theta}u = Eu$, then $v_n^x := e^{2\pi in\theta}\hat{u}(x+n\alpha)$ solves

$$H_{\alpha,x}v^x = Ev^x \qquad (1)$$

for a.e. x.

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v and $\bar{v}$ linearly dependent if and only if $2\theta + k\alpha \in \mathbb{Z}$.

Hopelessly breaks down for $2\theta + k\alpha \in \mathbb{Z}$.

New transform

Given $u \in \ell^2(\mathbb{Z})$, set

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{\pi i n(\theta + n\alpha - 2x)} \tag{2}$$

and, for a.e. x,

$$u_n^x = u(x + n\alpha) e^{\pi i n(x + \frac{n\alpha - 3\theta}{2})} \tag{3}$$

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Set $\tilde{H}_\alpha^x : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, x \in \mathbb{R}/2\mathbb{Z}$, by

$$(\tilde{H}_\alpha^x v)_n = 2\cos\pi(x + n\alpha) v_{n-1} + 2\cos\pi(x + (n+1)\alpha) v_{n+1} \tag{4}$$

Lemma

If $u \in \ell^2(\mathbb{Z})$ solves $H_{\alpha,\theta} u = Eu$, then $u^x \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of

$$\tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x = E u^x \tag{5}$$

for a.e. x.

Let $u \in \ell^2(\mathbb{Z})$, $\|u\|_2 = 1$ solve $H_{\alpha,\theta} u = Eu$. Then, for a.e. x,

$$\tilde{H}_\alpha^{x+\frac{\theta-\alpha}{2}} u^x = E u^x \tag{6}$$

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$$\tilde{H}_{\alpha}^{x+\frac{\theta-\alpha}{2}} \bar{u}^x = E \bar{u}^x \tag{7}$$

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The Wronskian of u^x and $\bar{u}^x$ is constant in n.

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Thus for a.e. x

$$\cos \pi x(u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x)e^{-\pi ix-ia(\alpha,\theta)}) = c. \tag{9}$$

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For a.e. x, we have $u(x) \neq 0$.

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Set $\phi(x)$ on on $\mathbb{R}/2\mathbb{Z}$ by

$$\phi(x) := \frac{u(x)}{\bar{u}(x)} e^{\pi ix+ia(\alpha,\theta)} \quad (12)$$

Then, $|\phi(x)| = 1$ and for a.e. x,

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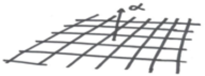
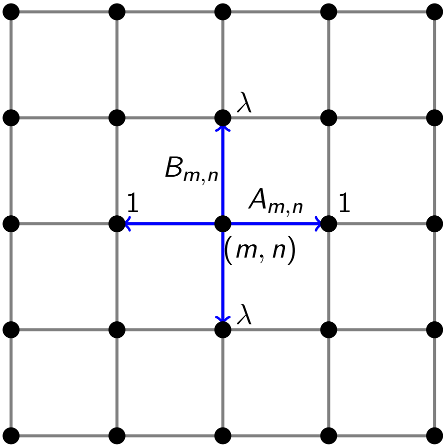
So if $\phi(x) = \sum_{k=-\infty}^{\infty} a_k e^{\pi ikx}$, we have $|a_{k+2}| = |a_k|$, a contradiction. □

The Harper's model

The tight-binding model of 2D Bloch electrons in magnetic fields



Standard Landau gauge



Choice of gauge

$$A_{m,n} + B_{m+1,n} - A_{m,n+1} - B_{m,n} = 2\pi\alpha,$$

here $2\pi\alpha$ is the total flux through each cell.

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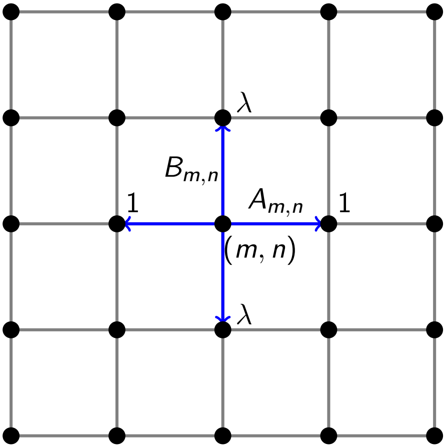

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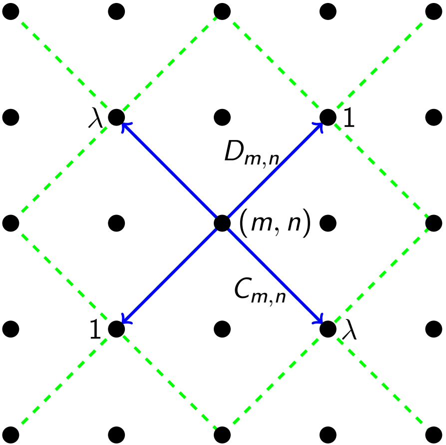

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$$C_{m,n} + D_{m+1,n} - C_{m,n+1} - D_{m,n} = 2\pi \cdot 2\alpha,$$

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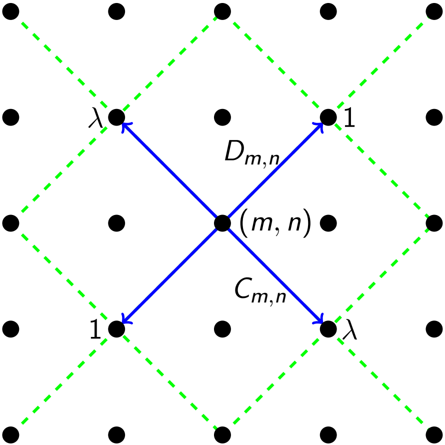

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Theorem (J-Krasovsky, preprint)

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almost immediate from the transform.

Hausdorff dimension

Thouless' argument:

- $|\sigma(H_{p/q})| \approx \frac{c}{q}$
- If $\sigma(H_\alpha)$ is "economically covered" by $\sigma(H_{p_n/q_n})$ and if all bands are of about the same size, the dimension is bounded by $1/2$.

"economically covered" $=$ continuity of spectra.

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J-Krasovsky, 19: representation as a **singular** Jacobi matrix allows to prove almost Lipshitz continuity of spectra - enough for all irrational α!

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As $p_n/q_n \to \alpha$, we have $q_n|\sigma(p_n/q_n)| \to c$ where $c = 32C_c/\pi = 9.33\ldots$.
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Using the chiral representation and a technique of S. Becker-R. Han-J, 19