Invariant differential forms, graph complexes & Feynman integrals.

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Harvard Mathematical Picture Language,

June 22nd 2021

arXiv : 2101.04419

Overview.

Output

Maths:

Graph complexes
Graph homology

Canonical integrals assigned to graphs.

Moduli space of tropical curves / metric graphs

Moduli space of tropical abelian varieties

Input

Symmetric spaces for $GL_n(\mathbb{Z})$

Biinvariant diff. forms live here

Physics:

Feynman integrals in perturbative Quantum Field Theory

ϕ^4, QED, QCD

I). "Commutative" graph complex GC_2.

- G a connected graph
  - no self edges 
  - no vertices of deg. ≤ 2 

- An orientation on edges is

$$\eta = e_1 \wedge \dots \wedge e_n \qquad \in \qquad (\det \mathbb{Z}^{E_G})^{\times}$$

i.e., ordering on edges up to sign.

GC_2 = $\mathbb{Q}$ - vector space generated by

(G, η) oriented graphs

modulo
- $(G, -\eta) = -(G, \eta)$
- $(G, \eta) = (G', \sigma(\eta))$

$\sigma : G \xrightarrow{\sim} G'$ isom.

Denote equivalences $[G, \eta]$.

Differential

$$d\,[G, e_1\wedge\ldots\wedge e_n] = \sum_{i=1}^{n} (-1)^i \,[G/\!\!/e_i,\; e_1\wedge\ldots\wedge\hat{e}_i\wedge\ldots\wedge e_n]$$

contract edge e_i

Well-defined & satisfies $d^2=0$.

Definition: Graph homology (Kontsevich)

$$H(GC_2) = \frac{\ker d}{\operatorname{Im} d}$$

It is bigraded:

- $H = \bigoplus_n H_n(GC_2)$ $\qquad n = e_G - 2h_G$
- $H_n(GC_2)$ also graded by e_G (or h_G)

e_G: # edges $\qquad h_G$: # loops

Examples

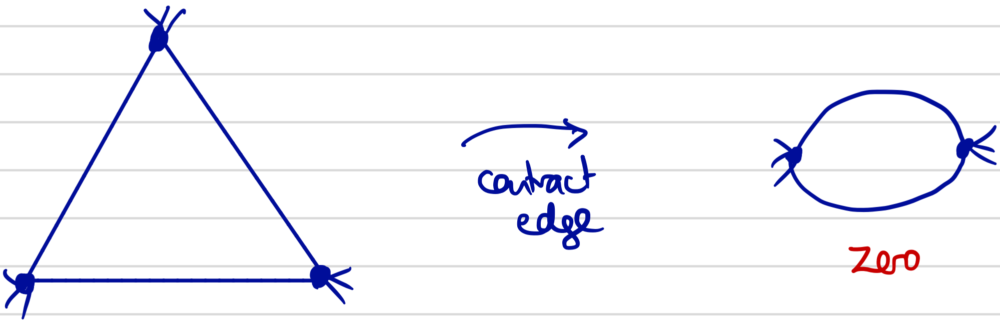
(i) If G admits an automorphism σ inducing an <u>odd</u> permutation of E_G, then

$$[G,\eta] = [G,\sigma\eta] = -[G,\eta] = 0$$

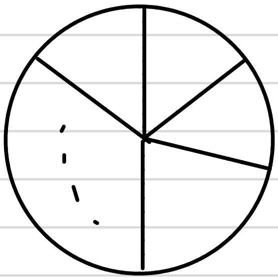
Ex: G has a bubble

then $[G,\eta]=0$.

(ii) If G such that every edge lies in a triangle then $dG=0$



- **Wheels**

W_n = 

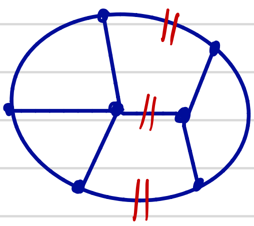
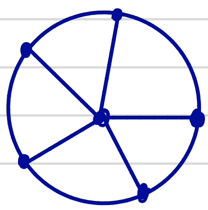
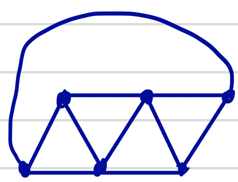
degree

$e_G - 2h_G = 0$

even: $W_{2n} = 0$ by (i)

odd: $d[W_{2n+1}] = 0$ by (ii)

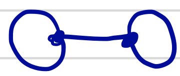
$[W_{2n+1}] \in H_0(GC_2)$ known to be non-zero.

- d  =  $-\,2$ 

Extra structures:

- Coalgebra structure

$$G \longmapsto \sum_{\substack{\gamma \subseteq G \\ \text{core}}} \gamma \wedge G/\gamma \qquad \text{Connes-Kreimer}$$

core $\Longleftrightarrow$ no bridges 

Graph cohomology is dual to graph homology, Lie algebra.

III/. Known results

- Chan - Galatius - Payne ('20)

$$H(GC_2) \cong \bigoplus_{g \geq 2} \mathrm{gr}^{W}_{6g-6} H^{\bullet}(\mathcal{M}_g ; \mathbb{Q})$$

- Willwacher ('15)
  - $H_n(GC_2) = 0 \qquad n < 0 \qquad (h_G > 0)$
  - $H_0(GC_2) \cong \mathfrak{grt}^{\vee}$ Lie coalg.

$\mathfrak{grt}$ = Grothendieck–Teichmüller Lie algebra (Drinfeld)

Deligne conj. '90, B. '12 !

conj. to be isom. (Drinfeld)

- $\mathrm{Lie}(\sigma_3, \sigma_5, \ldots, \;) \xhookrightarrow[\text{non.canonical}]{} \mathfrak{grt}$

  free graded Lie alg.

  $\deg \sigma_{2n+1} = -(2n+1)$

Q: How to interpret graph homology?

- How to think about $\mathfrak{grt}^\vee$?

There exists a natural map

$$\mathfrak{grt}^\vee \longrightarrow \mathcal{Z} \big/ (\zeta(2), \text{products})$$

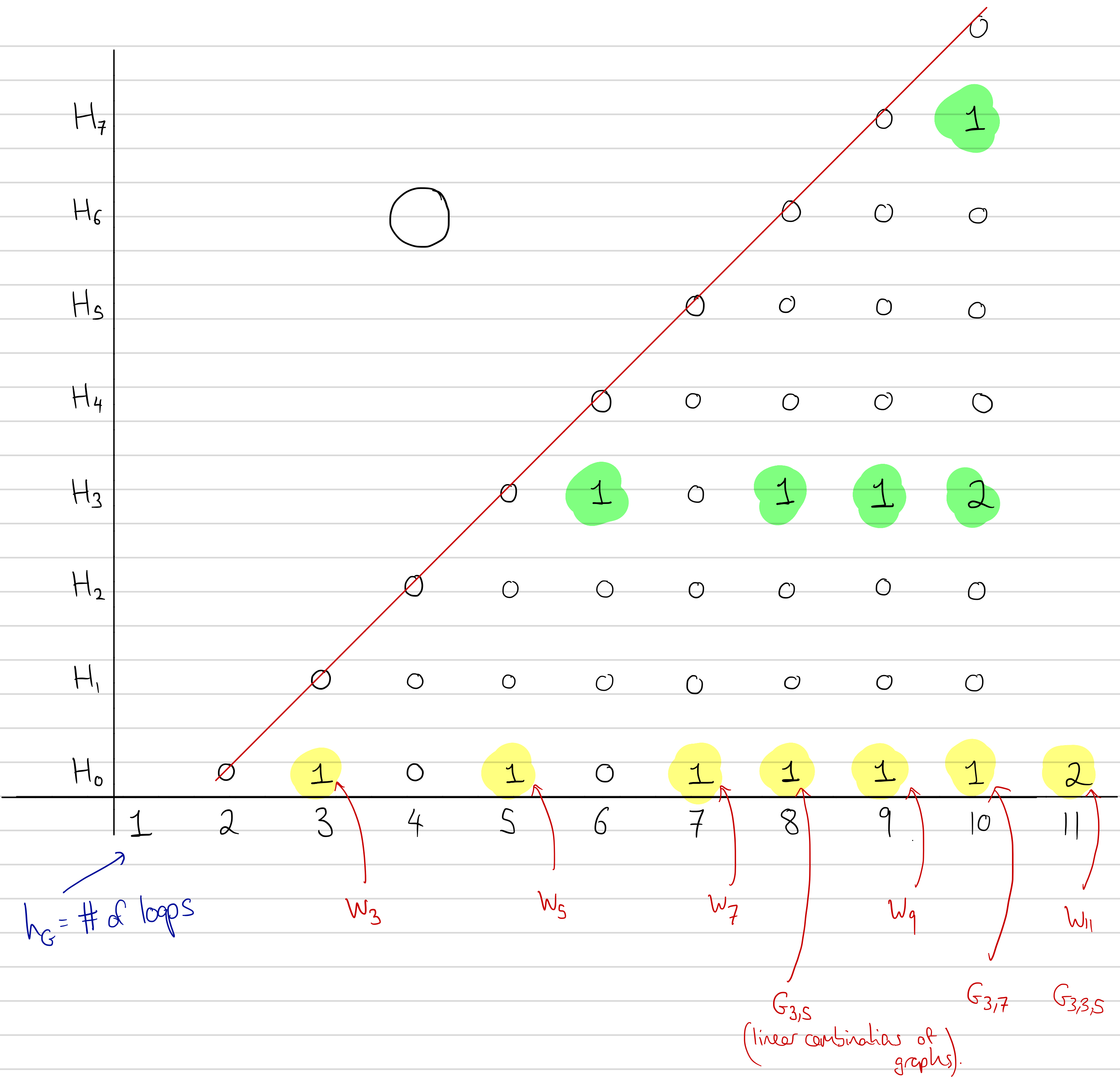
where $\mathcal{Z}$ = $\mathbb{Q}$ - v. space spanned by multiple zeta values

$$\zeta(n_1, .., n_r) = \sum_{1 \leq k_1 < .. < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

$n_1, .., n_r \geq 1$
$n_r \geq 2$

(Euler)

Many relations eg: $\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$



Yellow classes : $\mathfrak{grt}^\vee$ (formal MZV's mod products)

Green classes : ?? unknown.

(Numerical computations)

This suggests:

$$\text{graphs} \longrightarrow \text{numbers}$$

$$[W_{2n+1}] \rightsquigarrow \zeta(2n+1)$$

- $H_n(GC_2)$ = homology of some space M
- M should have differential forms (de Rham complex)

$$[\omega] \in H^n_{dR}(M)$$

$$H_n(GC_2) \longrightarrow \mathbb{C}$$
$$\gamma \longmapsto \int_\gamma \omega$$

Next:
- M moduli space of metric graphs
- ω invariant form.

bi-Invariant forms

Classical : Write down forms on spaces of matrices which are invariant under left & right mult. by $GL_n(\mathbb{Z})$.

- $X \in GL_n(R^0)$ $\qquad R = \bigoplus_{k \geq 0} R^k$ graded-comm. DGA over $\mathbb{Q}$

$$\mu_X = X^{-1} dX$$

left-invariant $\mu_{AX} = \mu_X$, $A \in GL_n(\mathbb{Z})$

- Define : $\beta_X^k = tr(\mu_X^k) \in R^k$ "k-form".

Properties :

- $\beta_X^{2k} = 0$
- $d\beta_X^{2k+1} = 0$
- $\beta_{X^T}^k = (-1)^{\frac{k(k-1)}{2}} \beta_X^k$

- The β^k_X are bi-invariant

$$\beta^k_{AX} = \beta^k_{XA} = \beta^k_X \qquad A \in GL_n(\mathbb{Z})$$

$\beta^1_X = d(\log \det(X))$ odd one out.

We get : $\beta^3_X, \beta^5_X, \beta^7_X, \ldots$ in general

If X symmetric, $0 = \beta^3_X = \beta^7_X = \ldots$, left with

$$\beta^5_X, \quad \beta^9_X, \quad \beta^{13}_X, \ldots$$

Examples.

- $$X = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}$$ generic

$$\beta^3_X = 3 \frac{\sum_{i=1}^{4} (-1)^i a_i \, da_1 \ldots \widehat{da_i} \ldots da_4}{(a_1 a_2 - a_3 a_4)^2}$$

$$\beta^S_X = 0$$

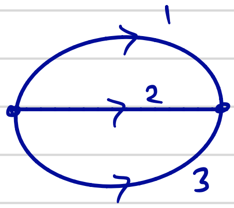
- $$X = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{pmatrix}$$ symmetric

$$\beta^3_X = 0$$

$$\beta^S_X = -10 \sum_{i=1}^{6} (-1)^i \frac{a_i \, da_1 \ldots \widehat{da_i} \ldots da_6}{\det(X)^2}$$

· Graph Laplacian

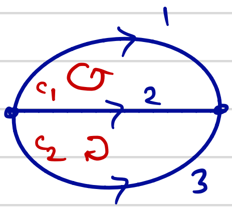
G connected graph.



- Number edges $e_1, \dots, e_N$

  Edge $i \rightsquigarrow$ parameter α_i "Schwinger parameter"

- Choose basis $c_1, \dots, c_h$ of homology cycles $H_1(G;\mathbb{Z}) \subseteq \mathbb{Z}^{E_G}$



$$c_1 = e_2 - e_1$$

$$c_2 = e_2 - e_3$$

- Incidence matrix $\mathcal{H}$

$$(\mathcal{H})_{e,c} = \text{coefficient of } e \text{ in } c$$

$$\mathcal{H} = \begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \end{array}$$

- a graph Laplacian matrix:

$$\Lambda_G = \mathcal{H}^T \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix} \mathcal{H}$$

$$\Lambda_\ominus = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 + \alpha_2 & -\alpha_2 \\ -\alpha_2 & \alpha_2 + \alpha_3 \end{pmatrix}$$

- $\psi_G = \det(\Lambda_G)$ graph polynomial (Kirchhoff)

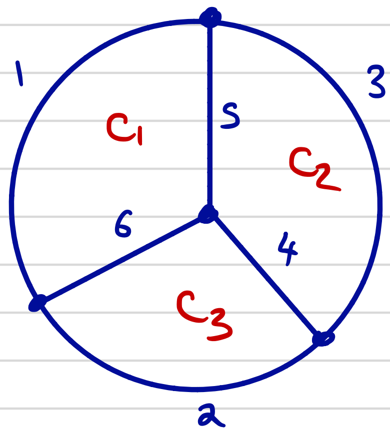
$$\psi_\ominus = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$

- <u>Define</u> : A (primitive) canonical graph form:

$$\omega_G^{4k+1} = \beta^{4k+1}_{\Lambda_G} = \operatorname{tr}\left(\left(\Lambda_G^{-1} d\Lambda_G\right)^{4k+1}\right)$$

Well-defined, by bi-invariance.

Ex :



$$\Lambda_G = \begin{pmatrix} \alpha_1+\alpha_5+\alpha_6 & -\alpha_5 & -\alpha_6 \\ -\alpha_5 & \alpha_3+\alpha_4+\alpha_5 & -\alpha_4 \\ -\alpha_6 & -\alpha_4 & \alpha_2+\alpha_4+\alpha_6 \end{pmatrix}$$

$$\omega^{5}_{W_3} = 10 \frac{\Omega_{W_3}}{\Psi^2_{W_3}}$$

$$\Omega_G = \sum_{i=1}^{N} (-1)^i \alpha_i \, d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_N$$

Canonical Integrals

- A canonical form is a product of primitive ones

eg. $\omega_G = \omega_G^5 \wedge \omega_G^9$

Ω^{can} = Exterior algebra on $\omega_\bullet^5, \omega_\bullet^9, \ldots$

- G with $e_G = d+1$ edges

$\omega \in \Omega^{can}$ of degree d

$$I_G(\omega) = \int_{\alpha_i \geq 0} \omega_G$$

Theorem: The integral is always finite.

Integrand of the form:

$$\omega_G^{4k+1} = \frac{N_G}{\psi_G^k} \cdot \Omega_G$$

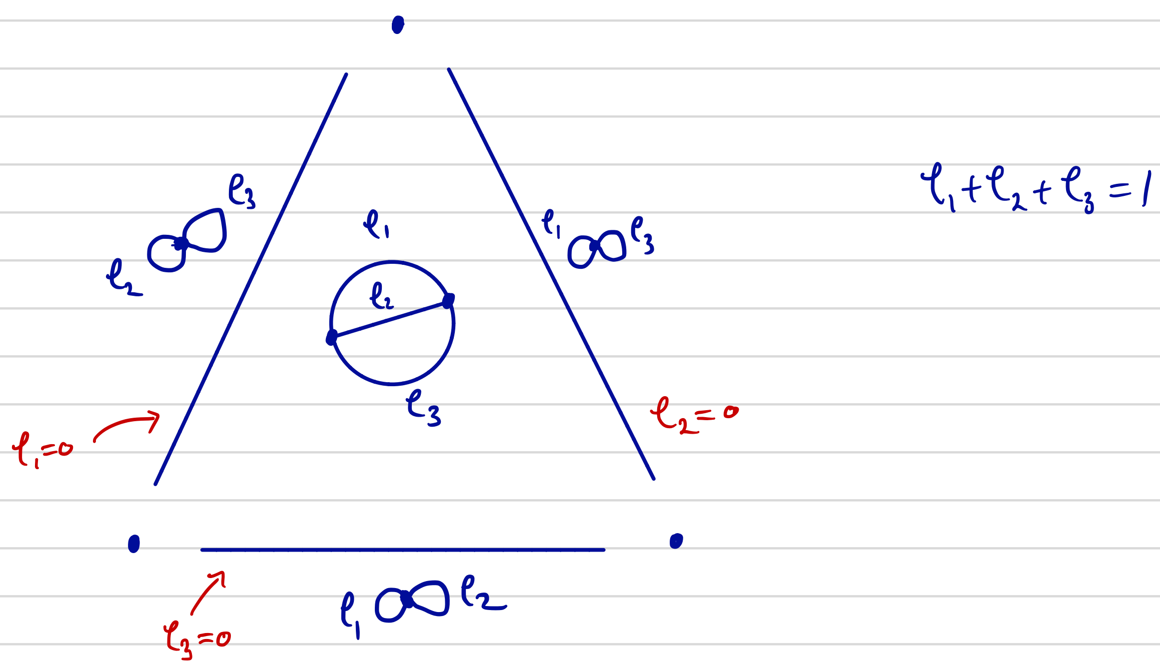
N_G = polynomial in α_i.

V) Metric graphs :

Metric graph G = graph G + $\ell_e \in \mathbb{R}_{>0}$, $\forall e$
length of each edge.

such that total length $\sum_{e \in E_G} \ell_e = 1$.

$\sigma(G)$ = space of metrics on G = open simplex $\{\ell_e > 0, \ \sum \ell_e = 1\}$



Faces of closed simplex $\overline{\sigma}(G)$: $\ell_e = 0 \longleftrightarrow \overline{\sigma}_{G/e}$

These simplices assemble to form Outer Space (Culler-Vogtmann), moduli of metric graphs (moduli of tropical curves)

(NB not all σ_G allowed, automorphisms)

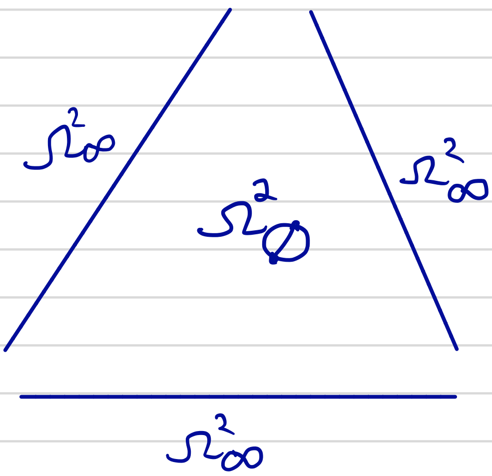
Canonical forms

Glue together

$$\begin{cases} \omega_G^{4k+1}\Big|_{\ell_e=0} = \omega_{G/e}^{4k+1} \\[2em] \omega_{\sigma(G)}^{4k+1} = \sigma^* \omega_G^{4k+1} \end{cases}$$

$$\forall \sigma \in \mathrm{Aut}(G)$$

+ many other nice properties.



← forms glue together.

Interpretation

In fact,

$$\left\{\begin{array}{c}\text{Moduli space of} \\ \text{metric graphs} \\ \text{tropical curves}\end{array}\right\} \xrightarrow[J]{\text{"tropical Torelli"}} \left\{\begin{array}{c}\text{tropical abelian} \\ \text{varieties}\end{array}\right\}$$

$$G \longmapsto [\Lambda_G]$$

$$\left\{\begin{array}{c}\text{+ve semi-definite matrices} \\ \text{of rank } g\end{array}\right\} \Big/ GL_g(\mathbb{Z})$$

$$\omega_G^{4k+1} = J^* \underbrace{\beta_X^{4k+1}}_{\text{invariant form.}}$$

Examples of canonical integrals

- $$I_{W_3}(\omega^5) = 10\,\zeta(3)$$

- $$I_{W_5}(\omega^9) = 1260\,\zeta(5)$$

- Conj : (w/ Schnetz)

$$I_{W_{2n+1}}(\omega^{4n+1}) = (2n+1)\binom{4n+2}{2n+1}\zeta(2n+1)$$

- Thm : There is $\Xi_{3,5} \in GC_2$

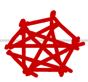
$$0 \neq [\Xi_{3,5}] \in H^3(GC_2)$$

$$I_{\Xi_{3,5}}(\omega^5 \wedge \omega^9) = \zeta(3)\zeta(5)$$

- (Schnetz-Borinsky)

$$I_{K_6}(\omega^5 \wedge \omega^9) = \text{MZV combination of } \zeta(3)\zeta(5),\ \zeta(3,5),\ \zeta(8)$$

Complete graph



Stokes Relation

$\omega \in \Omega^{can}$. Coproduct

$$\Delta\omega^{4k+1} = \omega^{4k+1}\otimes 1 + 1\otimes\omega^{4k+1}$$

$$\Delta\omega = \sum_i \omega_i' \otimes \omega_i''$$

$\forall$ $G \in GC_2$ with $\deg\omega + 1$ edges

$$\boxed{0 = \underbrace{\sum_{e\in E_G} I_{G/\!/e}(\omega)}_{\text{differential in } GC_2} + \underbrace{\sum_{e\in E_G} I_{G\setminus e}(\omega)}_{\text{"dual" differential}} + \underbrace{\sum_{\substack{\gamma\subseteq G\\ \text{core}}} \sum_i I_\gamma(\omega_i')\, I_{G/\gamma}(\omega_i'')}_{\text{Connes-Kreimer coalgebra str.}}}$$

Cor: If $X \in GC_2$, degree 0, $4k+2$ edges, $dX = 0$

$I_X(\omega^{4k+1}) \neq 0 \implies [X] \in H^0(GC_2)$ is non-zero.

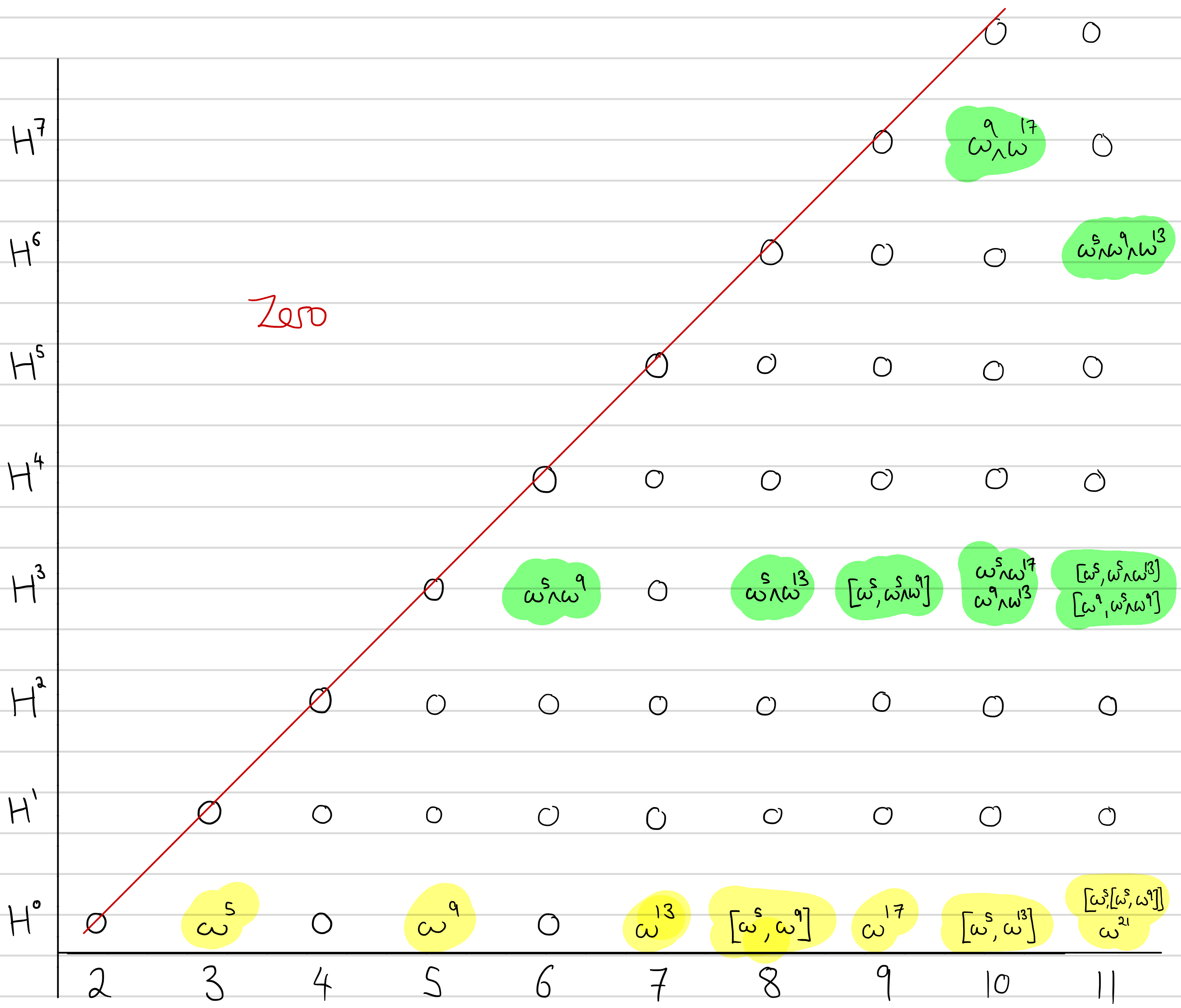
Canonical integrals can detect graph homology classes!

Conjecture: $\exists$ $\mathrm{Lie}\ \Omega^{\bullet}_{can} \xhookrightarrow[\text{non-can.}]{} \bigoplus_n H^n(GC_2)$

cohomology

Special case: $\mathrm{Lie}\left(\mathrm{Prim}\ \Omega^{\bullet}_{can}\right) = \mathrm{Lie}\left(\omega^5, \omega^9, \dots\right) \hookrightarrow \mathfrak{grt}$

$\mathrm{Lie}\left(\omega^5, \omega^9, \dots\right) \cong \mathrm{Lie}\left(\sigma_3, \sigma_5, \sigma_9, \dots\right) \hookrightarrow \mathfrak{grt}$ known



Feynman Integrals in ϕ^4_4.

Parametric form of residue in ε, $4-2\varepsilon$ spacetime dimensions

$$I_G^{Feyn} = \int_{\alpha_i \geq 0} \omega_G^{Feyn}$$

$$\omega_G^{Feyn} = \frac{\Omega_G}{\psi_G^2}$$

It is finite if:

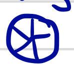
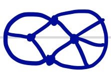
- $h_G = 2e_G$ degree is 0
- $h_\gamma < 2e_\gamma$ $\forall \ \gamma \subseteq G$

Many numbers I_G^{Feyn} also appear in QED, QCD,...

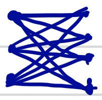
Studied intensively 1990's $\longrightarrow$ present

Broadhurst, Kreimer, Schnetz,....

"Amplitudes" community.

Feynman integrals I^{Feyn}			Canonical integrals	
W_3 	$6\,\zeta(3)$	$H_0(GC_2)$	W_3 	$60\,\zeta(3)$
W_4 	$20\,\zeta(5)$	$H_0(GC_2)$	W_5 	$1260\;\zeta(5)$
W_5 	$70\,\zeta(7)$	$H_0(GC_2)$	W_7	$24024\;\zeta(7)$
W_n	$\ast\;\zeta(2n-3)$	$H_0(GC_2)$	W_{2n+1}	$\ast\;\zeta(2n+1)$ (Conj.)
$W_3 : W_4$ 	$120\;\zeta(3)\zeta(5)$	$H_3(GC_2)$	$\Xi_{3,5}$	$\zeta(3)\zeta(5)$

$K_{3,4}$

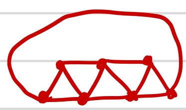


$I_{K_6}(\omega^5 \wedge \omega^9)$

complete graph

$$\ast\;\zeta(3)\zeta(5) + \ast\left(\zeta(3,5) - \frac{29}{12}\zeta(8)\right)$$

$$\ast\;\zeta(3)\zeta(5) + \ast\left(\zeta(3,5) - \frac{29}{12}\zeta(8)\right)$$

NB: W_n in ϕ^4, but zig-zag graphs  $= Z_n \in \phi^4$ play the same role.

$I^{\text{Feyn}}_{Z_n} = (\ast)\;\zeta(2n-3)$ (B., Schnetz)