Quantum scrambling with classical shadows

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Introduction

Quantum scrambling describes the delocalization of quantum information in quantum chaotic systems. Scrambling can be studied by probing the decay of out-of-time-ordered correlators (OTOCs). However, measurement of OTOCs is difficult due to their unusual time ordering. We present a protocol to measure higher-point out-of-time-ordered correlators using shadow tomography, a technique to measure functions of a state with few measurements. Our protocol avoids time-reversal and outperforms full quantum state tomography. We numerically demonstrate that higher-point OTOCs can reveal early-time scrambling behavior.

One-minute summary

Main idea

Measure dynamics of any higher-point OTOC using shadow tomography.

Protocol

Write the OTOC in terms of a state.

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Measure that state using shadow tomography.

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Use those measurements to estimate the OTOC.

Key result

Sufficiently many measurements produces a good estimation with a small variance.

Higher-point OTOCs

Quantum scrambling in a many-body system is quantified by commutator growth. The commutator between two initially commuting unitary operators, $[W(t), V]$, will grow as a system evolves chaotically. The Heisenberg operator

$$W(t) = U_H^\dagger(t) W U_H(t)$$

evolves according to chaotic Hamiltonian H with unitary $U_H(t)$. The information of $W(t)$ becomes scrambled as it spreads throughout the system. The commutator growth corresponds to the decay of out-of-time-ordered correlators (OTOCs). The system is said to become scrambled once the OTOCs decay to near-zero.

The Schatten 2n-norm of the commutator can be expanded in terms of OTOCs:

$$\Big|\Big|[W(t), V(0)]\Big|\Big|_{2n}^{2n} = \sum_{k=0}^{2n} a_k(n) \mathrm{Re}\{C_{4k}\}.$$

Higher-point OTOCs are

$$C_{4k} = \langle (W(t)V(0)W(t)V(0))^k \rangle.$$

The expectation value is taken with respect to the maximally mixed state. The time ordering makes the OTOC difficult to measure without implementing time reversal.

Shadow Tomography

What is shadow tomography?

Shadow tomography is a measurement technique which estimates functions of a target state.

How does it work?

1. Prepare an N-qubit target state ρ.
2. Randomly sample N independent, single-qubit Clifford unitaries u_i.
3. Apply Clifford unitaries to state $U = \otimes_{i=1}^N u_i$.
4. Measure state and store output $|\hat{b}\rangle\langle\hat{b}|$.
5. Construct the **classical snapshot** $$\hat{\rho} = \mathcal{M}^{-1}(U^\dagger |\hat{b}\rangle\langle\hat{b}| U).$$
6. Repeat K times to construct a set of K snapshots called the **classical shadow** $$S = \{\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_K\}.$$
7. Use the shadow to estimate functions of ρ.

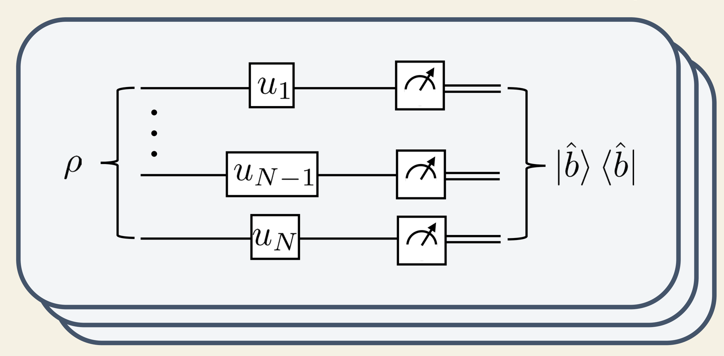

Figure 1: Quantum circuit to create a snapshot of target state ρ. N single-qubit, random Clifford unitaries are applied, then each qubit is measured. The output is used to construct a snapshot $\hat{\rho}_i$.

Properties of snapshots

Snapshots reproduce the exact state in expectation

$$\mathbb{E}(\hat{\rho}) = \rho.$$

$\mathbb{E}$ - average over random unitaries and outcomes.

Functions of states, $O = \mathrm{Tr}\{A\rho \otimes \rho\}$, can be estimated by introducing snapshots from the shadow

$$\hat{O} = \mathrm{Tr}\{A\hat{\rho}_1 \otimes \hat{\rho}_2\}.$$

The estimator satisfies $\mathbb{E}\{\hat{O}\} = O$.

Protocol

We apply shadow tomography to construct an estimator for the OTOC C_{4k}. Take $W = Z_1$ and $V = Z_N$. We write the OTOC in terms of target state ρ_V

$$C_{4k} = \frac{1}{d}\mathrm{Tr}\{(d^2\rho_V W \rho_V W - d\rho_V - dW\rho_V W + I^{\otimes N})^k\}$$

where the target state is

$$\rho_V = \frac{1}{2^{N-1}} U_H(t) I^{\otimes N_1} \otimes |0\rangle\langle 0| U_H^\dagger(t).$$

We use shadow tomography to create a shadow of ρ_V.

Protocol

1. Prepare an N-qubit state where qubit N is in the pure $|0\rangle\langle 0|$ state and the remaining qubits are in the maximally mixed state.
2. Evolve with unitary $U_H(t)$. Resulting state is ρ_V.
3. Create a shadow of size $K \geq 2k$ for ρ_V.
4. Use the shadow to construct an estimator for C_{4k}.

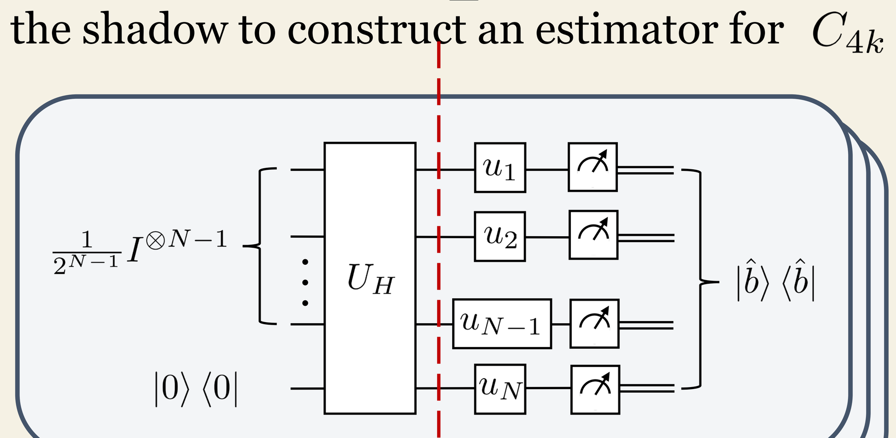

Figure 2: Quantum circuit to generate a classical snapshot of state ρ_V. Left of the dashed line gives preparation of ρ_V. Right of the dashed line shows shadow tomography.

Example: 4-point OTOC

Each pair of snapshots from the shadow is used to compute an estimator for C_4. An average over all estimators is used to construct the unbiased estimator of C_4.

$$S = \{\hat{\rho}_V^{(1)}, \hat{\rho}_V^{(2)}, \ldots, \hat{\rho}_V^{(K)}\}$$

$$\hat{C}_4 = \frac{1}{2}\binom{K}{2}^{-1} \sum_{i_1 \neq i_2} d\mathrm{Tr}\Big\{\hat{\rho}_V^{(i_1)} W \hat{\rho}_V^{(i_2)} W\Big\} - 1$$

Numerical Simulations

Dynamics of higher-point OTOCs

The first three OTOCs are

$k = 1 \quad C_4 = \langle W(t)V(0)W(t)V(0) \rangle$

$k = 2 \quad C_8 = \langle (W(t)V(0)W(t)V(0))^2 \rangle$

$k = 3 \quad C_{12} = \langle (W(t)V(0)W(t)V(0))^3 \rangle$

We plot the dynamics of these three OTOCs under evolution with a non-integrable, mixed-field Ising model. Higher point OTOCs exhibit faster and earlier decay, indicative of premature scrambling. The system becomes maximally scrambled as the OTOCs saturate to near-zero.

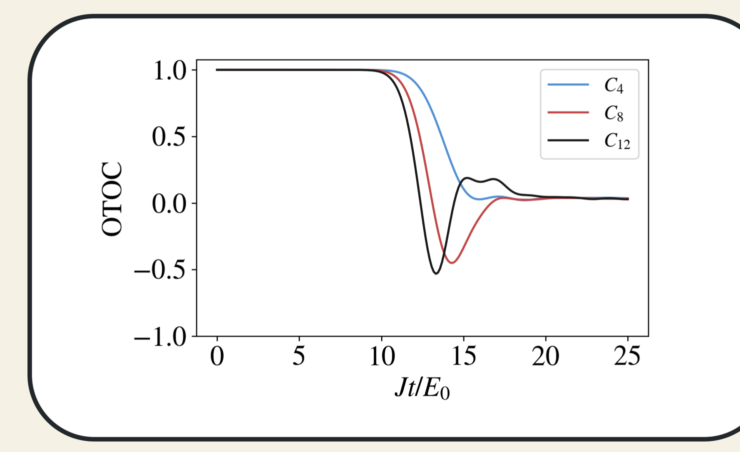

Figure 3: Plot of first three OTOCs for a mixed-field Ising model for N=10 qubits.

Estimation of OTOCs

Using the protocol, the first two OTOCs can be written as

$$C_4 = d\mathrm{Tr}\Big\{\rho_V W \rho_V W\Big\} - 1$$

$$C_8 = L_8 - 4C_4 - 3$$

The leading-order term for the eight-point OTOC is

$$L_8 = d^3\mathrm{Tr}\Big\{\rho_V W \rho_V W \rho_V W \rho_V W\Big\}$$

We numerically simulate our protocol and construct estimators for C_4 and L_8 using a mixed-field Ising model. We find that for a sufficiently large shadow size, agreement between the estimator and the exact values is good.

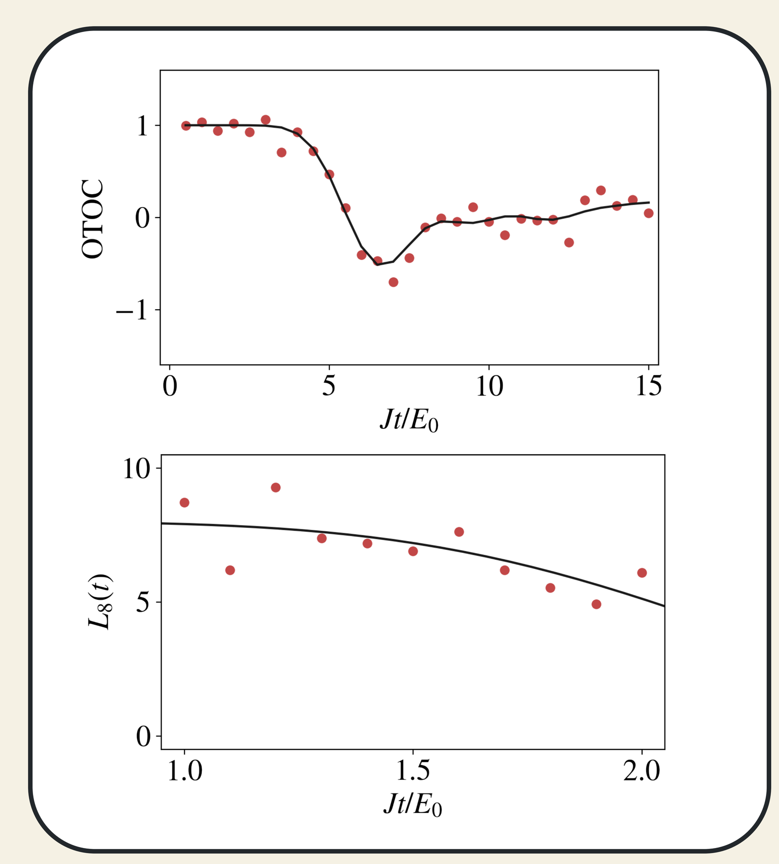

Figure 4: (Top) Red dots plot estimated values of C_4 for a shadow size of 15,000 and system size of N=4. Black line is exact value. (Bottom) Estimator for leading order term L_8 plotted against exact value for an average of 25 unbiased estimators, each computed with a shadow size of 150 (system size N=2).

Conclusion

We present a definition of higher-point out-of-time-ordered correlators to describe the dynamics of quantum scrambling in chaotic systems. In the early-time limit, higher-point correlators exhibit faster decay and reveal information delocalization earlier than the four-point OTOC. We present protocols using classical shadows to estimate these correlators and show they can outperform full quantum state tomography. For sufficiently many measurements, good agreement between the predicted and estimated values can be achieved.

References

[1] Roy J. Garcia, You Zhou, and Arthur Jaffe. Quantum scrambling with classical shadows, arXiv:2102.01008.