Prolate spheroidal functions and zeta

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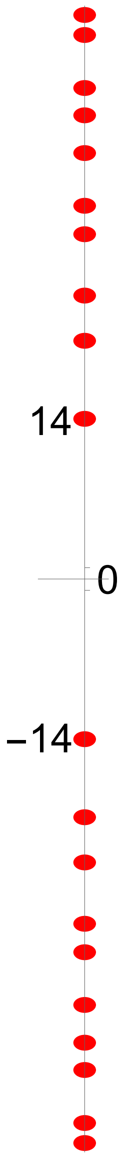
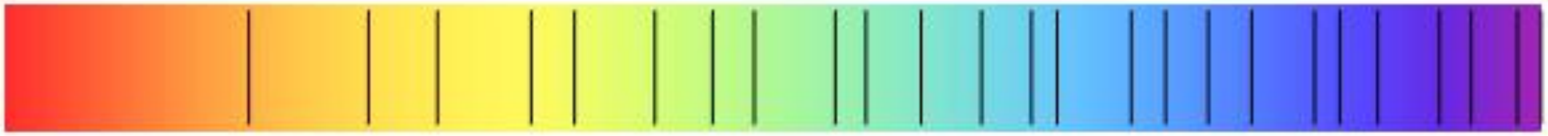
▶ The first part (joint work with C. Consani) will concern the low lying part of the spectrum and will give an operator theoretic incarnation of the Riemann-Siegel formula.

▶ The second part (joint work with H. Moscovici) will handle the ultraviolet part of the spectrum


14
0
−14

Mysterious Spectrum



Ultraviolet behavior

$$N(E) = \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}$$

$$+O(\log E)$$

Weil positivity

Riemann-Weil explicit formula

$$\widehat{f}(\frac{i}{2}) - \sum_{\frac{1}{2}+is\in Z} \widehat{f}(s) + \widehat{f}(-\frac{i}{2}) = \sum_{v} W_v(f)$$

$$QW_\lambda(f,g) := \sum_{1/2+is\in Z} \overline{\widehat{f}(\bar{s})}\,\hat{g}(s)$$

support f, g contained in $[\lambda^{-1}, \lambda]$

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma) f(1) +$$

$$+ \int_1^\infty \left(f(x) + f(x^{-1}) - 2x^{-1/2} f(1) \right) \frac{x^{1/2}}{x - x^{-1}} d^*x$$

$$W_p(f) = (\log p) \sum_{m=1}^{\infty} p^{-m/2} \left(f(p^m) + f(p^{-m}) \right)$$

Strong form of Weil positivity $(W_\infty := -W_{\mathbb{R}})$

$$W_\infty(f * f^*) \geq \mathrm{Tr}(\vartheta(f)\, S\, \vartheta(f)^*)$$

$$\forall f \in C_c^\infty(\mathbb{R}_+^*) \ \text{support}(f) \subset [2^{-1/2}, 2^{1/2}]$$

$$\widehat{f}(\frac{i}{2}) = 0, \widehat{f}(0) = 0$$

S = Projection on Sonin's space

$\vartheta(f)$ scaling action, $W_\infty = -W_{\mathbb{R}}$

$$\mathrm{tr}(\vartheta(f)\mathrm{S}) = W_\infty(f) + \int f(\rho)\epsilon(\rho)d^*\rho,$$

$\epsilon(\rho^{-1}) = \epsilon(\rho)$, $\rho \in \mathbb{R}^*_+$, $\epsilon(\rho)$, for $\rho \geq 1$, in terms of Prolate Wave functions

$$\epsilon(\rho) = \sum \frac{\lambda(n)}{\sqrt{1-\lambda(n)^2}} \langle \xi_n \mid \vartheta(\rho^{-1})\zeta_n\rangle.$$

A. Connes, C. Consani, *Weil positivity and trace formula, the archimedean place*. Selecta Math. (N.S.) 27 (2021) no 4.

A. Connes, C. Consani, *Quasi-inner functions and local factors*, Journal of Number Theory, 226 , pp. 139–167, 2021.

A. Connes, C. Consani, *Spectral triples and ζ-cycles*. Preprint (2021).

Minuscule eigenvalues

- For $\lambda^2 = 11$ the smallest positive eigenvalue is 2.389×10^{-48}

- The presence of these minuscule positive eigenvalues is explained conceptually by the fact that the radical of

Weil's quadratic form contains the range of the map $\mathcal{E}$, for $f(0) = \widehat{f}(0) = 0$

$$\mathcal{E}(f)(x) = x^{1/2} \sum_{n>0} f(nx)$$

- $f \in P_\lambda \Rightarrow$ support of $\mathcal{E}(f)$ is contained in $(0, \lambda]$

- $f \in \widehat{P}_\lambda \Rightarrow$ support of $\mathcal{E}(f)$ is contained in $[\lambda^{-1}, \infty)$

The pair P_λ, $\widehat{P}_\lambda$

$$P_\lambda = \{f \in L^2(\mathbb{R})^{\text{even}} \mid f(q) = 0,\ \forall q \ \text{ with } \ |q| > \lambda\}.$$

Fourier transform :

$$\mathbb{F}_{e_\mathbb{R}}(f)(y) := \int f(x) e^{-2\pi i x y} dx$$

$$\widehat{P}_\lambda = \mathbb{F}_{e_\mathbb{R}} P_\lambda \mathbb{F}_{e_\mathbb{R}}^{-1}.$$

Pairs of projections

Pair of projections P_i, $i = 1, 2$, $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, free product of two copies of the group $\mathbb{Z}/2\mathbb{Z}$.

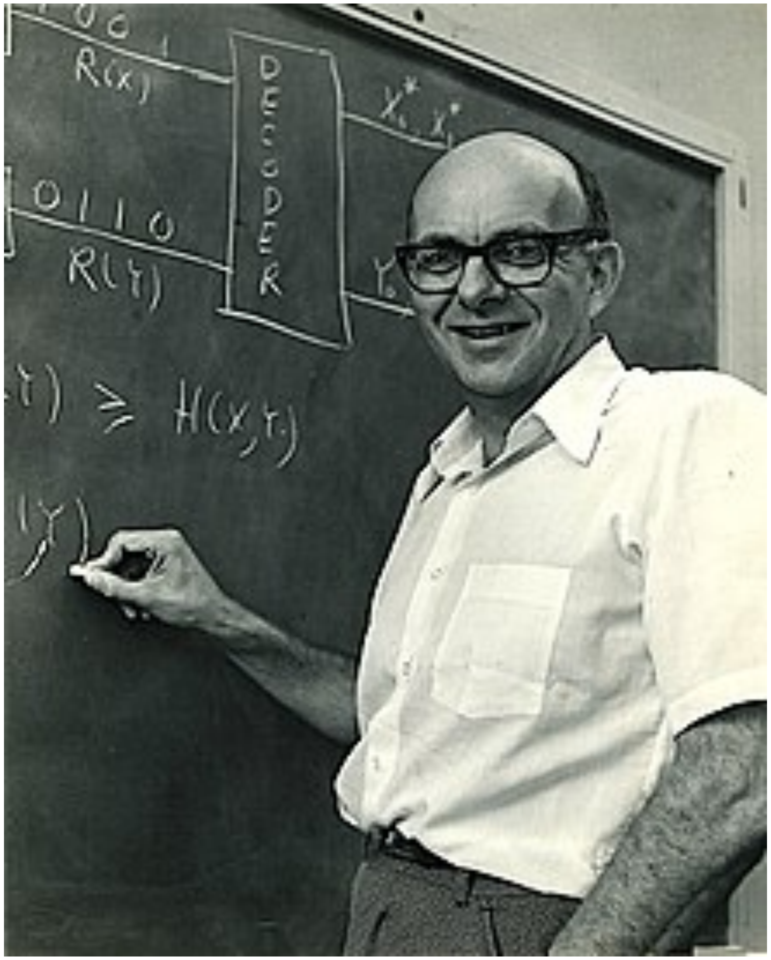
$$\mathcal{U}_1 = 1 - 2P_1, \qquad \mathcal{U}_2 = 1 - 2P_2$$

The irreducible representations are at most two dimensional since $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, $\alpha \in [0, \frac{\pi}{2}]$

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} \cos^2\alpha & \cos\alpha \sin\alpha \\ \cos\alpha \sin\alpha & \sin^2\alpha \end{pmatrix}.$$

$$P_1 P_2 P_1 = \cos^2(\alpha) P_1, \; (P_1 - P_2)^2 = \sin^2(\alpha)$$

D. Slepian et all, Bell-labs, 1960-1965



D. Slepian, H. Pollack, *Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty*, The Bell System technical Journal (1961), 43–63.

D. Slepian, *Some asymptotic expansions for prolate spheroidal wave functions*, J. Math. Phys. Vol. **44** (1965), 99–140.

D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, Siam Review. Vol. 23 (1983), 379–393.

Paradox in communication of signals

- Duration of signal is limited
- Range of frequencies is limited

Prolate spheroidal wave functions

$$\int_{-1}^{1} PS_{2m,0}(\gamma,\eta)\,\exp(i\gamma\eta\omega)\,d\eta = 2(-1)^m S^{(1)}_{2m,0}(\gamma,1)\,PS_{2m,0}(\gamma,\omega)$$

$$\psi_m(y) := PS_{2m,0}(2\pi\mu,\frac{y}{\lambda}), \quad \mu = \lambda^2$$

$$\chi(\mu,m) = 2\lambda S^{(1)}_{2m,0}(2\pi\mu,1), \quad \xi = \lambda\,\eta$$

$$\int_{-\lambda}^{\lambda} \psi_m(\xi)\,\exp(i2\pi\xi y)\,d\xi = (-1)^m\chi(\mu,m)\psi_m(y), \;\; \forall y \in [-\lambda,\lambda]$$

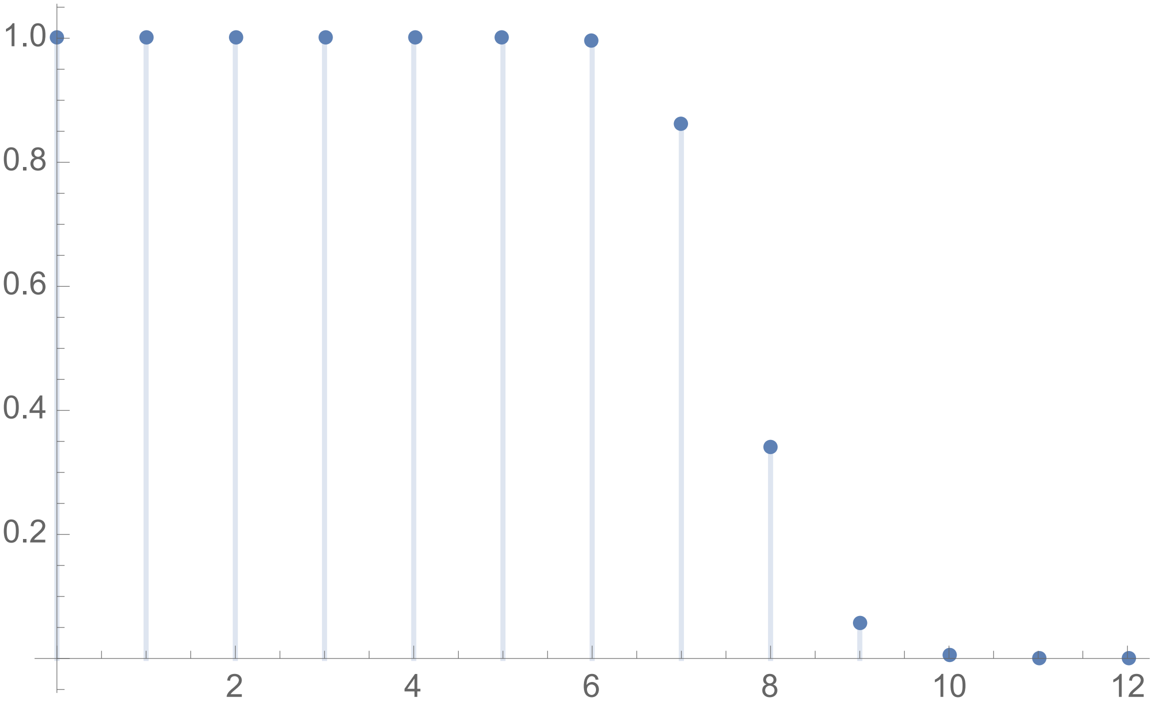
Cosine of angle, $\alpha = \angle(P_\lambda, \widehat{P}_\lambda)$

$$P_\lambda \mathbb{F}_{e_{\mathbb{R}}} P_\lambda \psi_m = (-1)^m \chi(\mu, m) P_\lambda \psi_m$$

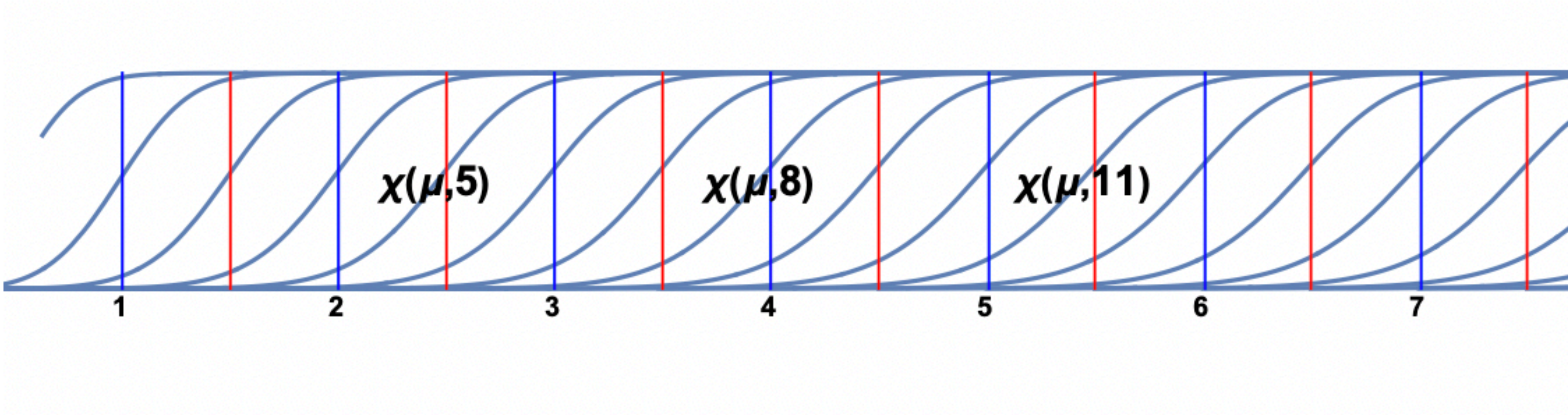
$$P_\lambda \widehat{P}_\lambda P_\lambda \psi_m = \chi(\mu, m)^2 P_\lambda \psi_m$$

$$\cos^2(\alpha_m) = \chi(\mu, m)^2$$

Behavior of $\chi(\mu, m)$

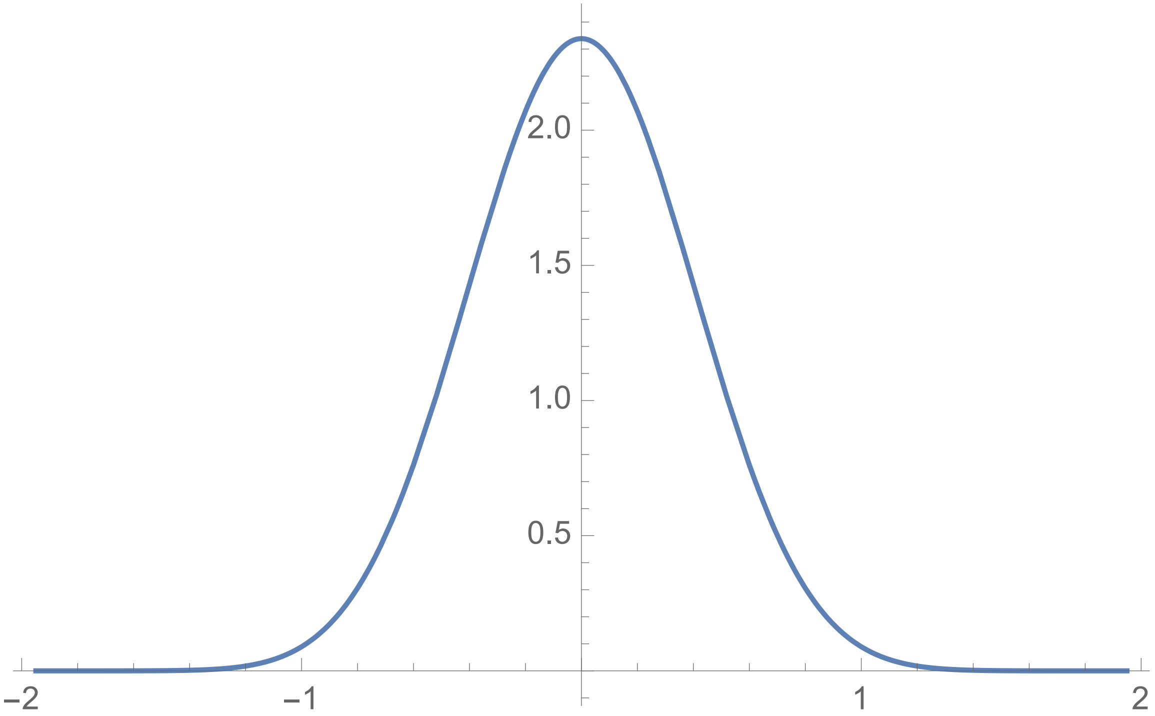


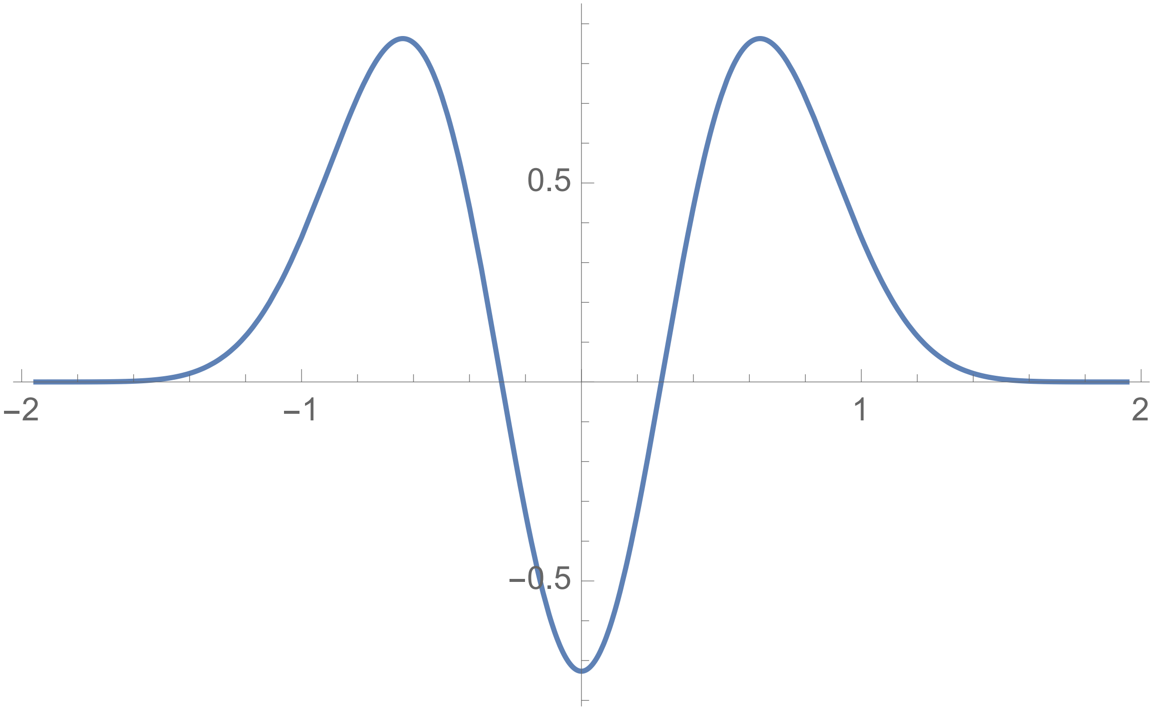
$\chi(\mu, m) \sim 1$ for $m \leq \nu(\mu) \sim 2\mu - 1$

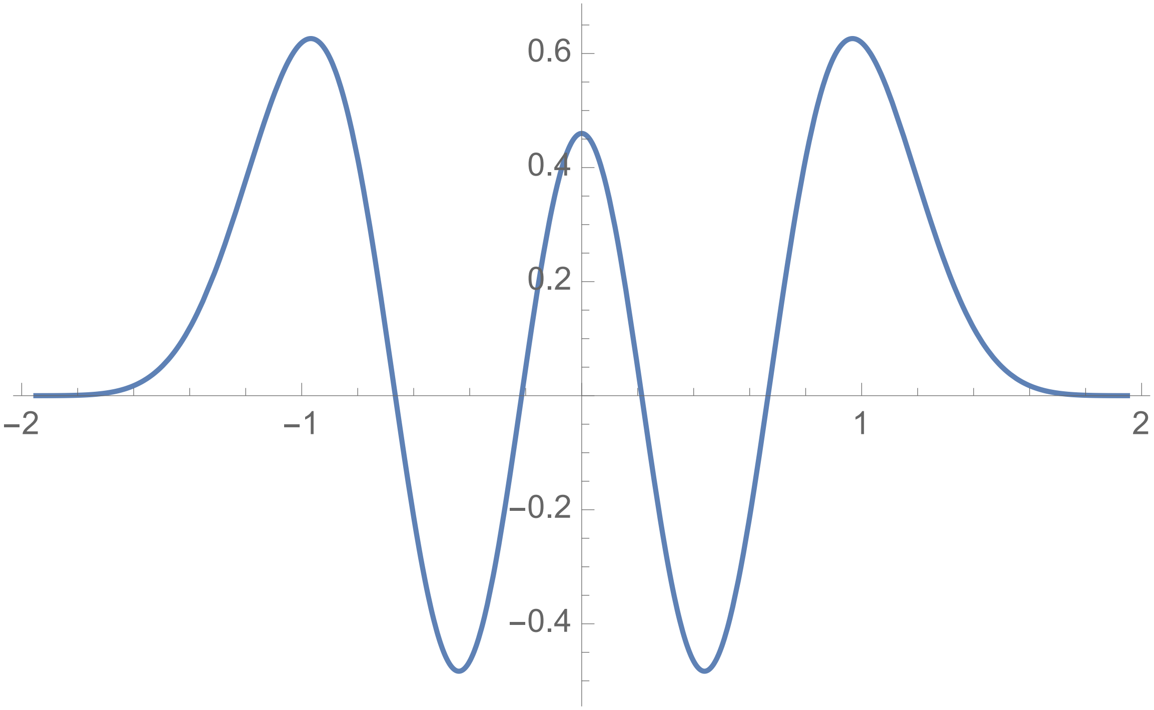

χ(μ,5)
χ(μ,8)
χ(μ,11)
1
2
3
4
5
6
7

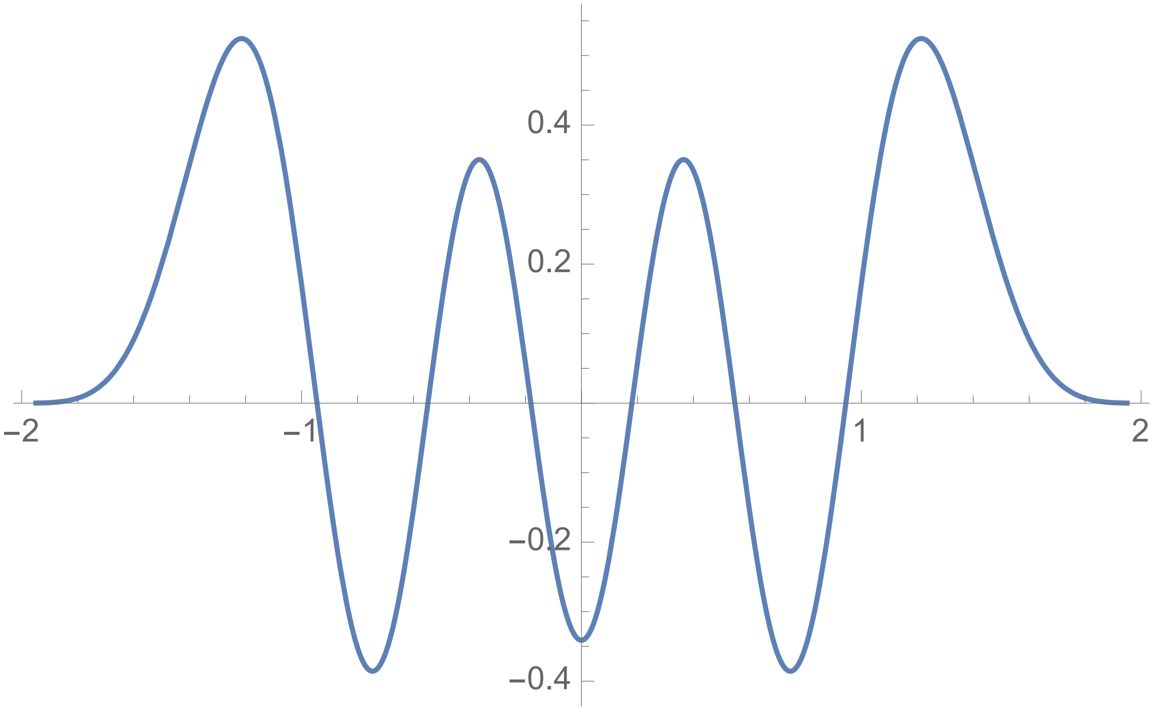
$\chi(\mu, m)$ **for** $\mu = 3.8$

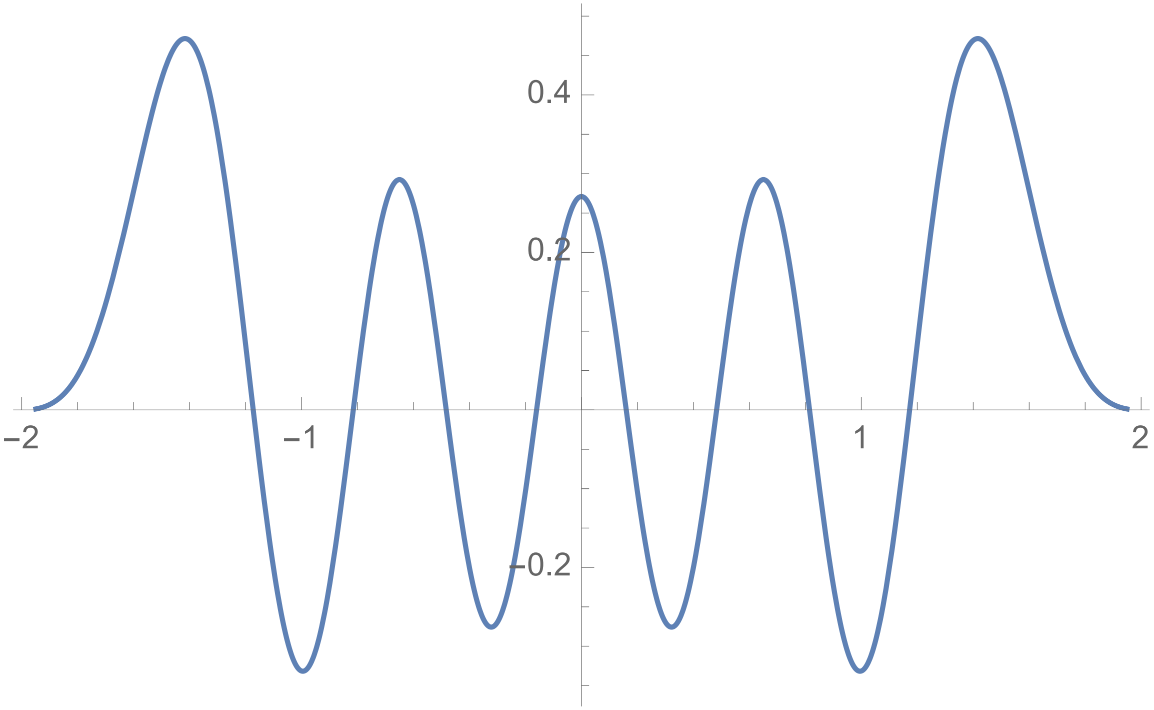
0.99999999999999999997, 0.99999999999999948822,
0.99999999999878471440, 0.99999999902401560803,
0.99999965576640391865, 0.99994101604887038003,
0.99514777747935401414

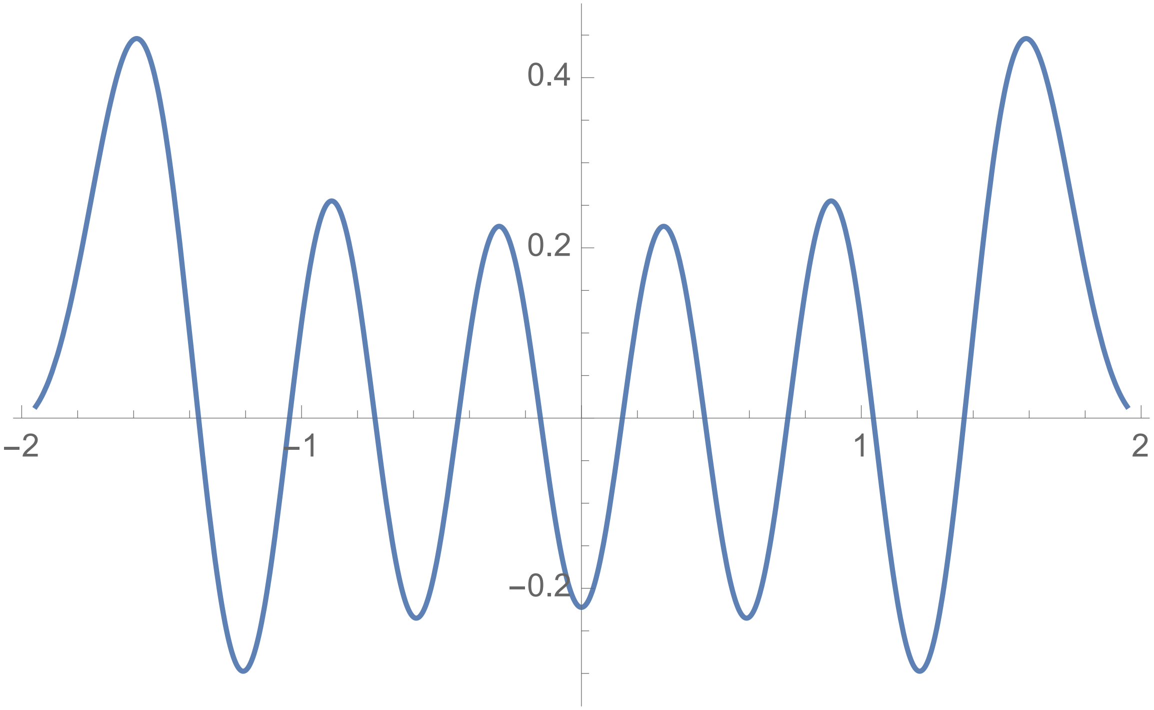

2.0
1.5
1.0
0.5
−2
−1
1
2

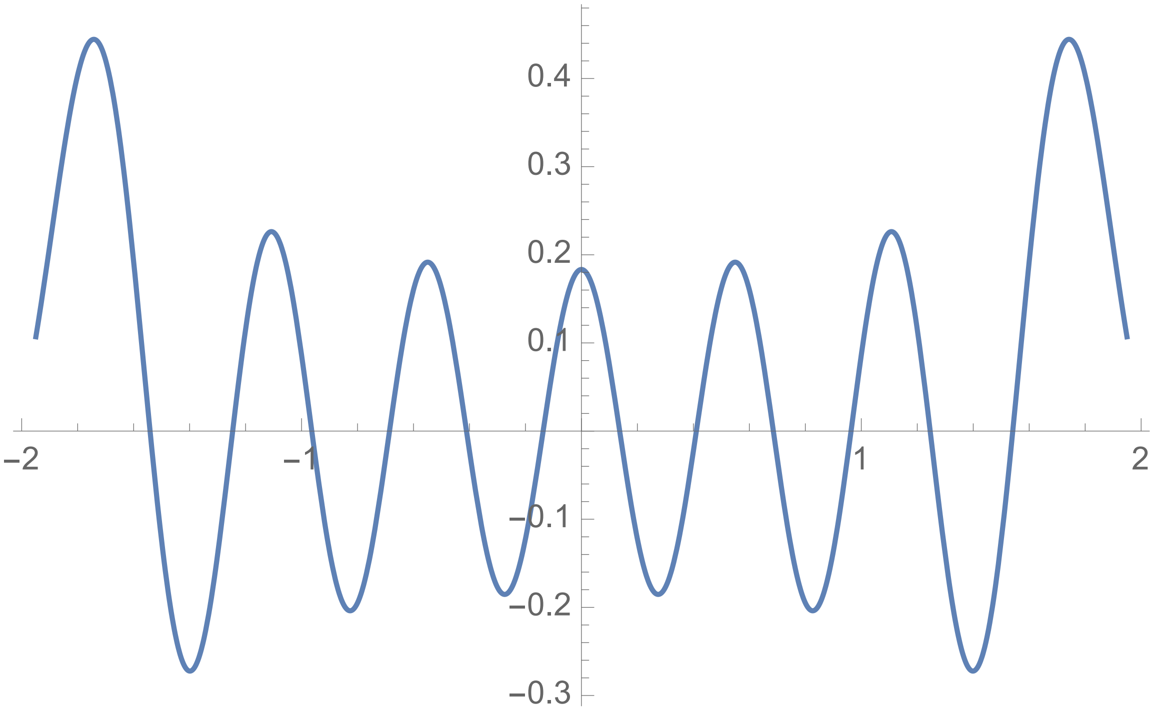

0.5
−0.5
−2
−1
1
2

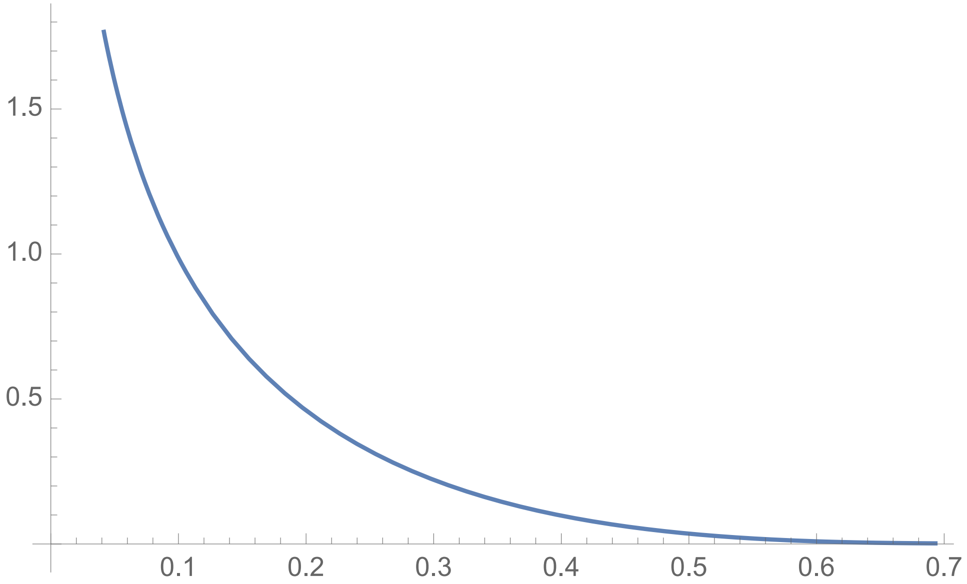

0.6
0.4
0.2
-0.2
-0.4
-2
-1
1
2


0.4
0.2
−0.2
−0.4
−2
−1
1
2

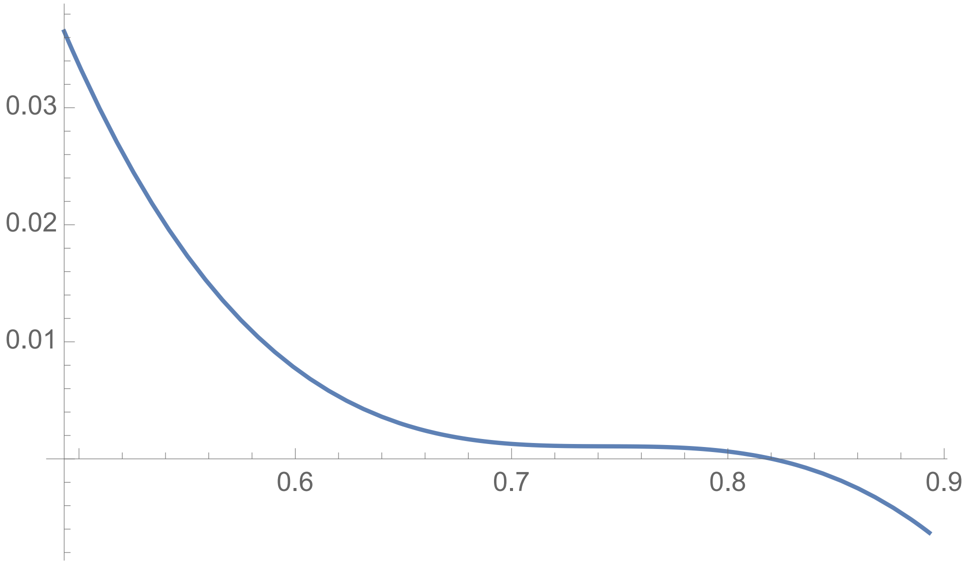

0.4
0.2
-0.2
-2
-1
1
2


0.4
0.2
−0.2
−2
−1
1
2

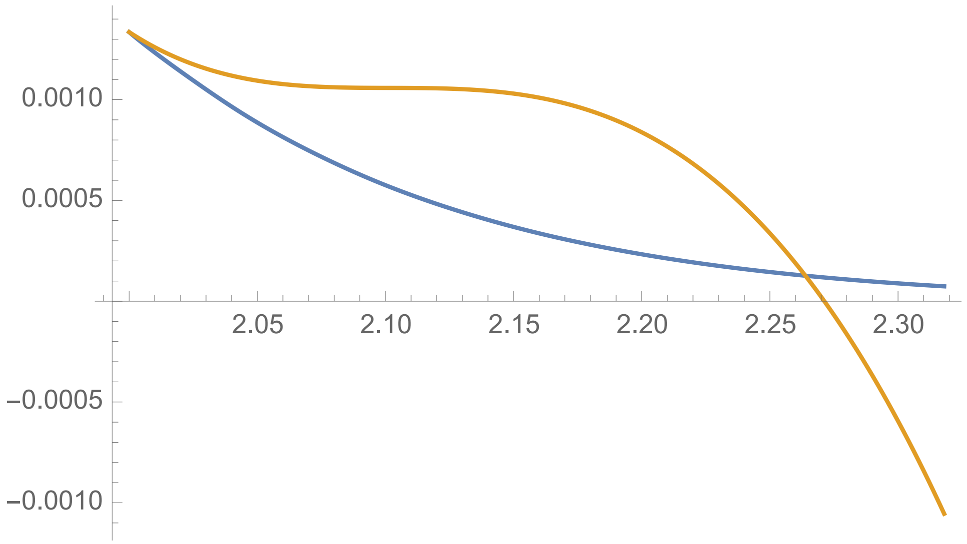

0.4
0.3
0.2
0.1
−0.1
−0.2
−0.3
−2
−1
1
2



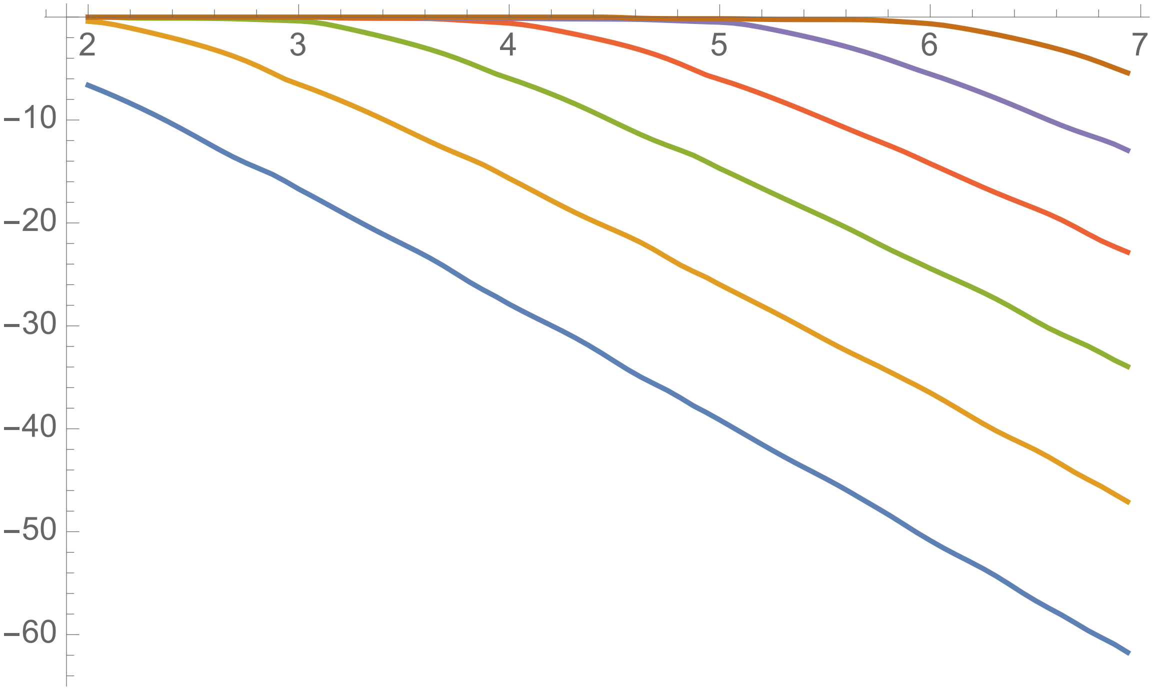
Positivity of the archimedean contribution to the even matrix for $L \in [0, \log 2]$. The smallest eigenvalue when $L = \log 2$ is ~ 0.00133



Change of sign of the smallest eigenvalue of the archimedean contribution to the even matrix for $L \in [\log 2 - 0.2, \log 2 + 0.2] \sim [0.493, 0.893]$

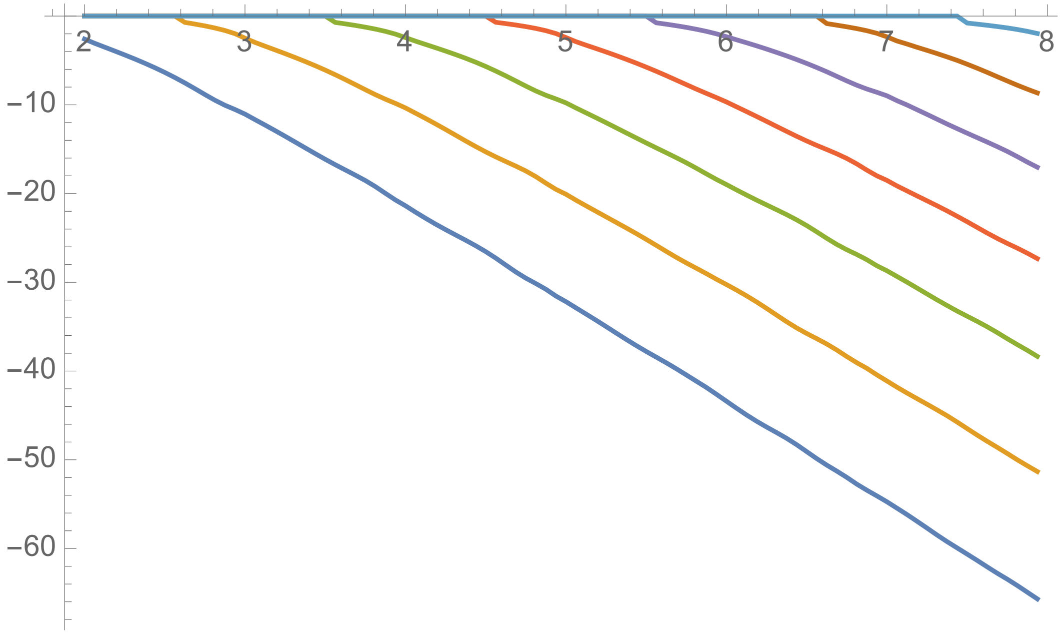


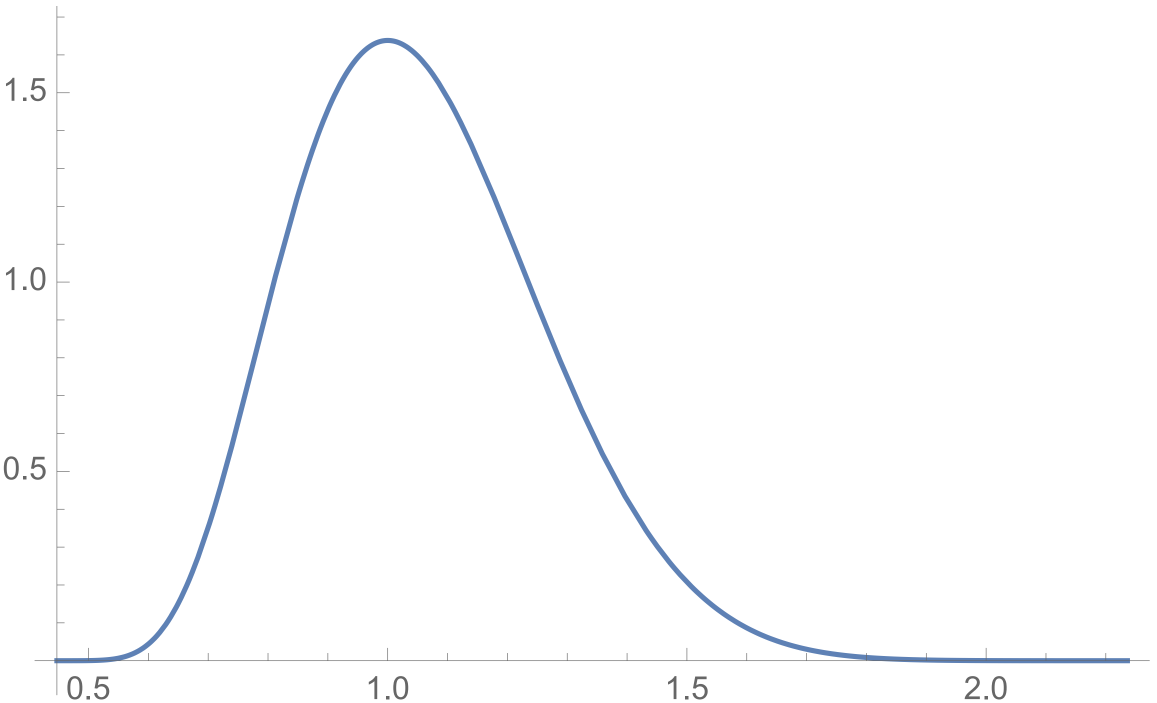
Change of sign of the smallest eigenvalue for the archimedean contribution alone, as a function of $\mu := \exp L$, near $\mu = 2$ (in yellow). After adding the contribution of the prime 2 the smallest eigenvalue of the even matrix is > 0 (in blue)



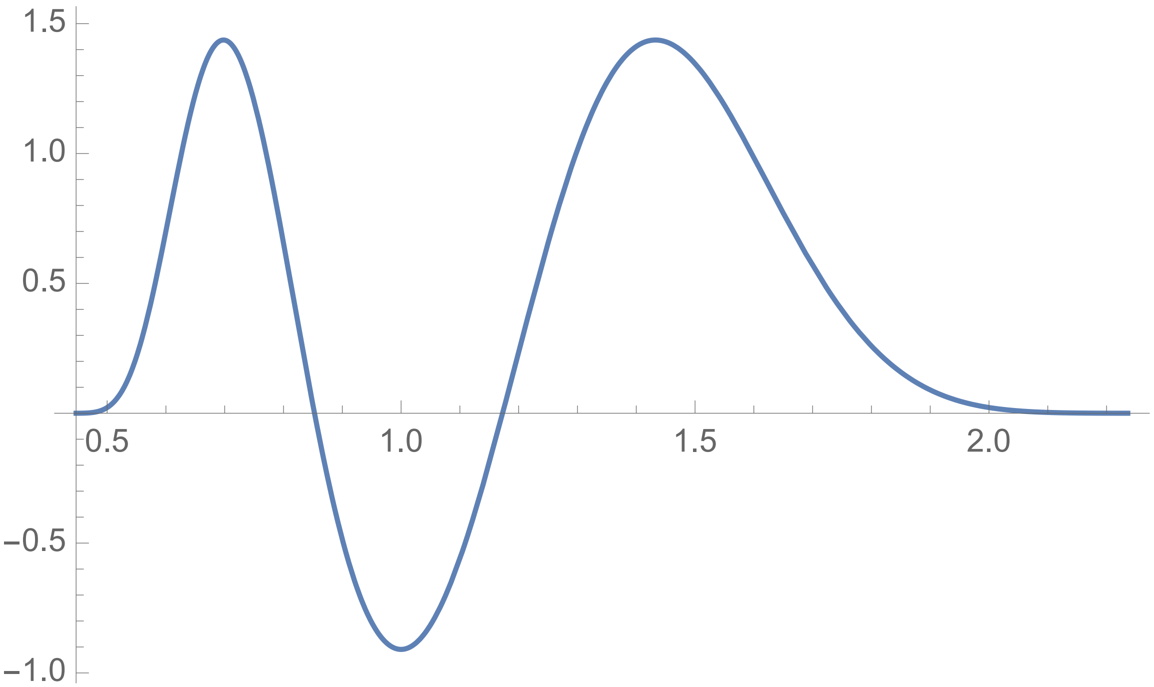
Decay of the log of the smallest eigenvalues of the even matrix σ^+ as a function of $\mu = \exp L$, their number increases roughly like $\mu = \exp L$.

For the odd matrix σ^{-}, the behavior is similar but with one less small eigenvalue,

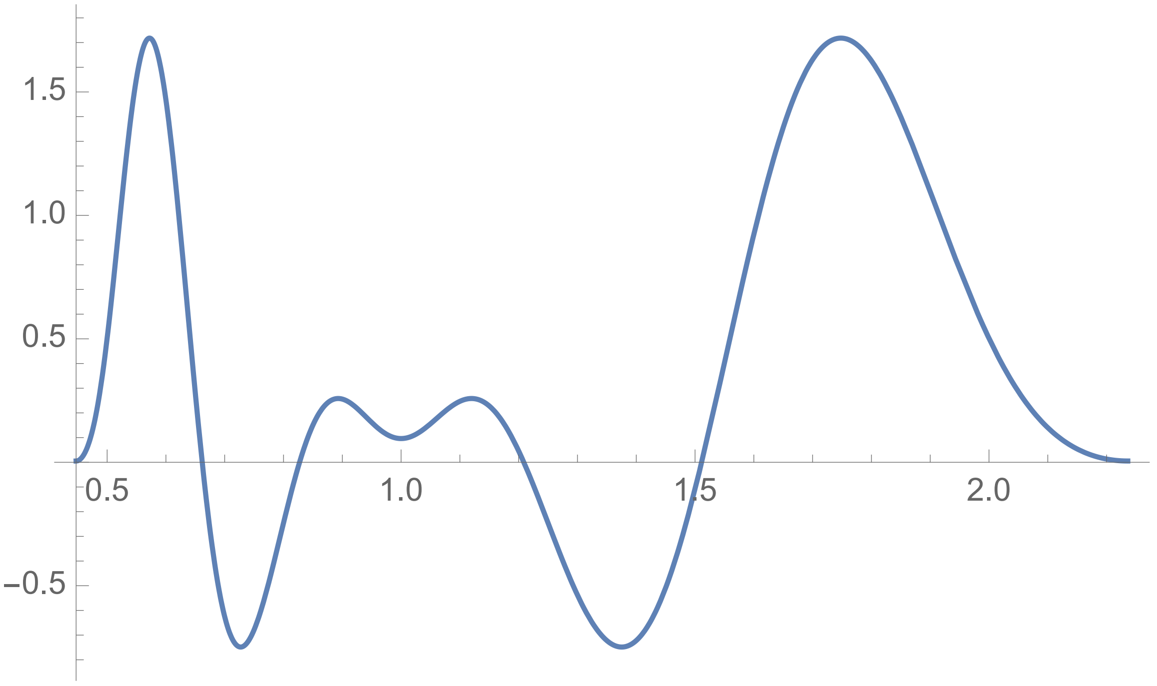




Eigenvector for the smallest eigenvalue of QW_λ^+ as function on $\mathbb{R}_+^*$



Eigenvector for the second smallest eigenvalue of QW_λ^+ as function on $\mathbb{R}_+^*$



Eigenvector for the third smallest eigenvalue of QW_λ^+ as function on $\mathbb{R}_+^*$

The functions ϕ_n

$E(\lambda, k)$ codimension two subspace of linear span of the ψ_m for $m \leq k+1$ determined by the conditions $f(0) = \widehat{f}(0) = 0$. For $n \leq \nu(\mu)$ one may approximate $\mathbb{F}_{e_{\mathbb{R}}}(\psi_n)$ by $(-1)^n \psi_n$ and impose the vanishing at 0 using the following functions

$$\phi_{2n}(x) := \psi_{2n}(x)\psi_0(0) - \psi_0(x)\psi_{2n}(0),$$

$$\phi_{2n+1}(x) := \psi_{2n+1}(x)\psi_1(0) - \psi_1(x)\psi_{2n+1}(0)$$

Prolate projection $\Pi(\lambda, k)$

$$\mathcal{E}(f)(x) = x^{1/2} \sum_{n>0} f(nx)$$

Riemann-Roch = Poisson Formula

$$f(0) = \widehat{f}(0) = 0 \Rightarrow \mathcal{E}(\widehat{f})(x) = \mathcal{E}(f)(x^{-1})$$

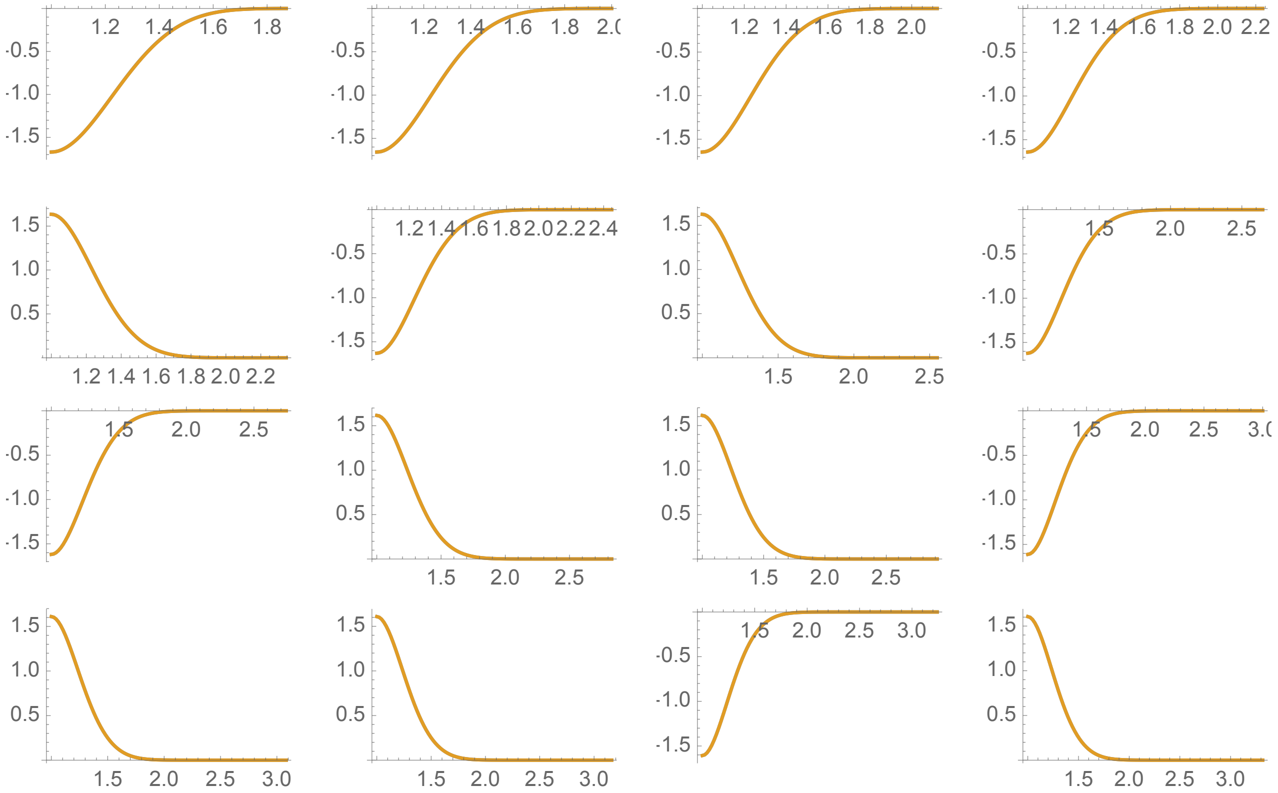
$\Pi(\lambda, k)$ orth. proj. on $\mathcal{E}(E(\lambda, k))$

$\{\eta_j, j \in \mathbb{Z}\}$ canonical orthonormal basis of real functions in $\mathcal{H}$ (cosines and sines)

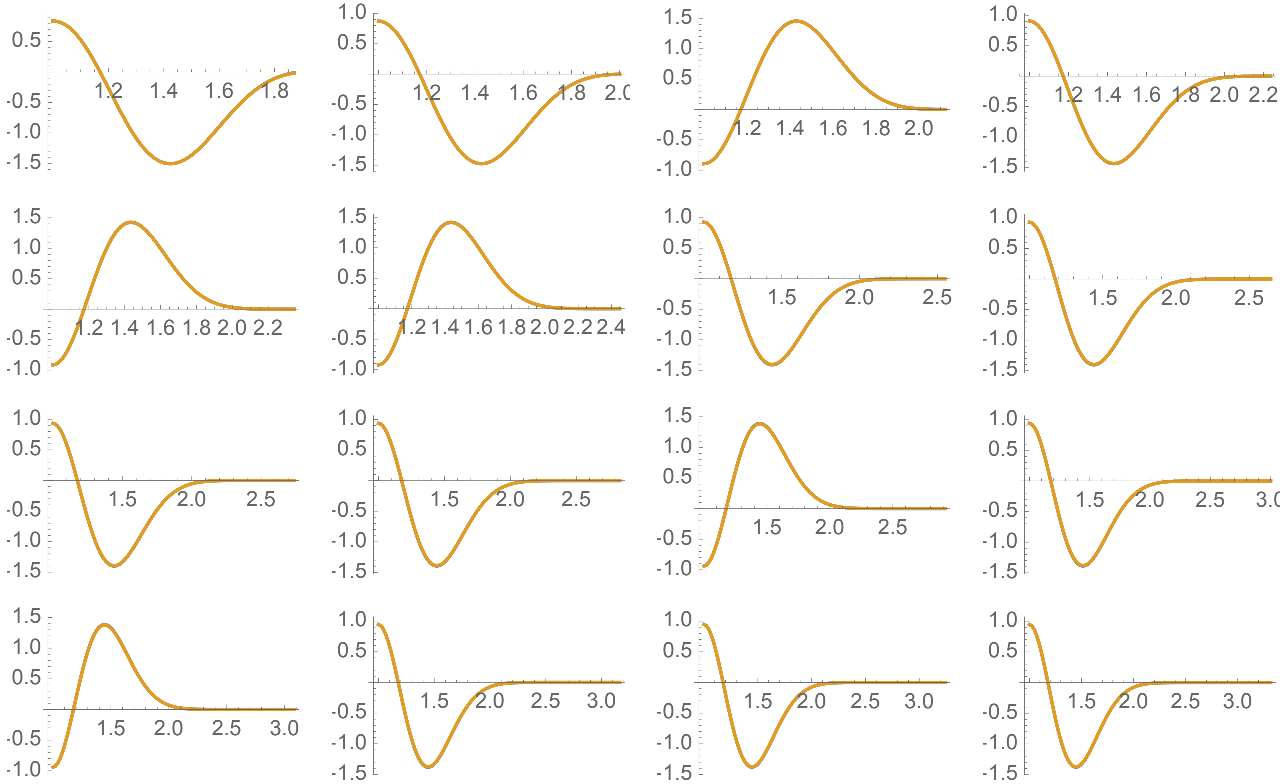
$$\mathcal{E}(\phi_n)_j \simeq 2 \sum_{1 \leq r < \lambda} \int_1^{\lambda/r} u^{1/2} \phi_n(ru) \eta_j(u) d^*u.$$

Fundamental Fact

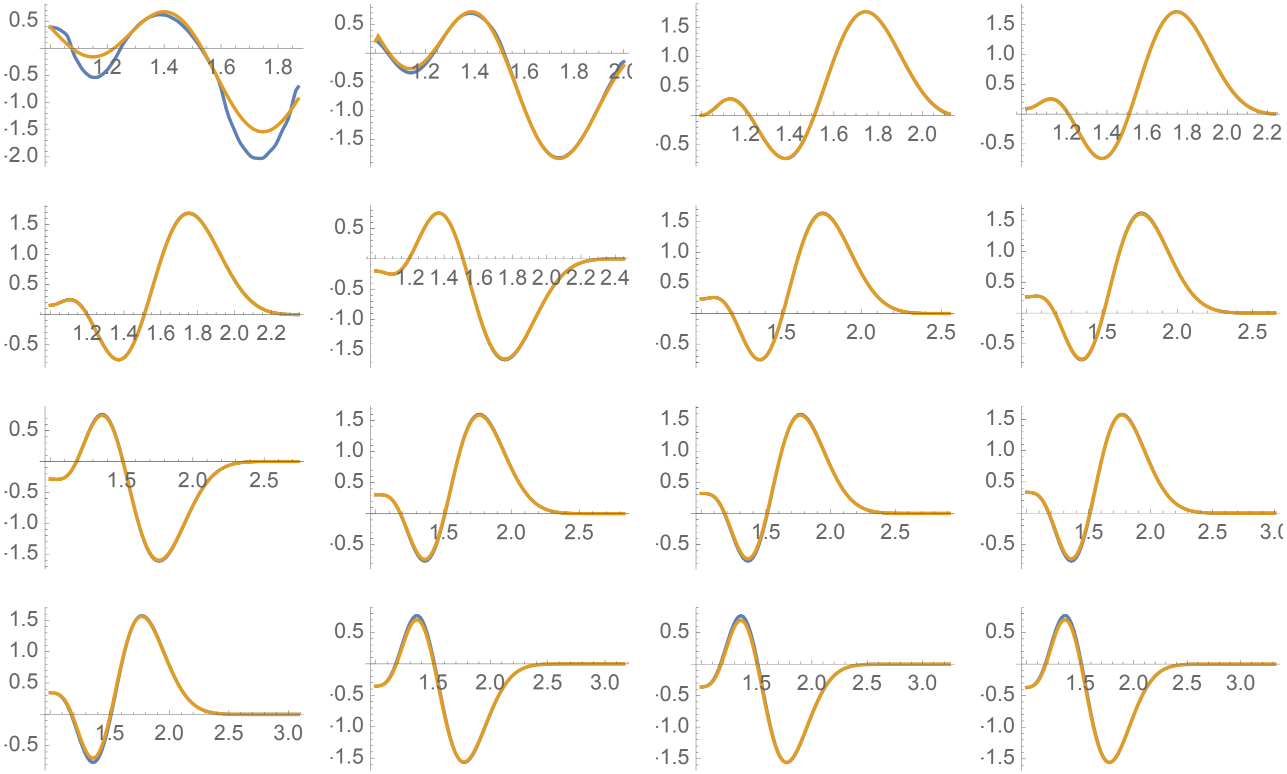
The space of eigenvectors of k lowest eigenvalues for the Weil quadratic form QW_λ corresponds to the prolate projection $\Pi(\lambda, k)$



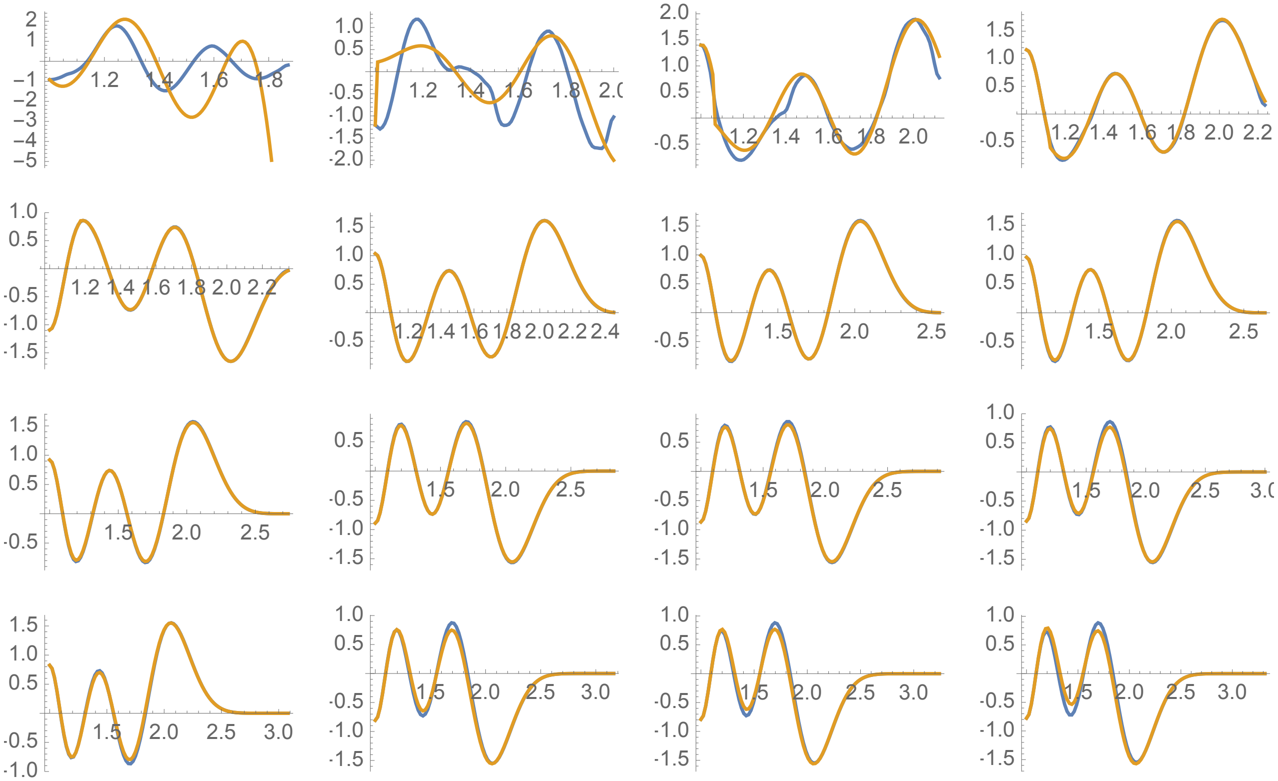
agreement of eigenfunctions for the even matrix and the smallest eigenvalue, for the 16 values of μ between 3.5 and 11. For $\mu = 11$ the eigenvalue is 2.389×10^{-48}



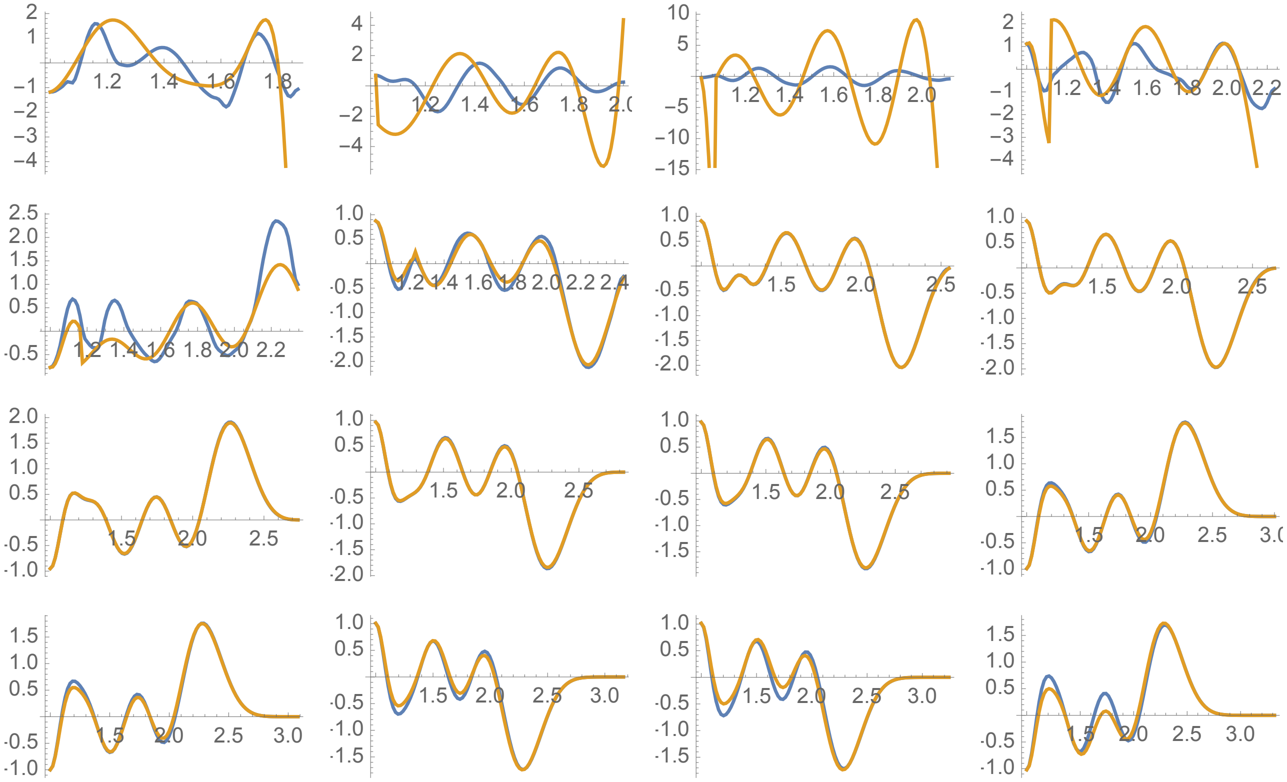
agreement of eigenfunctions for the even matrix and the second smallest eigenvalue for the 16 values of μ between 3.5 and 11



agreement of eigenfunctions for the even matrix and the third smallest eigenvalue for the 16 values of μ between 3.5 and 11. They begin to agree around $\mu = 4$



agreement of eigenfunctions for the even matrix and the 4-th smallest eigenvalue for the 16 values of μ between 3.5 and 11. They begin to agree around $\mu = 5$



agreement of eigenfunctions for the even matrix and the 5-th smallest eigenvalue for the 16 values of μ between 3.5 and 11. They begin to agree around $\mu = 6$

Spectral triple $\Theta(\lambda, k) = (\mathcal{A}(\lambda), \mathcal{H}(\lambda), D(\lambda, k))$

- $\mathcal{A}(\lambda) := C^\infty(\mathbb{R}_+^*/\mu^{\mathbb{Z}})$, $\mu = \lambda^2$

- $\mathcal{H} = L^2(\mathbb{R}_+^*/\mu^{\mathbb{Z}}, d^*u) \simeq L^2([\lambda^{-1}, \lambda], d^*u)$

- $D(\lambda, k) := Q \circ D_0(\lambda) \circ Q$,

$Q = 1 - \Pi(\lambda, k)$, $D_0(\lambda) := -iu\partial_u$

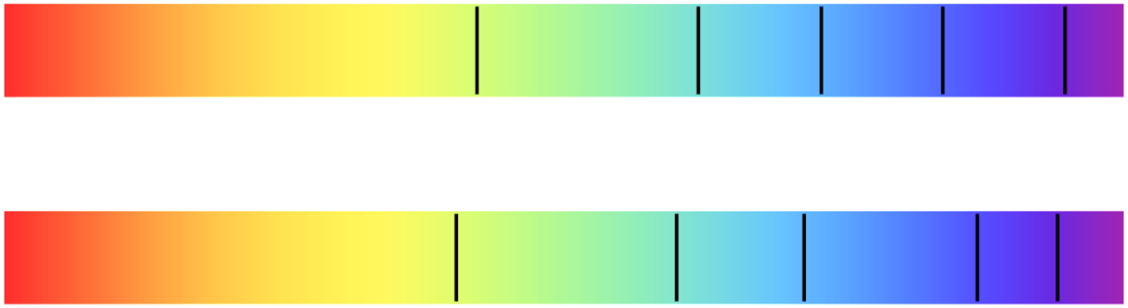
$\mu = 5.5$

Cosine eigenvalues $\chi(5.5, n)$ are extremely close to 1 when $n = 0, 1, 2, 3, 4$ and

n	$\chi(5.5, n)$
5	0.99999999999647719857
6	0.99999999894391115741
7	0.99999980631702676769
8	0.99997809227622865324
9	0.99852183576050441685
10	0.95065832620623051607
11	0.57197061534624863399
12	0.139174533954574303539

$$\nu(5.5) = 10,\ 2\pi 5.5 \sim 34.5575.$$

λ_j	ζ_j
14.781	14.1347
21.701	21.022
25.547	25.0109
29.345	30.4249
33.168	32.9351



$\mu = 6.5$

n	$\chi(6.5, n)$
7	0.99999999998668315975
8	0.99999999731589077585
9	0.99999963978717981581
10	0.99996808936687677767
11	0.99821407841789989100
12	0.94788066237037484836
13	0.57534099083086049406
14	0.14710511279564130503

One has $\nu(6.5) = 12$, $2\pi 6.5 \sim 40.8407$.

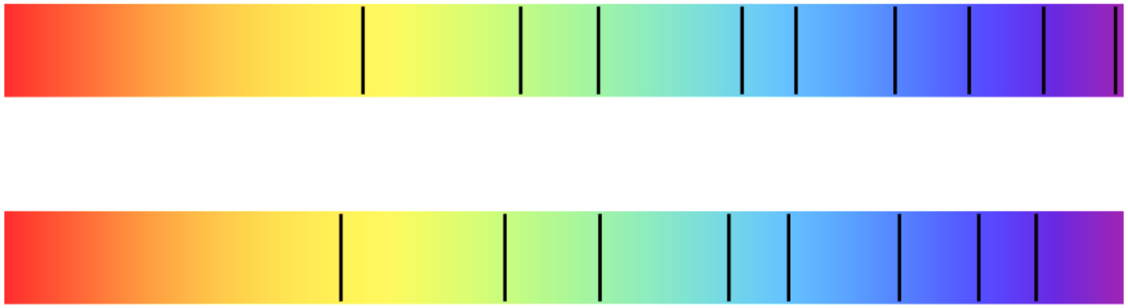
λ_j	ζ_j
13.936	14.1347
20.580	21.022
24.690	25.0109
30.194	30.4249
33.454	32.9351
36.826	37.5862
40.259	40.9187

$\mu = 7.5$

n	$\chi(7.5, n)$
9	0.99999999996397226733
10	0.999999999453062631606
11	0.99999941709770526957
12	0.99995709581648305854
13	0.99792322303841470726
14	0.94552083061302325507
15	0.57809629788957190907
16	0.15383636015962926720

One has $\nu(7.5) = 14$, $2\pi 7.5 \sim 47.1239$.

λ_j	ζ_j
15.060	14.1347
21.683	21.022
24.948	25.0109
30.979	30.4249
33.243	32.9351
37.406	37.5862
40.514	40.9187
43.643	43.3271
46.658	48.0052

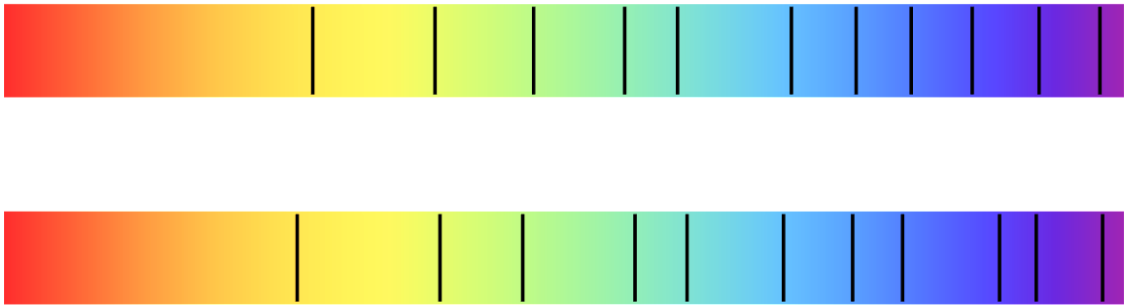


$\mu = 8.5$

n	$\chi(8.5, n)$
11	0.99999999992101000288
12	0.99999999034148375362
13	0.99999913999089362040
14	0.99994536408530411219
15	0.99764801726717553636
16	0.94347292951033144975
17	0.58041289343441020661
18	0.15967051202562674536

One has $\nu(8.5) = 16$, $2\pi 8.5 \sim 53.4071$.

λ_j	ζ_j
14.887	14.1347
20.778	21.022
25.535	25.0109
29.928	30.4249
32.473	32.9351
37.965	37.5862
41.088	40.9187
43.741	43.3271
46.685	48.0052
49.910	49.7738
52.845	52.9703

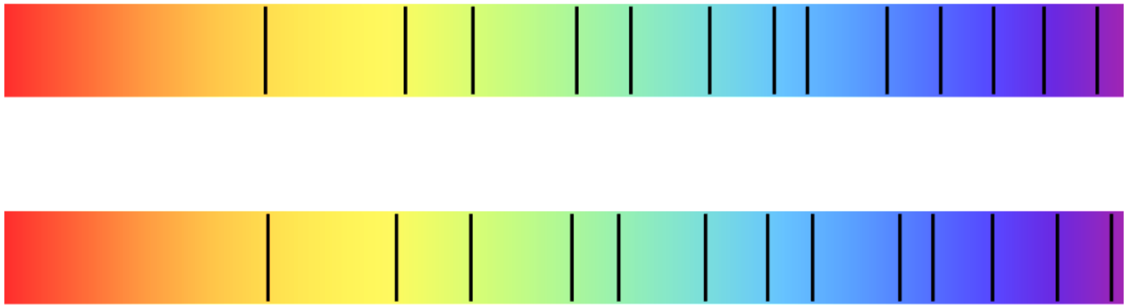


$\mu = 9.5$

n	$\chi(9.5, n)$
13	0.99999999984990646525
14	0.99999998455736228573
15	0.99999881131048713492
16	0.99993308190344158164
17	0.99738707752987412262
18	0.94166650390462098514
19	0.58240244869697875785
20	0.16480962032526478957

One has $\nu(9.5) = 18$, $2\pi 9.5 \sim 59.6903$.

λ_j	ζ_j
13.998	14.1347
21.501	21.022
25.121	25.0109
30.689	30.4249
33.583	32.9351
37.813	37.5862
41.272	40.9187
43.050	43.3271
47.319	48.0052
50.190	49.7738
53.026	52.9703
55.731	56.4462
58.581	59.347

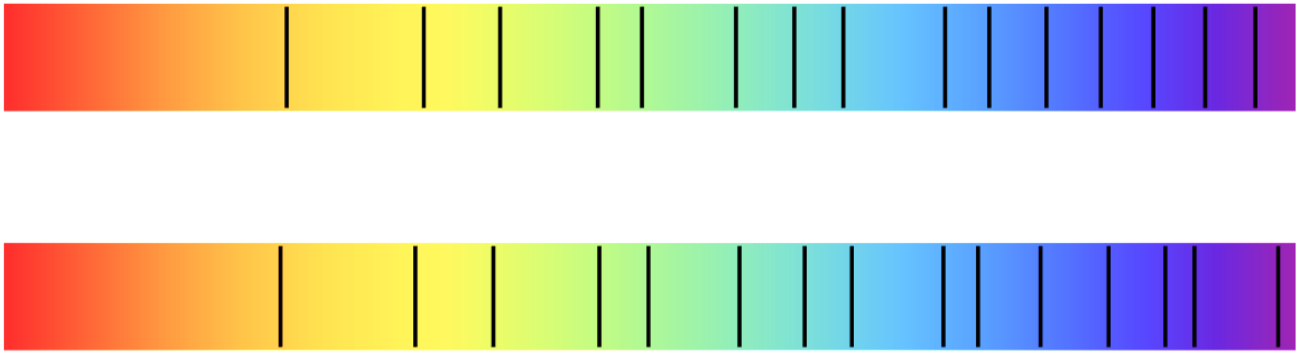


$\mu = 10.5$

n	$\chi(10.5, n)$
15	0.99999999974270022369
16	0.99999997703659571104
17	0.99999843436641476606
18	0.99992039045021729410
19	0.99713907784499135361
20	0.94005235637340584775
21	0.58413979804862029634
22	0.16939519615152177689

One has $\nu(10.5) = 20$, $2\pi 10.5 \sim 65.9734$.

λ_j	ζ_j
14.450	14.1347
21.455	21.022
25.356	25.0109
30.345	30.4249
32.600	32.9351
37.410	37.5862
40.387	40.9187
42.895	43.3271
48.095	48.0052
50.346	49.7738
53.272	52.9703
56.050	56.4462
58.737	59.347
61.386	60.8318
63.949	65.1125



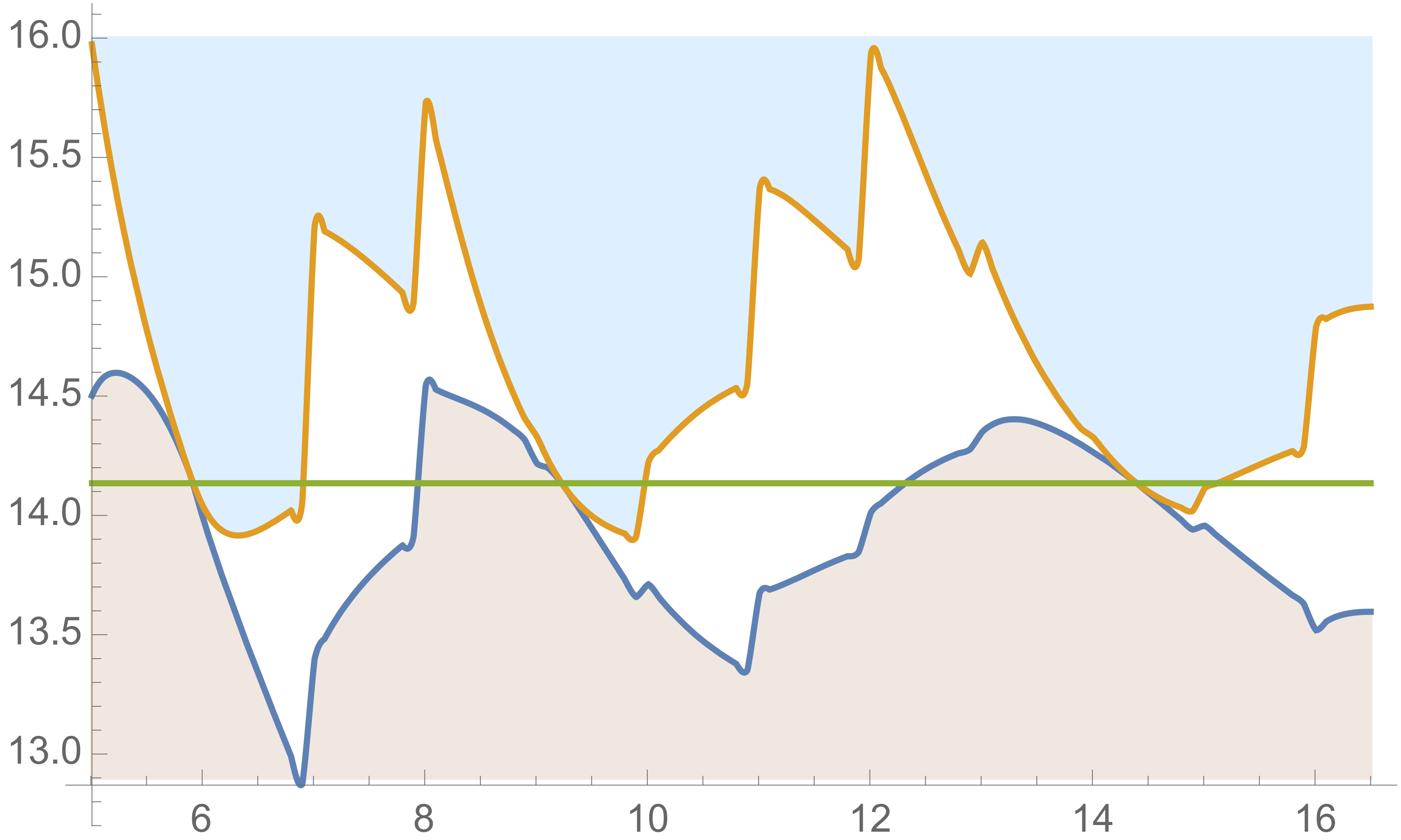
Spectral Similarity

— Comparison of $\lambda_n(D(\lambda, 2\ell))$ with $\lambda_n(D(\lambda, 2\ell + 1))$

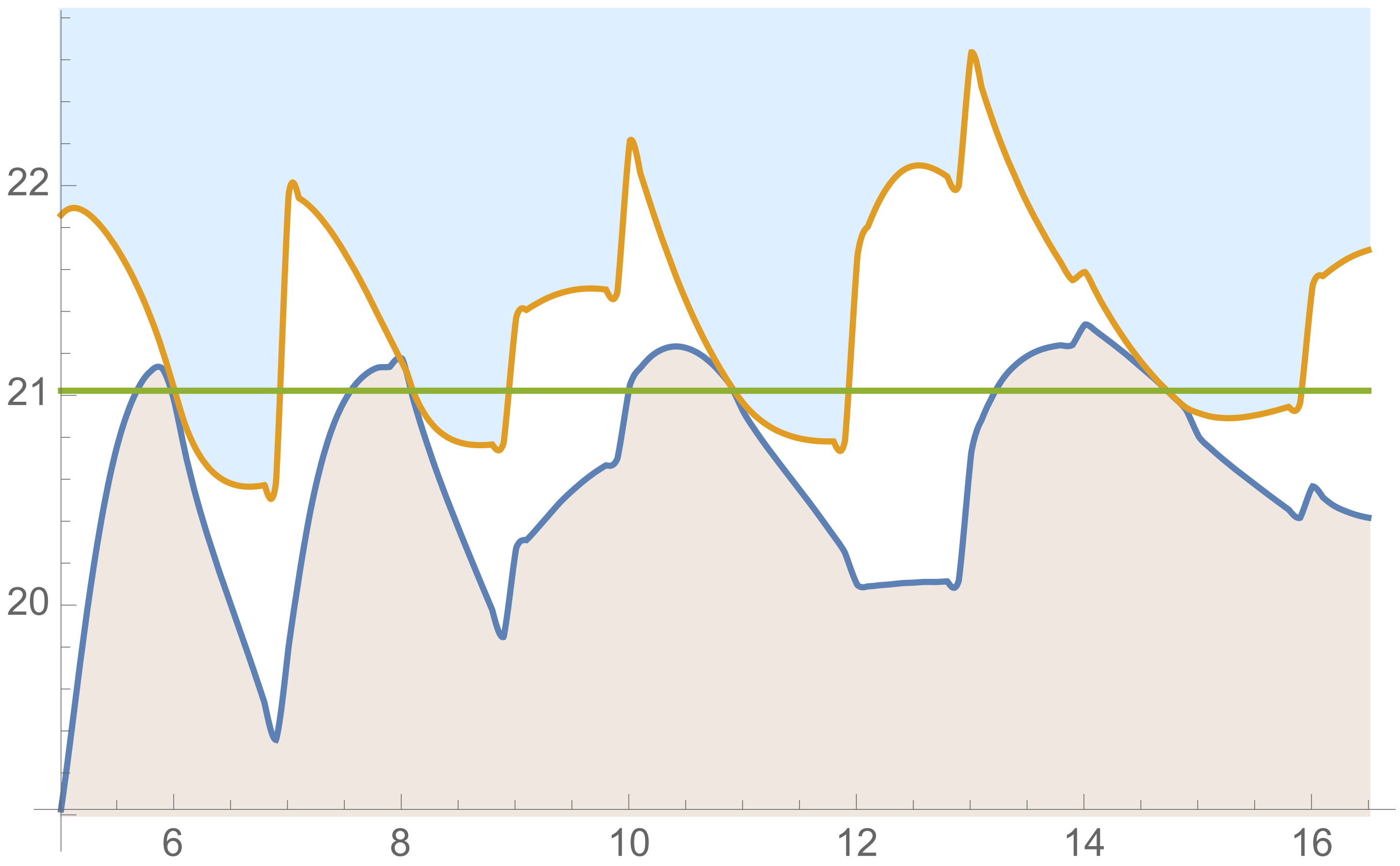
— Evolution of $\lambda_n(D(\lambda, k))$ as a function of λ

— Quantization criterion $x^{2iy} = 1$ applied to the point $(\lambda, \lambda_n(D(\lambda, k)))$

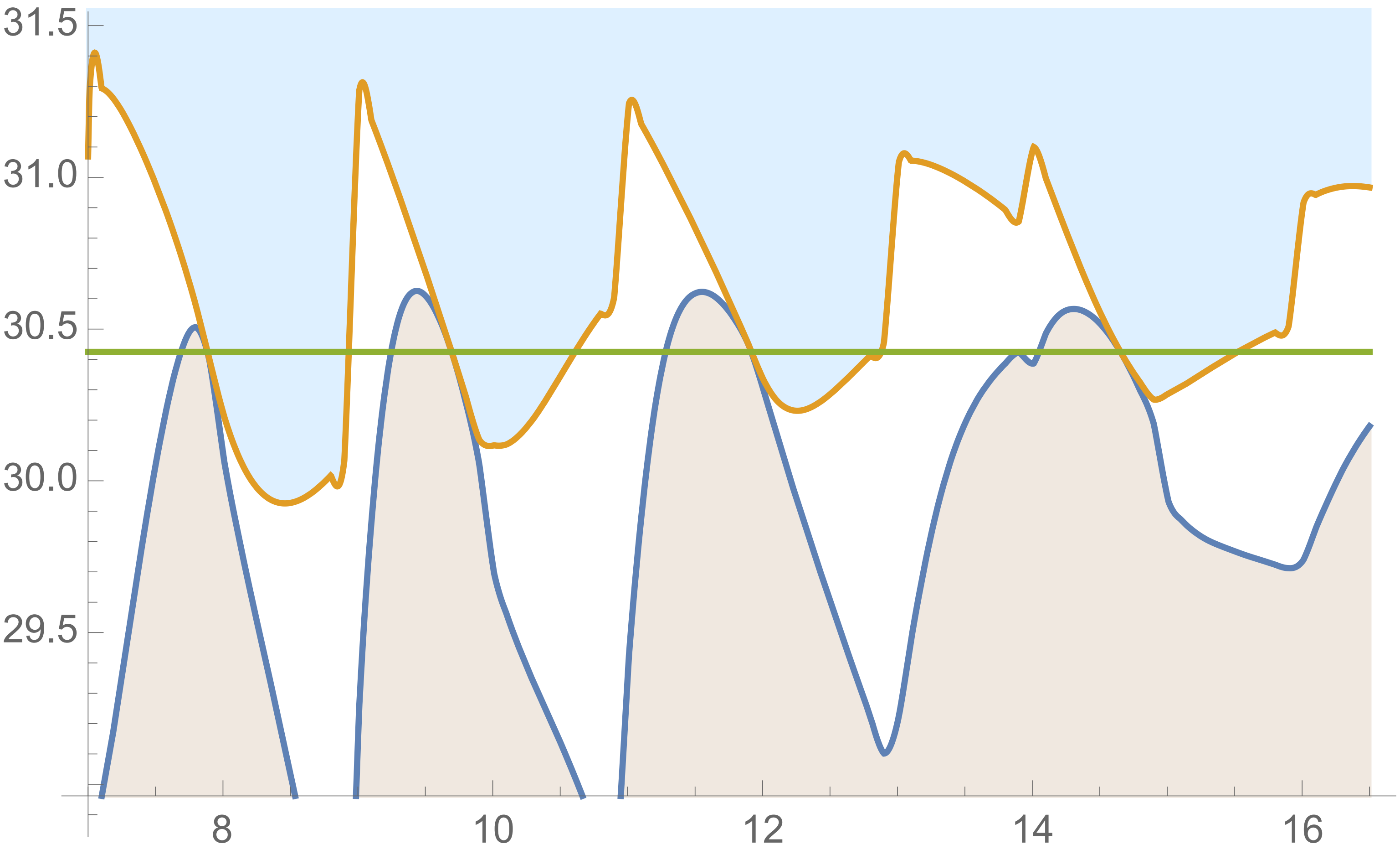
— How far the eigenvector $\xi_n(D(\lambda,k))$ for $D(\lambda,k)$ is from being an eigenvector of $D_0(\lambda)$



1-st eigenvalue

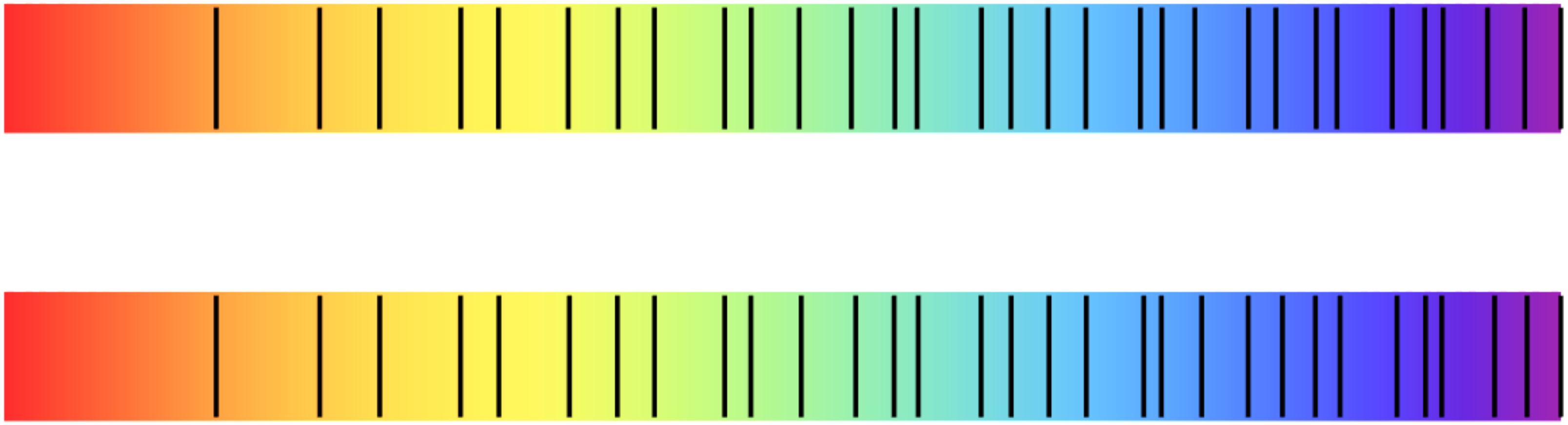


2-nd eigenvalue



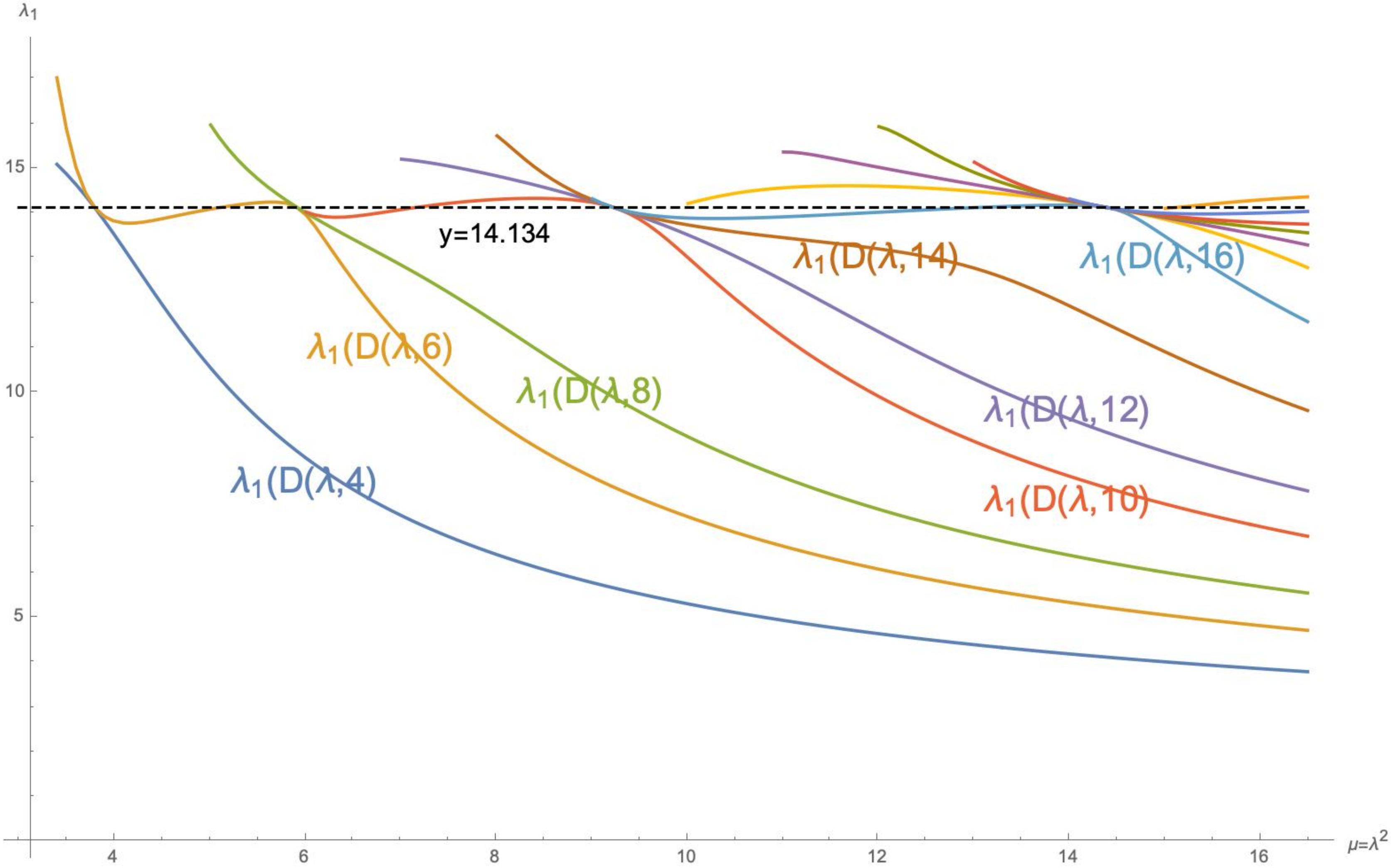
4-th eigenvalue

Criterion $\lambda_n(D(\lambda, 2\ell)) \sim \lambda_n(D(\lambda, 2\ell + 1))$



Evolution of eigenvalues

The curves represent as a function of $\mu = \lambda^2$ the first positive eigenvalue $\lambda_1(D(\lambda, 2k))$ of $D(\lambda, 2k)$. The ordinate of the points where the graphs touch each other is constant and coincides with the imaginary part $\zeta_1 \sim 14.134$ of the first zero of zeta.

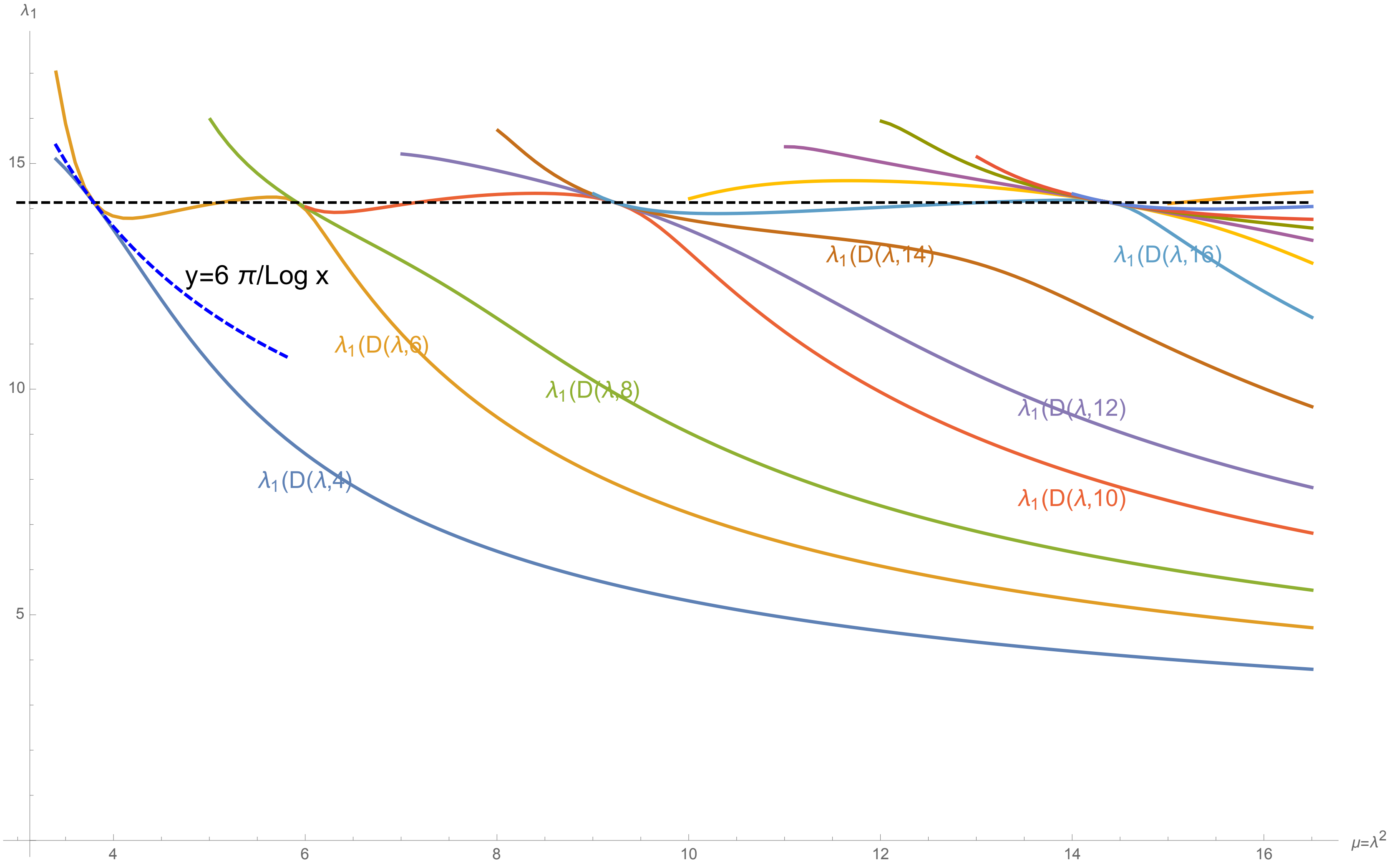

λ_1
15
10
5
y=14.134
$\lambda_1(D(\lambda,4)$
$\lambda_1(D(\lambda,6)$
$\lambda_1(D(\lambda,8)$
$\lambda_1(D(\lambda,10)$
$\lambda_1(D(\lambda,12)$
$\lambda_1(D(\lambda,14)$
$\lambda_1(D(\lambda,16)$
4
6
8
10
12
14
16
$\mu=\lambda^2$

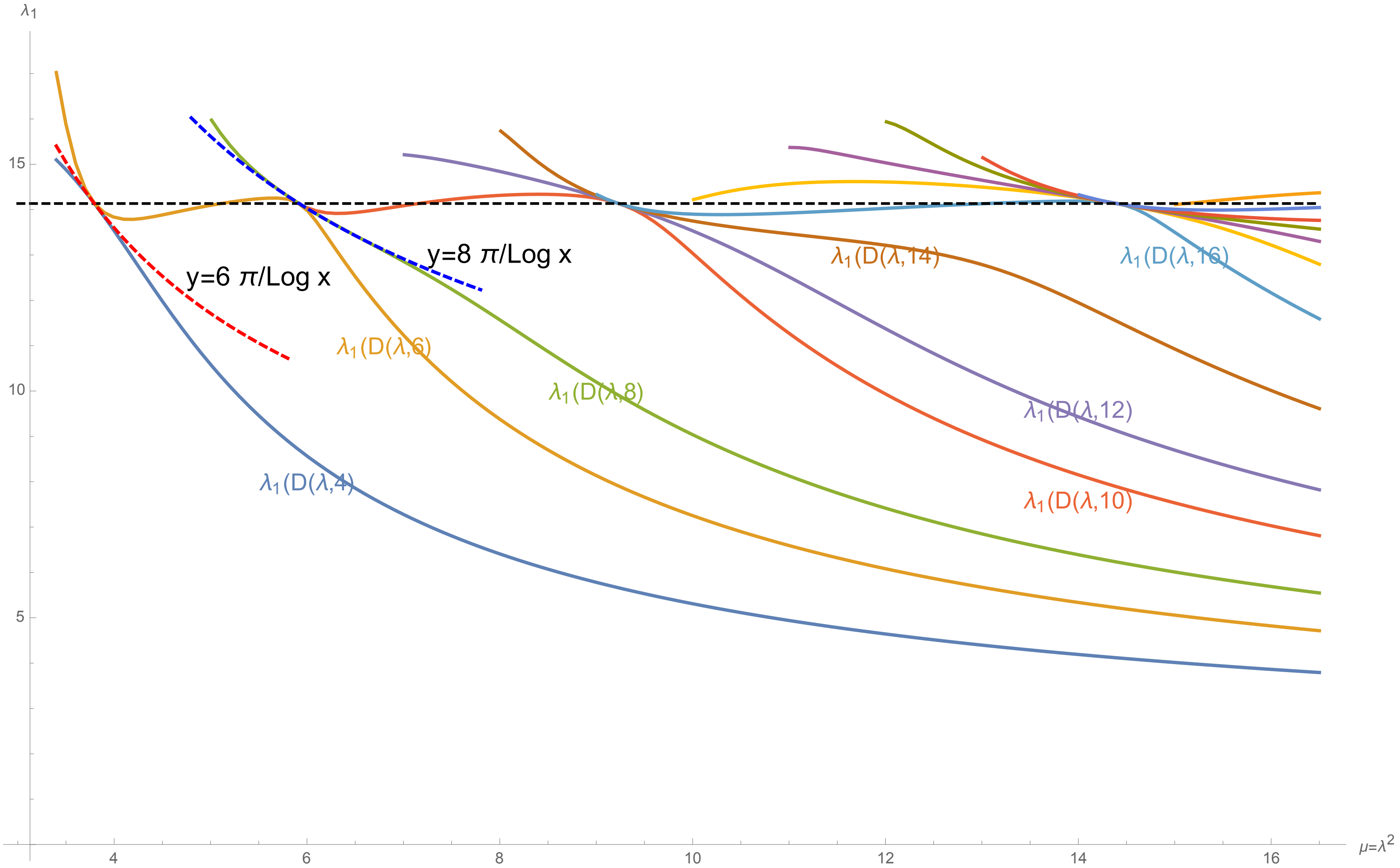
Quantization condition

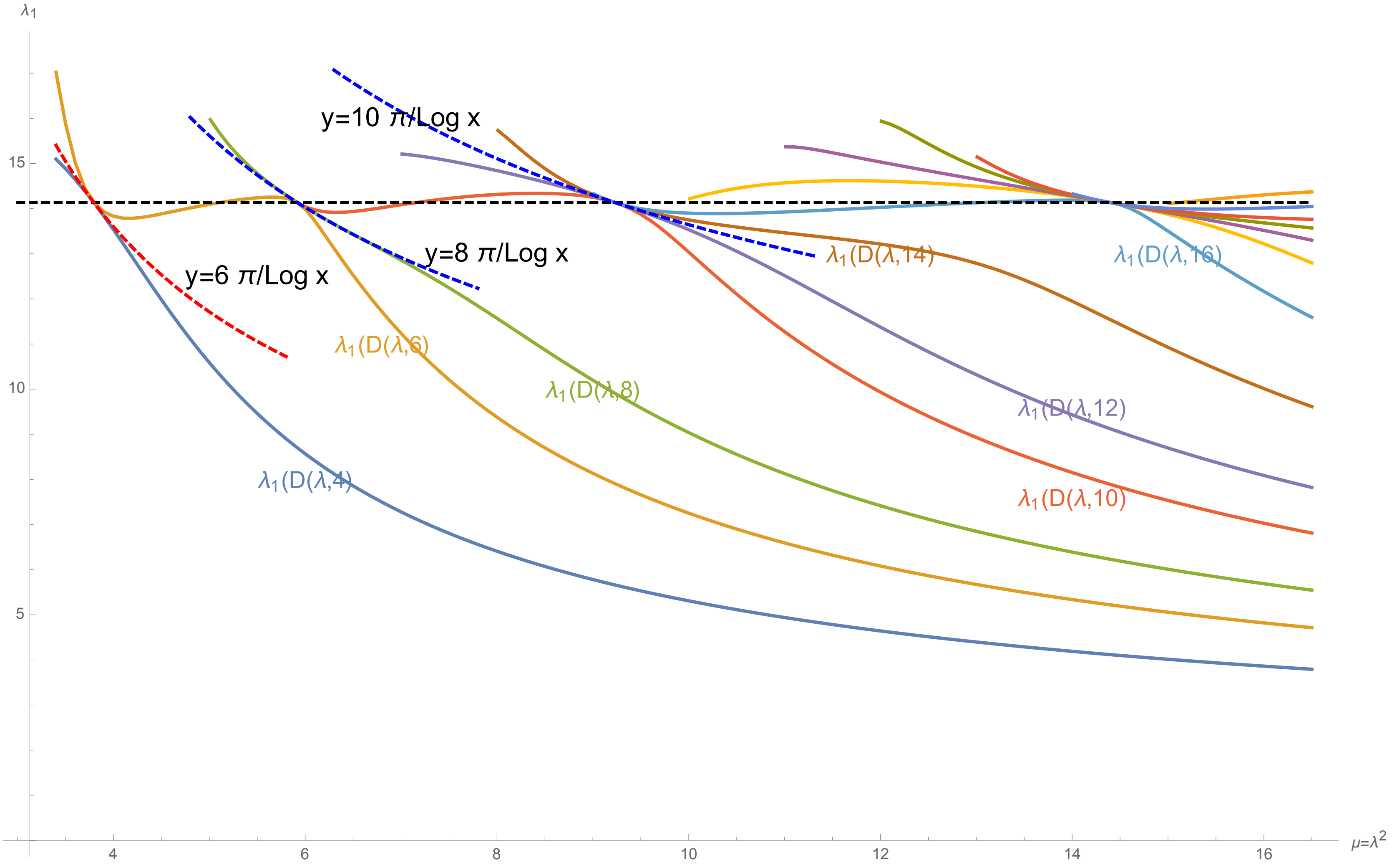
- The points of contact for λ_n fulfill $\log x \in \frac{2\pi}{\zeta_n}\mathbb{Z}$

- The coordinates (x, y) of the points of contact fulfill

$$x^{iy} = 1$$


λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=6 π/Log x
$\lambda_1(D(\lambda,4))$
$\lambda_1(D(\lambda,6))$
$\lambda_1(D(\lambda,8))$
$\lambda_1(D(\lambda,10))$
$\lambda_1(D(\lambda,12))$
$\lambda_1(D(\lambda,14))$
$\lambda_1(D(\lambda,16))$


λ_1
15
10
5
4
6
8
10
12
14
16
$\mu=\lambda^2$
y=6 π/Log x
y=8 π/Log x
λ_1(D(λ,4))
λ_1(D(λ,6))
λ_1(D(λ,8))
λ_1(D(λ,10))
λ_1(D(λ,12))
λ_1(D(λ,14))
λ_1(D(λ,16))


λ_1
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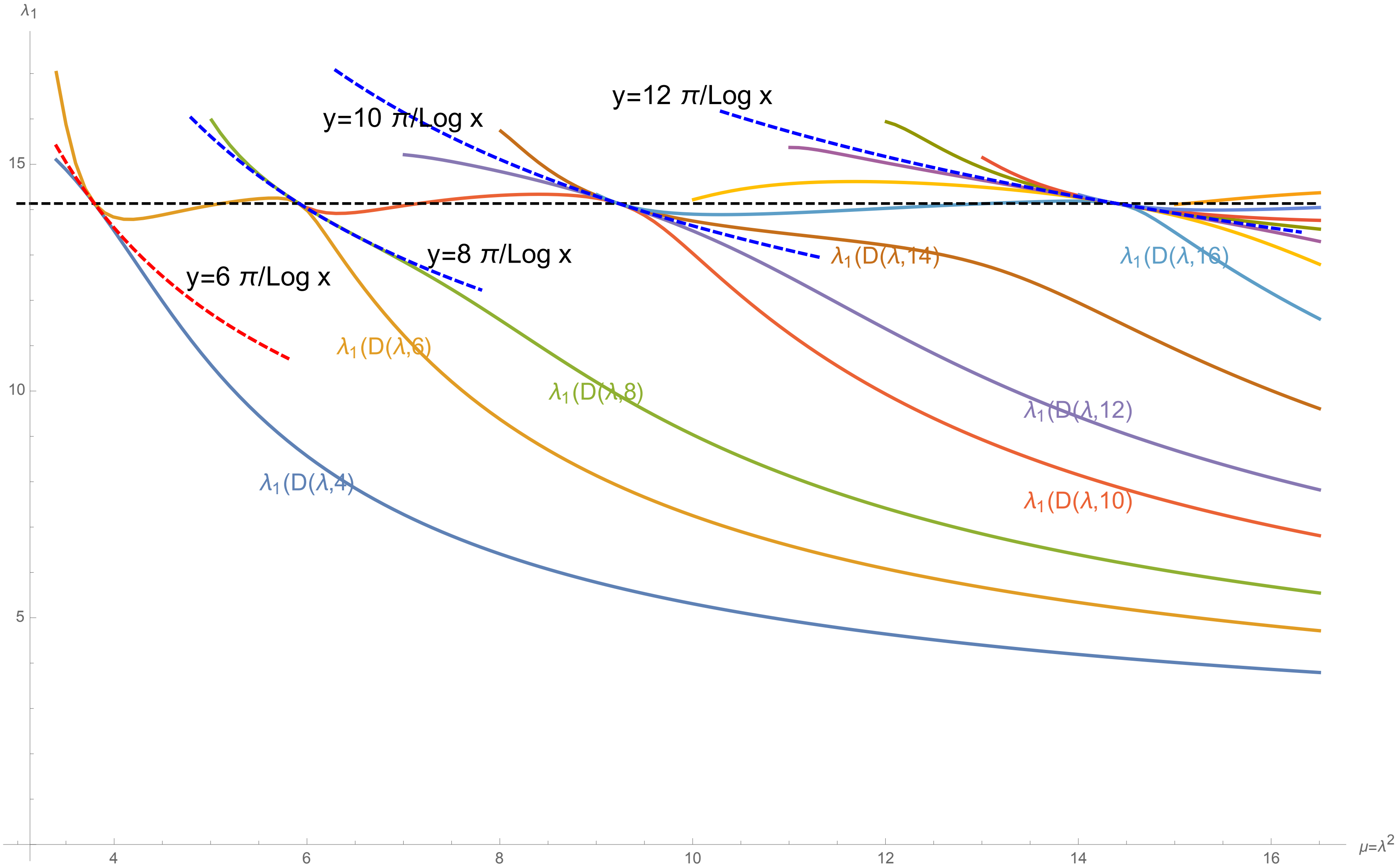
16
$\mu=\lambda^2$
y=10 π/Log x
y=8 π/Log x
y=6 π/Log x
λ_1(D(λ,4))
λ_1(D(λ,6))
λ_1(D(λ,8))
λ_1(D(λ,10))
λ_1(D(λ,12))
λ_1(D(λ,14))
λ_1(D(λ,16))


λ_1
15
10
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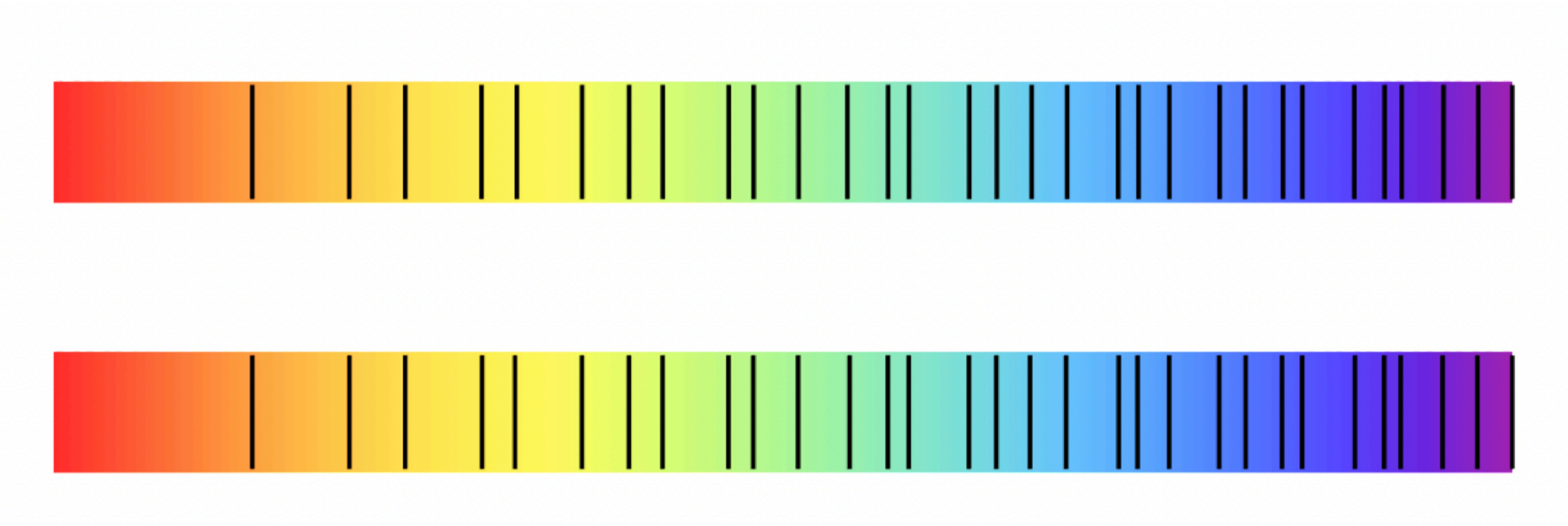
12
14
16
$\mu=\lambda^2$
y=10 π/Log x
y=12 π/Log x
y=8 π/Log x
y=6 π/Log x
λ_1(D(λ,14))
λ_1(D(λ,16))
λ_1(D(λ,6))
λ_1(D(λ,8))
λ_1(D(λ,12))
λ_1(D(λ,4))
λ_1(D(λ,10))

Eigenvectors

▶ The quantization condition is fulfilled by the eigenvalues of $D_0(\lambda)$, the coincidence suggests to compare the eigenvectors.

▶ One compares the eigenvector ξ_n of $D(\lambda, k)$ for the eigenvalue $\lambda_n(D(\lambda, k))$ with the eigenvector of $D_0(\lambda)$ having the same rotation number.

Criterion $\xi_n(D)$ eigenvector of D_0



Conceptual explanation

- Riemann sums in integration
- Scale invariant Riemann sums
- Zeta-cycles

Scale invariant Riemann sums

$$\int f(u)du = 0, \ \ (f(0) = 0)$$

$$(\mathcal{E}f)(u) := u^{1/2} \sum_{n>0} f(nu)$$

$$(\Sigma_\mu g)(u) := \sum_{k\in\mathbb{Z}} g(\mu^k u).$$

Zeta-cycles

A ζ-**cycle** is a circle C of length

$L = \log\mu$ such that the subspace $\Sigma_\mu\mathcal{E}(\mathcal{S}_0)$

is not dense in the Hilbert space $L^2(C)$.

Theorem

(i) Let C be a ζ-cycle. Then the spectrum of the action of the multiplicative group $\mathbb{R}_+^*$ on the orthogonal complement of $\Sigma_\mu\mathcal{E}(\mathcal{S}_0)$ in $L^2(C)$ is formed by imaginary parts of zeros of zeta on the critical line.

Conversely :

(ii) Let $s > 0$ be such that $\zeta(\frac{1}{2} + is) = 0$, then any circle C of length an integral multiple of $2\pi/s$ is a zeta cycle and its spectrum, for the action of $\mathbb{R}_+^*$ on $\Sigma_\mu\mathcal{E}(\mathcal{S}_0) \subset L^2(C)$, contains is

This Theorem provides the theoretical explanation for the above coincidence of spectral values. Indeed, the special values of $\lambda^2 = \mu = \exp L$ at which the k dependence of the eigenvalue $\lambda_n(D(\lambda, k))$ disappear, signal that the related circle of length L is a ζ-cycle and that $\lambda_n(D(\lambda, k))$ is in its spectrum. This explains why the low lying part of the spectrum of the spectral triple $\Theta(\lambda, k)$ possesses a tantalizing resemblance with the low lying zeros of the Riemann zeta function.

Zeta-cycles $\sim$ closed geodesics !

$C = \zeta$-cycle then for any integer $n > 0$

The n-fold cover of C is a ζ-cycle.

Ultraviolet behavior spectrally realized

A. Connes and H. Moscovici

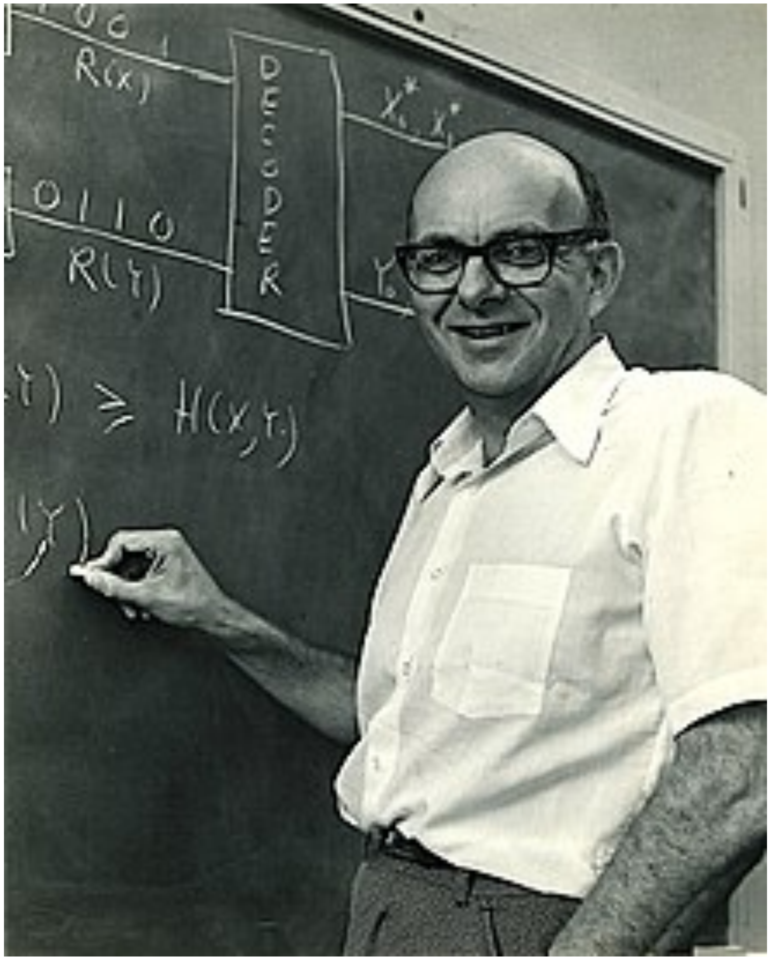
- For zeta one has (Riemann)

$$N(E) \sim \frac{E}{2\pi}\left(\log \frac{E}{2\pi} - 1\right)$$

Commutation with differential operator

- $x\partial_x$ commutes with $1_{[0,\infty]}$
- $(\lambda^2 - x^2)\partial_x$ commutes with $1_{[-\lambda,\lambda]}$
- $\partial_x(\lambda^2 - x^2)\partial_x$ commutes with $1_{[-\lambda,\lambda]}$

D. Slepian et all, Bell-labs, 1960-1965



Commutation with P_λ and $\widehat{P}_\lambda$

- The operator

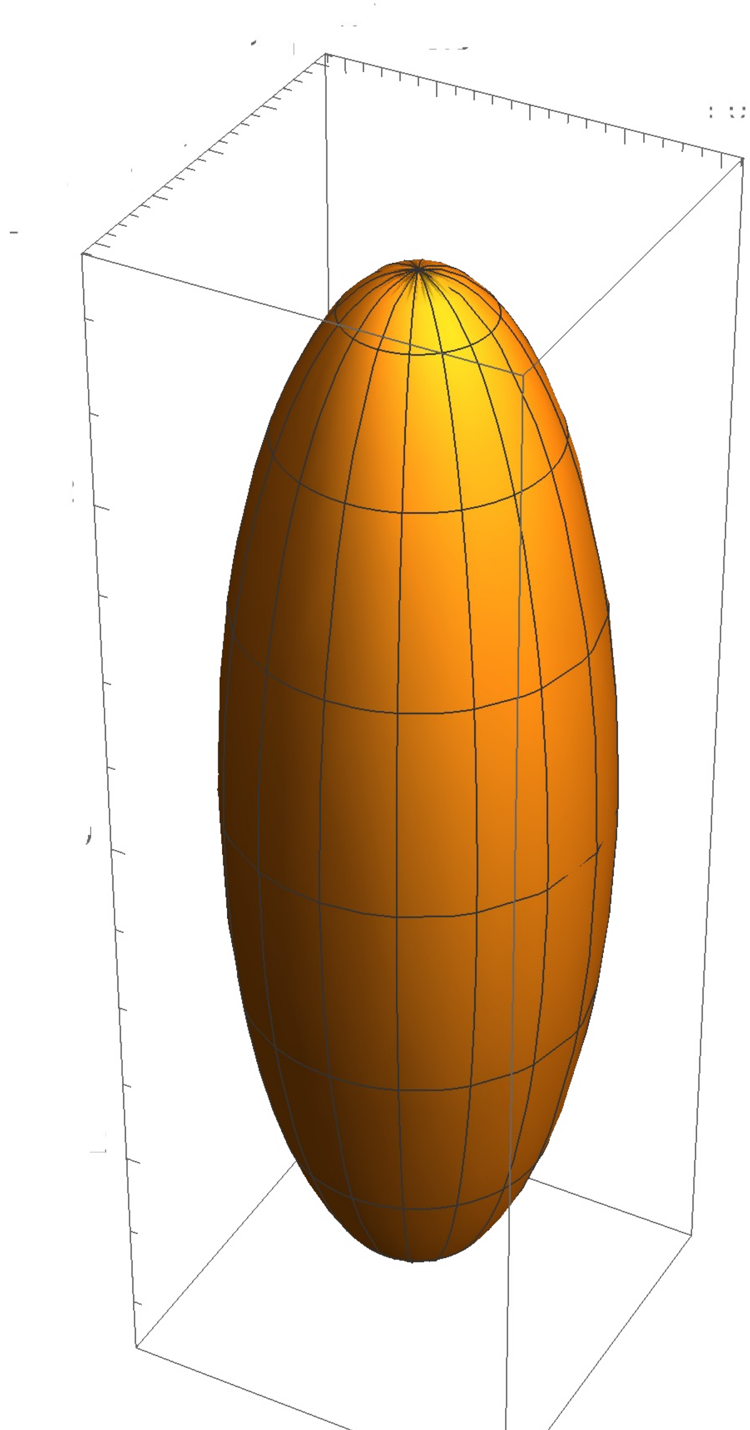
$$\mathbf{W}_\lambda := -\partial_x((\lambda^2 - x^2)\partial_x) + (2\pi\lambda x)^2$$

is invariant under $\mathbb{F}_{e_\mathbb{R}}$.

- $\mathbf{W}_\lambda$ commutes with P_λ and $\widehat{P}_\lambda$

Prolate spheroidal wave functions

The operator W_λ appears from separation of variables in the Laplacian Δ for the prolate spheroid :



Prolate coordinates

$$x = \sqrt{\left(a^2 - 1\right)\left(1 - b^2\right)}\,\cos(c)$$

$$y = \sqrt{\left(a^2 - 1\right)\left(1 - b^2\right)}\,\sin(c)$$

$$z = ab$$

Laplacian in Prolate coordinates

$$\Delta = (a^2 - b^2)^{-1} \left(\partial_a (a^2 - 1) \partial_a + \partial_b (1 - b^2) \partial_b \right)$$

$$+ (a^2 - 1)^{-1} (1 - b^2)^{-1} \partial_c^2$$

$$(a^2 - b^2)(\Delta + k^2) \to \partial_b (1 - b^2) \partial_b - k^2 b^2 + k^2 a^2$$

Self-adjoint extension

▶ The minimal domain is the Schwartz space $\mathcal{S}(\mathbb{R})$

▶ The deficiency indices are (4,4).

▶ Unique self-adjoint extension W_λ commuting with P_λ and $\widehat{P}_\lambda$.

- W_λ commutes with Fourier

- The selfadjoint operator W_λ has discrete spectrum.

- ϕ eigenfunction of $W_\lambda \Rightarrow$

$$\phi(x) \sim c \, \frac{\sin(2\pi\lambda x)}{x}, \quad x \to \infty$$

if ϕ is even and $\frac{\cos(2\pi\lambda x)}{x}$ if ϕ is odd.

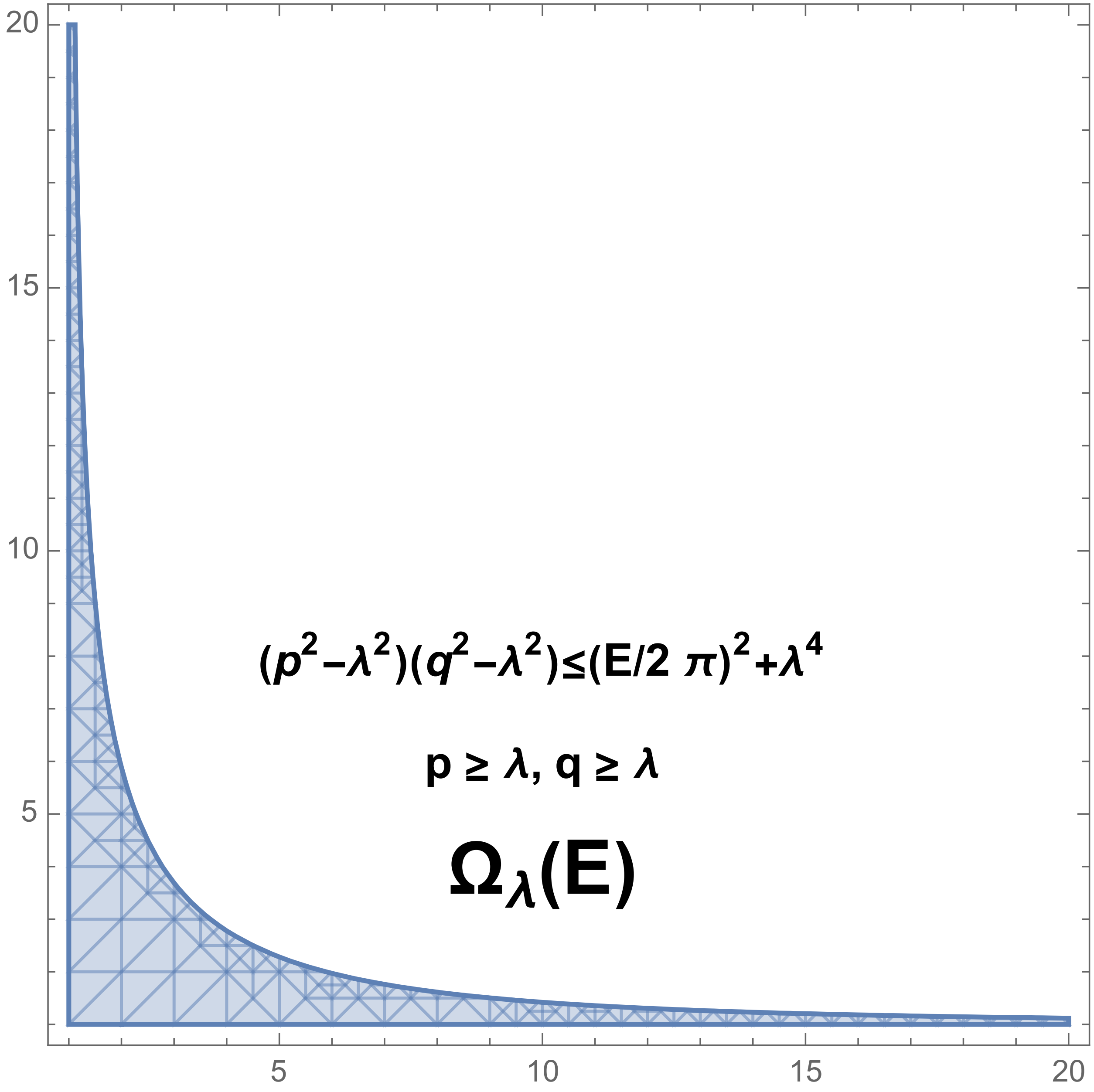
Semiclassical approximation

$$H_\lambda(p,q) = (p^2 - \lambda^2)(q^2 - \lambda^2)$$

$$W_\lambda = -4\pi^2 H_\lambda + 4\pi^2\lambda^4$$

$$\Omega_\lambda(E) := \{(q,p) \mid q \geq \lambda, p \geq \lambda, H_\lambda(p,q) \leq a\}$$

$$a = \left(\frac{E}{2\pi}\right)^2$$


20
15
10
5
5
10
15
20
$(p^2-\lambda^2)(q^2-\lambda^2)\leq(E/2\pi)^2+\lambda^4$
$p \geq \lambda, q \geq \lambda$
$\Omega_\lambda(E)$

The area $\sigma(E)$ of $\Omega_\lambda(E)$ is given, with $a = \left(\frac{E}{2\pi}\right)^2$, by the convergent integral

$$I_\lambda(a) = \int_\lambda^\infty \left(\frac{\sqrt{a + \lambda^2 x^2 - \lambda^4}}{\sqrt{x^2 - \lambda^2}} - \lambda \right) dx$$

$$\sigma(E) \sim \frac{E}{2\pi} \left(\log\left(\frac{E}{2\pi}\right) - 1 + \log(4) - 2\log(\lambda) \right)$$

$$+\lambda^2 + o(1)$$

In fact one has

$$I_\lambda(a) = \lambda^2 I_1(a\,\lambda^{-4})$$

and in terms of elliptic integrals

$$I_1(a) = aK(1-a) - E(1-a) + 1$$

$$\sim \frac{1}{2}\sqrt{a}(\log(a) - 2 + 2\log(4)) + 1 + o(1)$$

Dirac operator

- We found Dirac operator with Laplacian two copies of W_λ, using the Darboux method.

- We explore associated geometry.

Darboux method

$$p(x) = x^2 - \lambda^2,\ V(x) = 4\pi^2\lambda^2 x^2,\ W_\lambda = \partial(p(x)\partial) + V(x),$$

$$U : L^2([\lambda,\infty), dx) \to L^2([\lambda,\infty), p(x)^{-1/2}dx)$$

$$U(\xi)(x) := p(x)^{1/4}\xi(x), \quad (\delta f)(x) := p(x)^{1/2}\partial f(x)$$

$$\delta w(x) + w(x)^2 = -V(x) + \left(\frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)}\right), \qquad \forall x \in [\lambda,\infty)$$

$$W_\lambda = U^*\,(\delta + w)(\delta - w)\,U$$

Solution of Riccati equation

For $z \in \mathbb{C}$ and $u = u_1 + zu_2$ the solution u has no zero in (λ, ∞) if $z \notin \mathbb{R}$ and an infinity of zeros otherwise.

Solutions of the Riccati equation

$$w_z(x) = \frac{(x^2 - \lambda^2)^{1/4} \partial \Big((x^2 - \lambda^2)^{1/4} u(x) \Big)}{u(x)}$$

where $u = u_1 + zu_2$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

Dirac operator

$$D = \begin{pmatrix} 0 & \delta + w(x) \\ \delta - w(x) & 0 \end{pmatrix}$$

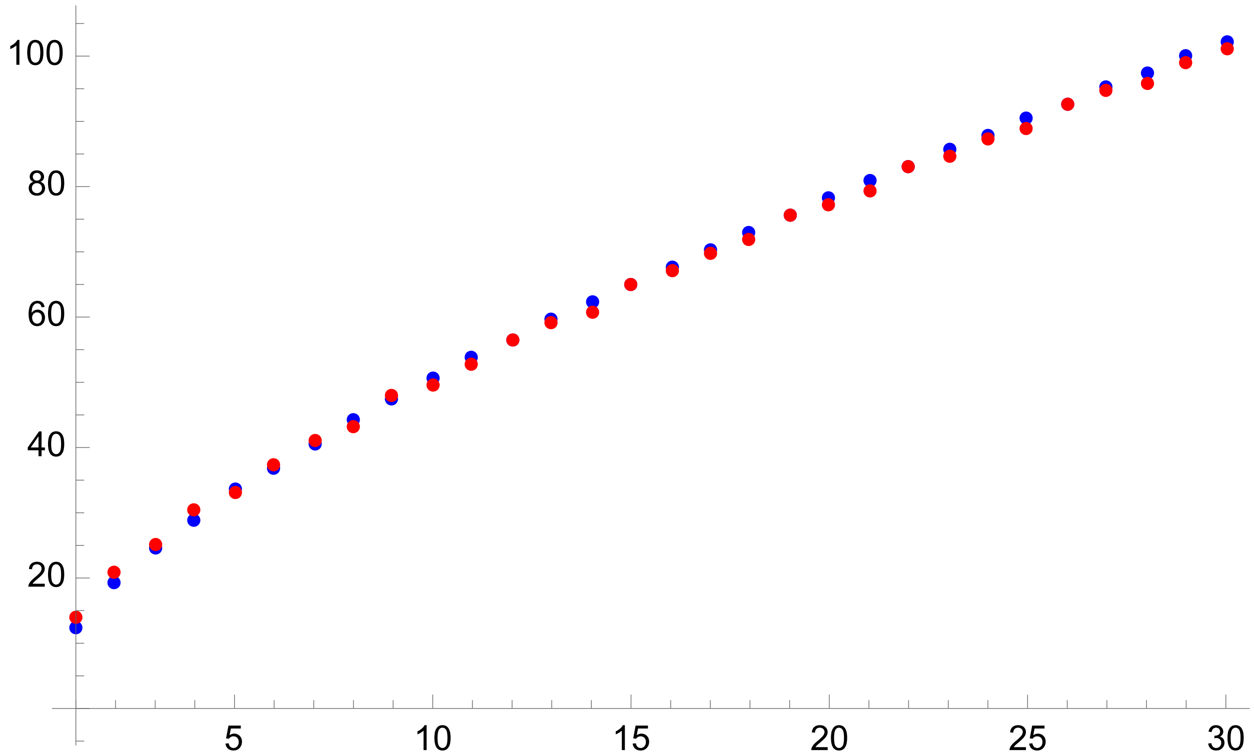
Then the square of D is diagonal with each diagonal term spectrally equivalent to W_λ,

$$U^* D^2 U = \begin{pmatrix} W_\lambda & 0 \\ 0 & W_\lambda + 2\delta w(x) \end{pmatrix}$$

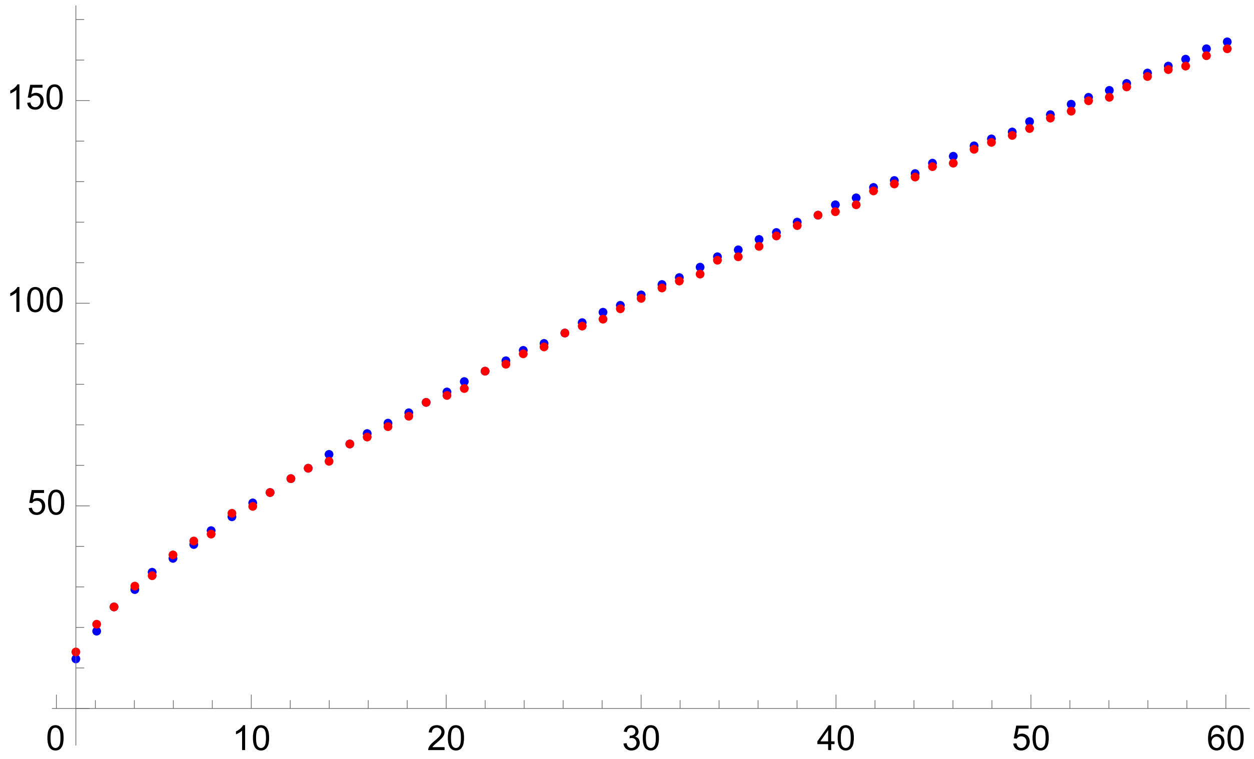
Ultraviolet∼Zeta

The operator $2D$ has discrete simple spectrum contained in $\mathbb{R} \cup i\mathbb{R}$. Its imaginary eigenvalues are symmetric under complex conjugation and the counting function $N(E)$ counting those of positive imaginary part less than E fulfills

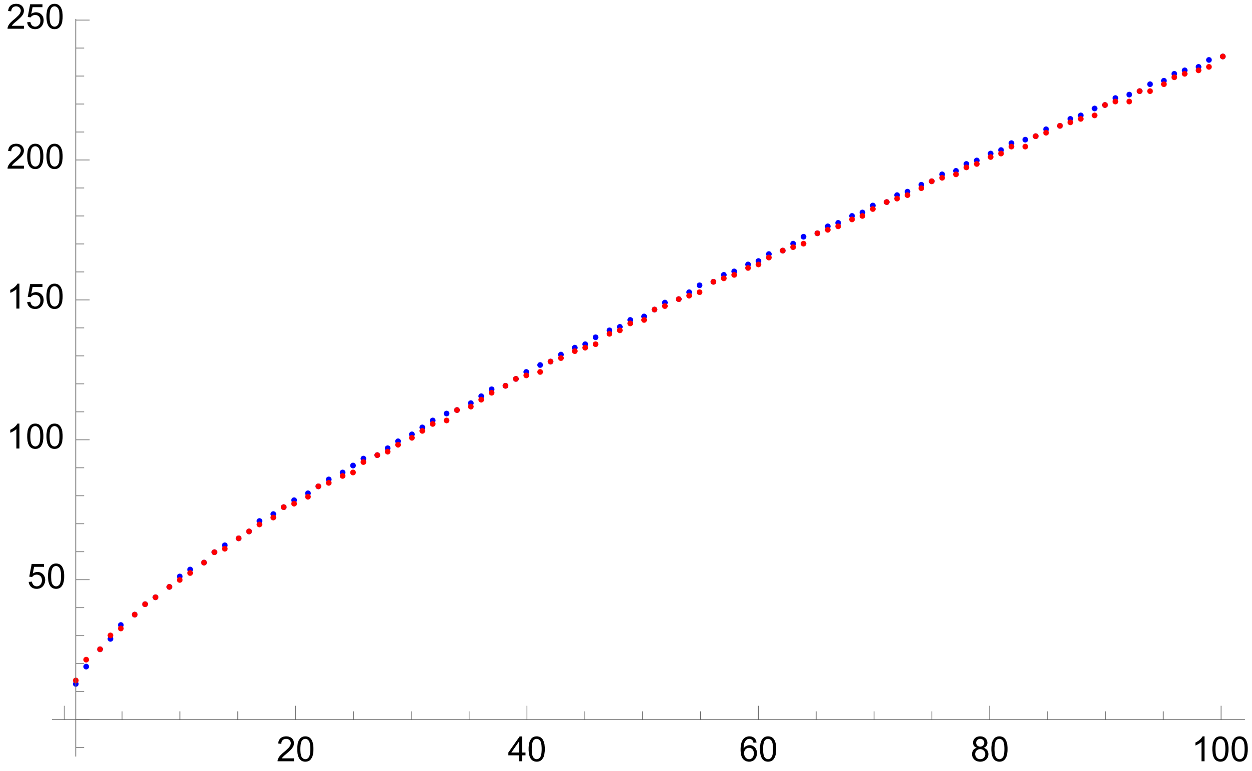
$$N(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + O(\log E)$$



100
80
60
40
20
5
10
15
20
25
30



150
100
50
0
10
20
30
40
50
60


250
200
150
100
50
20
40
60
80
100

Geometry= spectral triple

The metric associated to the spectral triple is

$$ds^2 = -\frac{1}{4} dx^2/(x^2 - \lambda^2) = \frac{1}{\alpha(x)} dx^2$$

Geometry extends to $\mathbb{R}$ and two dimensional

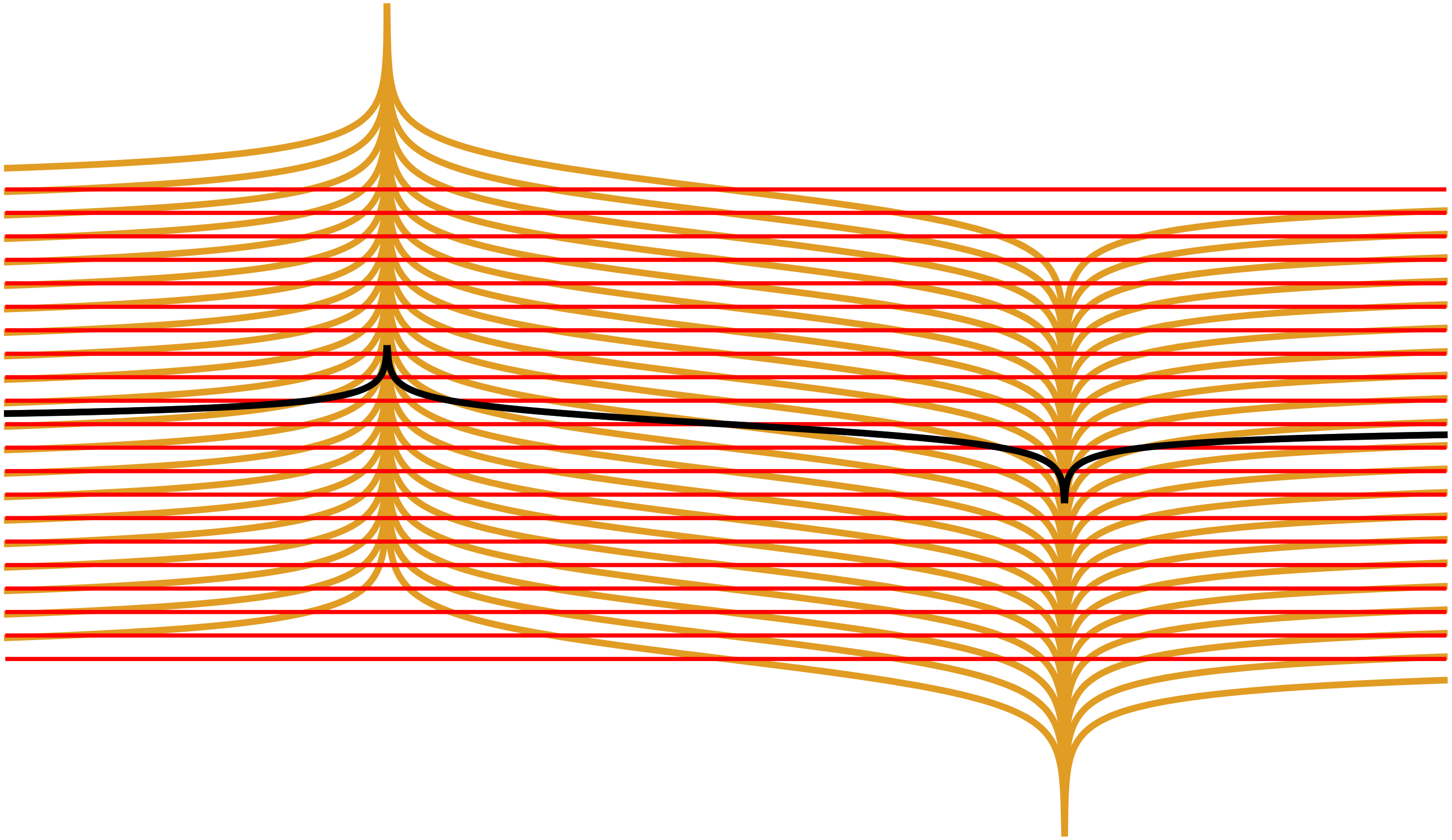
$$ds^2 = -\alpha(x) dt^2 + \frac{1}{\alpha(x)} dx^2$$

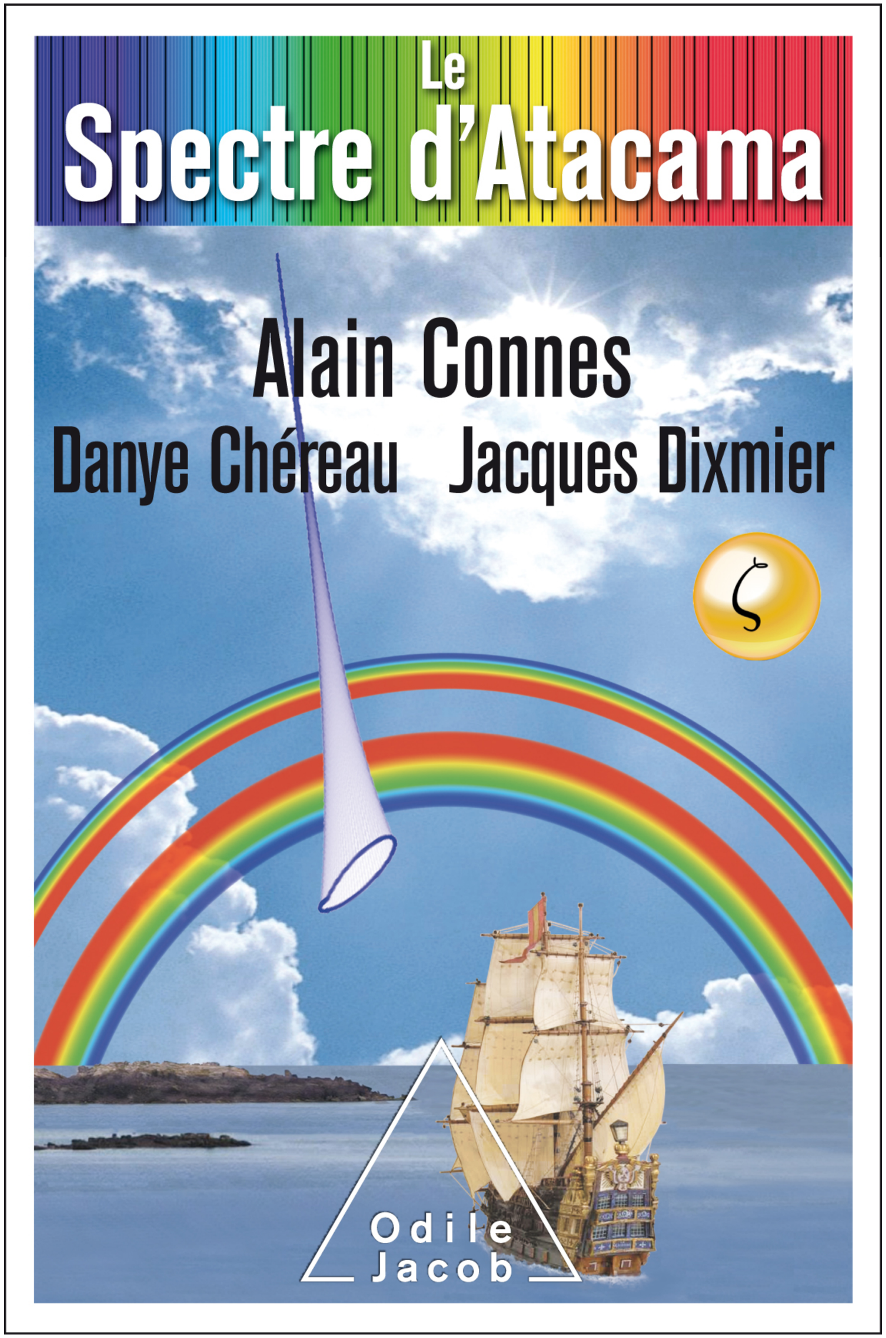
which after changing coordinates to $v = t - t(x)$ with

$$t(x) = \frac{1}{8\lambda} \log\left((\lambda + x)/(x - \lambda)\right)$$

becomes (black hole trick)

$$ds^2 = 4\left(x^2 - \lambda^2\right) dv^2 - 2dvdx$$





Le Spectre d'Atacama

Alain Connes

Danye Chéreau Jacques Dixmier

Odile Jacob