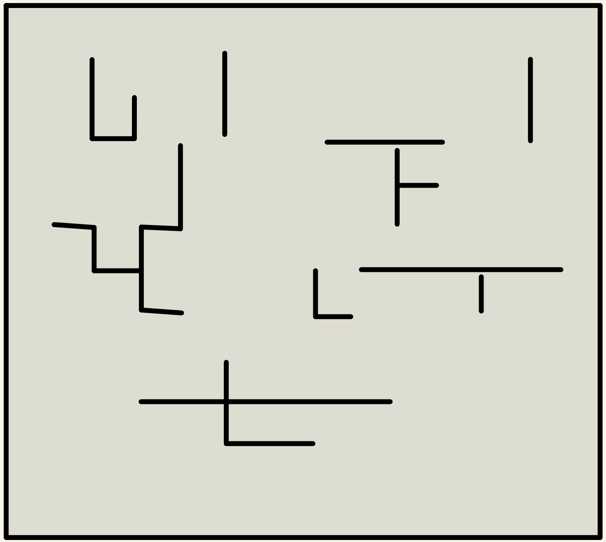
Random forests and the OSp(1|2) nonlinear sigma model



Roland Bauerschmidt (Cambridge)

with N. Crawford, T. Helmuth, A. Swan

Nonlinear sigma models (NLSM)

Fields $u: x \in \Lambda \longrightarrow u(x) \in T$

Lattice $\Lambda \subset \mathbb{Z}^d$
Continuum $\Lambda \subset \mathbb{R}^2$

Target (nonlinear)
Symmetric space
E.g. S^n

Action "$\frac{1}{2}\int_\Lambda (\nabla u)^2 \, dx$".

Models that realise symmetry in a minimal way.

Nonlinear sigma models (NLSM)

Examples ($\Lambda \uparrow \mathbb{Z}^d$):

- $T = \{\pm 1\}$ Ising model discrete / abelian
- $T = \mathbb{S}^1$ classical XY model flat / abelian
- $T = \mathbb{S}^2$ classical Heisenberg model curved / nonabelian

$$\langle F \rangle_\beta \propto \int_{(\mathbb{S}^n)^\Lambda} F(u)\, e^{-\frac{\beta}{2}(u,-\Delta u)_\Lambda} \prod_{x\in\Lambda} du_x ,$$

Haar

$$(u,-\Delta u)_\Lambda = \sum_{x\sim y} (u_x - u_y)\cdot(u_x - u_y)$$

Conjectures (and results) $d \geq 3$

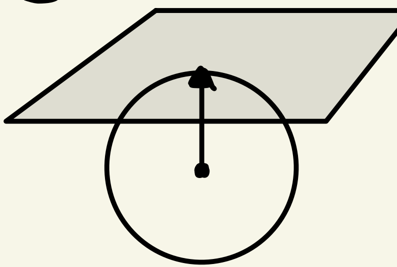
symmetry spontaneously broken

Lattice models have ordered low temperature phases

Proofs only available in special cases
and either fragile or very hard.

(reflection positivity)
Fröhlich-Simon-Spencer

(Balaban)



Goldstone mode — strong correlations

Continuum models do not exist.

Conjectures $d=2$

Predicted behaviour depends on curvature.

"Polyakov conjecture": positive curvature

$\Rightarrow$ Mass gap for all $\beta > 0$.

Continuum theory exists.

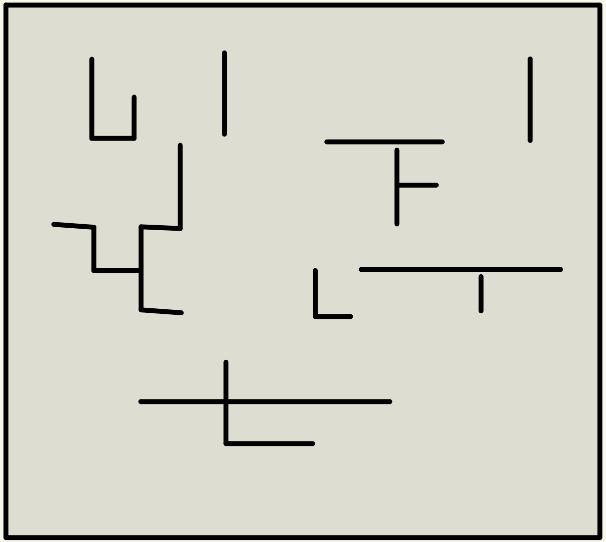
Similar to Yang-Mills in $d=4$.

The arboreal gas

$G=(\Lambda,E)$ finite graph

$F=(\Lambda,E(F))\subset G$ is a forest if it has no cycles

$$\mathbb{P}_\beta^G(F)=\frac{1}{Z_\beta}\beta^{|E(F)|}\,\mathbb{1}(F\text{ is a forest})$$



The arboreal gas

unrooted

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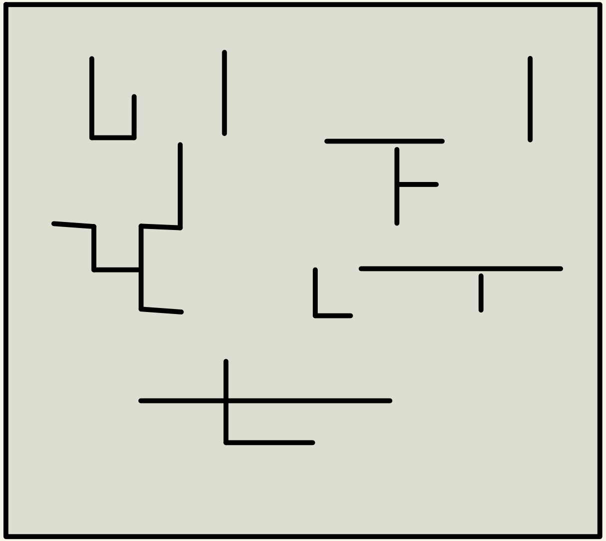

The arboreal gas

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- Fortuin-Kasteleyn: $q \to 0$ limit of random cluster model
- Lubensky-Isaacson: model for gelation of polymers
- Caracciolo-Jacobsen-Saleur-Sokal-Sportiello
- Luczak-Pittel, Martin-Yeo: analysis on complete graph
- Pemantle, Kahn, Grimmett-Winkler: Conj. negative dependence
- Brändén-Huh, Anari et al.: a main example of Lorentzian polyn.

Matrix tree theorem

delete a row and column

$$\#\text{spanning trees on } G = \det(-\Delta^{\circ})$$

$$\mathbb{P}^{UST}(e_1 \in T, \dots, e_k \in T) = \det(K(e_i, e_j))_{i,j}$$

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spanning trees $\equiv$ free (symplectic) fermions

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spanning trees $\equiv$ free (symplectic) fermions

$\xi_x, \eta_x, x \in \Lambda$ anticommuting variables

$\#ST = \int \partial_\eta \partial_\xi \, e^{(\xi, -\Delta^\circ \eta)}$ Grassmann int.

project onto top degree coefficient

expand $e^{\sum \xi_x (-\Delta^\circ)_{xy} \eta_y}$

Matrix tree theorem

delete a row and column

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$$\mathbb{P}^{UST}(e_1 \in T, \dots, e_k \in T) = \left\langle \prod_{i=1}^{k} (\nabla_{e_i} \xi)(\nabla_{e_i} \eta) \right\rangle_{FF}$$

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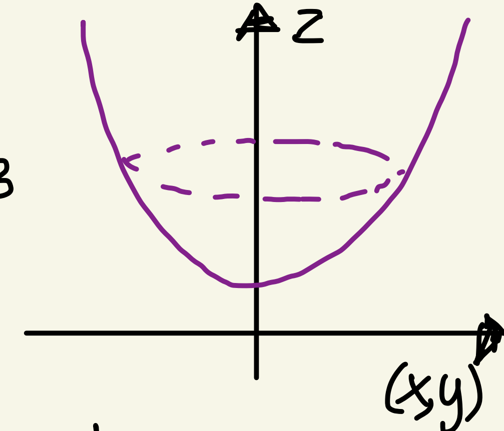
$$\boxed{\begin{array}{l}\text{spanning trees} \equiv \text{free (symplectic) fermions} \\ \xi_x, \eta_x, \; x \in \Lambda \quad \text{anticommuting variables} \\ \#ST = \int \partial_\eta \partial_\xi \, e^{(\xi, -\Delta^\circ \eta)} \quad \text{Grassmann int.} \\ \mathbb{P}^{UST}(e_1 \in T, \dots, e_k \in T) = \left\langle \prod_{i=1}^{k} (\nabla_{e_i} \xi)(\nabla_{e_i} \eta) \right\rangle_{FF}\end{array}}$$

Hyperbolic plane $\mathbb{H}^2$



Vectors $u_i = (x_i, y_i, z_i)$ in $\mathbb{R}^3$

$z_i > 0$

with $u_i \cdot u_i = -1$

where $u_i \cdot u_j = x_i x_j + y_i y_j - z_i z_j$

$\Big\}$ $SO^+(2,1)$ invariant

Concretely $z_i = \sqrt{1 + x_i^2 + y_i^2}$

$$\int_{\mathbb{H}^2} F = \int dx_i \, dy_i \, \frac{1}{z_i} F$$

Lebesgue integral

$SO(2,1)$ inv. volume form

Fermionic hyperbolic plane $\mathbb{H}^{0|2}$

Supervectors $u_i = (\underbrace{\xi_i, \eta_i}_{\text{odd}}, \underbrace{z_i}_{\text{even}})$ in $\mathbb{R}^{1|2}$

$$\left.\begin{aligned} &\text{with} \quad u_i \cdot u_i = -1 \\ &\text{where} \quad u_i \cdot u_j = \eta_i \xi_j + \eta_j \xi_i - z_i z_j \end{aligned}\right\} \text{OSp}(1|2) \text{ invariant}$$

Concretely $z_i = \sqrt{1 - 2\xi_i \eta_i} = 1 - \xi_i \eta_i$

$$\int_{\mathbb{H}^{0|2}} F = \int \partial_{\eta_i} \partial_{\xi_i} \frac{1}{z_i} F$$

Grassmann integral

OSp(1|2) inv. volume form

Fermionic hyperbolic plane $\mathbb{H}^{0|2}$

Supervectors $u_i = (\underbrace{\xi_i, \eta_i}_{\text{odd}}, \underbrace{z_i}_{\text{even}})$ in $\mathbb{R}^{1|2}$

Infinitesimal supersymmetries:

$$\left.\begin{array}{l} T\xi_i = z_i, \quad T\eta_i = 0, \quad Tz_i = -\eta_i \\ \bar{T}\xi_i = 0, \quad \bar{T}\eta_i = z_i, \quad \bar{T}z_i = \xi_i \\ \text{plus symplectic symmetries.} \end{array}\right\} \mathfrak{osp}(1|2)$$

$\mathbb{H}^{0|2}$ model $\longrightarrow$ arboreal gas

$$\langle F \rangle_{\beta,h} \propto \int_{(\mathbb{H}^{0|2})^\Lambda} e^{\frac{\beta}{2}(u,\Delta u) - h\sum_i z_i} F$$

$(u, -\Delta u) = \sum_{ij} (u_i - u_j) \cdot (u_i - u_j)$

OSp(1|2) inner prod.

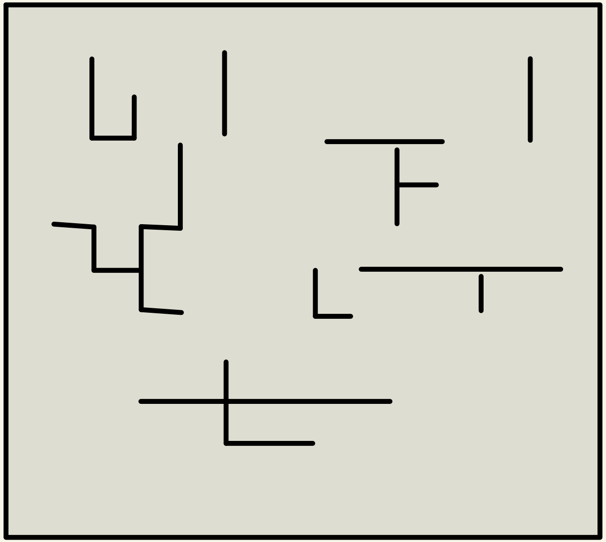
<u>Thm.</u> (CJSSS, BCHS) Nonlinear Matrix Tree Thm.

$$\mathbb{P}_{\beta,0}(0 \longleftrightarrow x) = -\langle u_0 \cdot u_x \rangle_{\beta,0}$$

$$\mathbb{P}_{\beta,0}(e_1 \in F, \dots, e_k \in F) = \left(\frac{-\beta}{2}\right)^k \left\langle \prod_{i=1}^k (\nabla_{e_i} u) \cdot (\nabla_{e_i} u) \right\rangle_{\beta,0}$$

$\vdots$

When does the arboreal gas percolate?



When does the arboreal gas percolate?

<u>Prop.</u> On any graph, the arboreal gas is stochastically dominated by bond percolation with $p = \beta/(1+\beta)$.

$\longrightarrow$ Arboreal gas does not percolate on $\mathbb{Z}^d$ if β small

When does the arboreal gas percolate?

Prop. On any graph, the arboreal gas is stochastically dominated by bond percolation with $p = \beta/(1+\beta)$.

→ Arboreal gas does not percolate on $\mathbb{Z}^d$ if β small

Thm (BCHS). In $d=2$, no percolation for any $\beta > 0$.

Thm (BCH). In $d \geq 3$, percolation for $\beta > \beta_0$.

When does the arboreal gas percolate?

<u>Thm.</u> (BCHS). Let $d=2$. For any $\beta>0$, $\exists\, c(\beta)>0$ s.t.

$$\mathbb{P}_\beta^\Lambda(0 \leftrightarrow x) \leq O(|x|^{-c(\beta)}) \quad \text{for all } \Lambda \subset \mathbb{Z}^2.$$

<u>Thm.</u> (BCH). Let $d\geq 3$. There are $\beta_0 \in (0,\infty)$ and $\zeta(\beta) = 1 - O(1/\beta)$ s.t. for $\beta \geq \beta_0$,

$$\mathbb{P}_\beta^\Lambda(0 \leftrightarrow x) = \zeta(\beta) + \frac{c(\beta)}{\beta |x|^{d-2}} + O\left(\frac{1}{\beta |x|^{d-2+\kappa}} + \frac{1}{\beta L^{\kappa N}}\right)$$

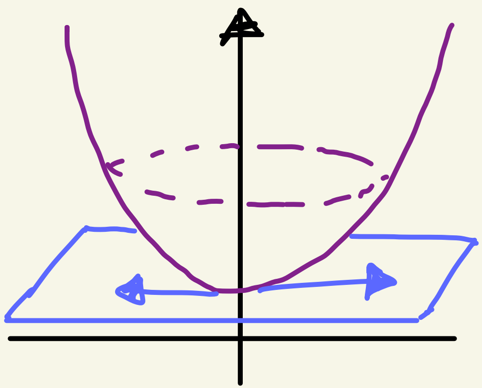
if Λ is a torus $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$, $L \geq L_0(d)$.

$\mathbb{H}^{0|2}$ model $\longrightarrow$ arboreal gas

$$\mathbb{P}_{\beta,0}(0 \longleftrightarrow x) = -\langle u_0 \cdot u_x \rangle_{\beta,0}$$

$\Rightarrow$ percolation $\leftrightarrow$ spontaneously broken symmetry

"spins point in same direction over large dist."



Continuous symmetry

$\rightarrow$ expect diffusive corrections

(Goldstone mode)

Ward identity: $$\frac{\langle z_i \rangle_{\beta,h}}{h} = \sum_j \langle \xi_i \eta_j \rangle_{\beta,h}$$

'Magic formula' $\longrightarrow$ No percolation in $d=2$

Thm (BCHS). For the arboreal gas,

$$\mathbb{P}_\beta(0 \leftrightarrow x) \propto \int e^{t_x} \, e^{-\sum_{i\sim j} \beta \cosh(t_i - t_j)} \left(\det(-\Delta^0_{\beta(t)})\right)^{3/2} \prod_{i \neq 0} e^{-3t_i} dt_i .$$

$\rightarrow$ Versions of Mermin-Wagner can be adapted to RHS.
(BCHS, Sabot, Kozma-Peled, Merkl-Rolles)

'Magic formula': Same measure but with $3 \rightsquigarrow 1$ is related to reinforced walks.

Symmetry breaking ($d \geq 3$)

Want to prove:

$$-\langle u_0 \cdot u_x \rangle_{\beta,0} = \theta(\beta)^2 + \frac{2c(\beta)}{\beta |x|^{d-2}} + O\big((L^{-\kappa N} + |x|^{-(d-2+\kappa)})\beta^{-1}\big)$$

Introduction of $h>0$ breaks symmetry:

$$\frac{1}{h}\langle z_0 \rangle_{\beta,h} = \sum_{x \in \Lambda_N} \langle \xi_0 \eta_x \rangle_{\beta,h}$$

(same Ward identity as in other NLSMs)

Two possible limits: '$N \to \infty, h \downarrow 0$' and '$h \downarrow 0, N \to \infty$'.

Symmetry breaking ($d \geq 3$)

Thm. (BCH). Let $d \geq 3$, $L \geq L_0(d)$, $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$. Then

$$-\langle u_0 \cdot u_x \rangle_{\beta,0} = \theta(\beta)^2 + \frac{2C(\beta)}{\beta |x|^{d-2}} + O\left(\left(L^{-\kappa N} + |x|^{-(d-2+\kappa)}\right)\beta^{-1}\right)$$

Symmetry breaking ($d \geq 3$)

<u>Thm.</u> (BCH). Let $d \geq 3$, $L \geq L_0(d)$, $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$. Then

$$-\langle u_0 \cdot u_x \rangle_{\beta,0} = \theta(\beta)^2 + \frac{2c(\beta)}{\beta |x|^{d-2}} + O\left(\left(L^{-\varkappa N} + |x|^{-(d-2+\varkappa)} \right) \beta^{-1} \right)$$

same constants $c(\beta)$, $\theta(\beta)$

Also, $\lim_{h \downarrow 0} \lim_{N \to \infty} \langle z_0 \rangle_{\beta,h} = \theta(\beta)$

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle \xi_0 \eta_x \rangle_{\beta,h} = \frac{c(\beta)}{\beta |x|^{d-2}} + O\left(\frac{1}{\beta |x|^{d-2+\varkappa}} \right)$$

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle z_0 ; z_x \rangle_{\beta,h} = \frac{-c(\beta)^2}{\beta^2 \theta(\beta)^2 |x|^{2d-4}} + O\left(\frac{1}{\beta^2 |x|^{2d-4+\varkappa}} \right)$$

Many questions

The arboreal gas arises as the $q \to 0$ limit of the random cluster model.

The $H^{0|2}$ model behaves in all ways as the "$q=0$" Potts model. Is it solvable in $d=2$?

The Potts model with $q \geq 2$ has symmetry S_q, while the $H^{0|2}$ has $OSp(1|2)$: "$\lim_{q\to 0} S_q = OSp(1|2)$"?

Parisi: "$\lim_{q\to 0} S_q$" is continuous

Many questions

- Universality. Is the order statistics of the component sizes on $\mathbb{Z}^d / L\mathbb{Z}^d$, $d \geq 3$, $\beta > \beta_0$, the same as on the complete graph?

Expected similarly to universality of Wigner-Dyson.

Distributions known explicitly for complete graph:

$$\left(T^{(1)} - \alpha N\right) N^{-2/3} \longrightarrow \text{limiting distribution}$$

k-th largest component

$$T^{(k)} N^{-2/3} \longrightarrow \text{limiting distribution}$$

Many questions

- Critical dimension should be $d_c = 6$

  Deng-Garoni-Sokal: Numerical evidence

  Cubic saddle point in mean-field theory

  Can one show this, at least for spread-out interaction (finite range spin coupling)?
- $6-\varepsilon$ and $2+\varepsilon$ expansions? Klebanov et. al.
- Mass gap in $d=2$?

Many questions

- Negative correlation conjecture. (Pemantle, Grimmett-Winkler, Kahn)

$$\mathbb{P}_\beta(e_1 \in F, e_2 \in F) \leq \mathbb{P}_\beta(e_1 \in F)\,\mathbb{P}_\beta(e_2 \in F)$$

Known only in limit $\beta \to \infty$ (UST) or $|\Lambda| \leq 8$.

Brändén-Huh and Anari et al. recently proved

$$\mathbb{P}_\beta(e_1 \in F, e_2 \in F) \leq 2\,\mathbb{P}_\beta(e_1 \in F)\,\mathbb{P}_\beta(e_2 \in F)$$

and the partition function with edge dependent weights is log concave (Lorentzian polynomial).

Thank you!

The arboreal gas

$G = (\Lambda, E)$ finite graph

$F = (\Lambda, E(F)) \subset G$ is a forest if it has no cycles

$$\mathbb{P}^G_{\beta,h}(F) = \frac{1}{Z_{\beta,h}} \beta^{|E(F)|} \prod_{T \in F} (1 + h|V(T)|) \, \mathbb{1}(F \text{ is a forest})$$

- Fortuin-Kasteleyn: $q \to 0$ limit of random cluster model
- Lubensky-Isaacson: model for gelation of polymers
- Caracciolo-Jacobsen-Saleur-Sokal-Sportiello
- Luczak-Pittel, Martin-Yeo: analysis on complete graph
- Pemantle, Kahn, Grimmett-Winkler: Conj. negative dependence
- Brändén-Huh, Anari et al.: a main example of Lorentzian polyn.

Symmetry breaking ($d \geq 3$)

In coordinates, $\xi = \bar\psi/\sqrt{\beta}$, $\eta = \psi/\sqrt{\beta}$, $z = 1 - \psi\bar\psi/\beta$,

$$\exp\left(-(\nabla\psi)(\nabla\bar\psi) - \frac{1}{\beta}(1+h)\psi\bar\psi - \frac{1}{\beta}\psi\bar\psi(\nabla\psi)(\nabla\bar\psi)\right)$$

same coupling

irrel. in $d>2$

relevant

- Couplings related → Use Ward identity to fix relevant coupling
- Limit $h \downarrow 0$ first → Propagator will have 0-mode.

Setup for renormalisation group

finite range decomposition

Gaussian propagator $C = (-\Delta + m^2)$

$$\exp\left(-s_0 (\nabla \psi)(\nabla \bar{\psi}) - a_0 \psi\bar{\psi} - b_0 \psi\bar{\psi}(\nabla\psi)(\nabla\bar{\psi})\right)$$

Stable manifold theorem: $a_0^c(b_0, m^2)$, $s_0^c(b_0, m^2)$

Corresponds to $\mathbb{H}^{0|2}$ if $a_0 = \frac{1+s_0}{\beta}(1+h) - m^2$

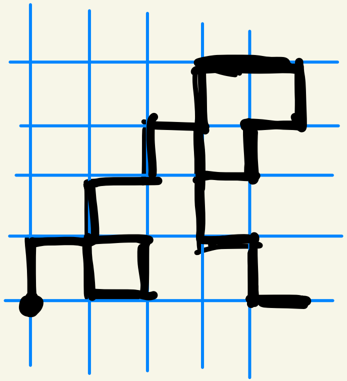
parameters (β, h)

$$b_0 = \frac{(1+s_0)^2}{\beta^2}$$

Word identity: $h = m^2 \beta (1 + O(b_0))$

Magic formula for ERRW and VRJP

ERRW: Edge e has weight $\alpha + N_e(t)$



# of crossings of edge e

parameter

α small: strong reinforcement

α large: weak reinforcement

Magic formula for ERRW and VRJP

ERRW: Edge e has weight $\alpha + N_e(t)$

random conductances

Diaconis-Freedman & Coppersmith-Diaconis: $\mathbb{P}_0^{ERRW(\alpha)} = \int \mathbb{P}_0^{SRW(c)} \, d\mu_\alpha(c)$

walk starting at vertex 0

mixing measure given by 'magic formula'

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VRJP: Vertex reinforced version

Sabot - Tarrès: $\mathbb{P}_0^{VRJP(\beta)} = \int \mathbb{P}_0^{SRW(c(t))} \, d\nu_\beta(t)$

Thm (Sabot-Tarrès). Mixing measure for VRJP:

$$d\nu_\beta(t) \propto e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^0(-\Delta_{\beta(t)})\right)^{1/2}\prod_{i\neq 0} e^{-t_i}\,dt_i$$

$$(\Delta_{\beta(t)}f)_i = \sum_{j\sim i}\beta e^{t_i+t_j}(f_j-f_i)$$

Thm (BCHS). For the arboreal gas,

$$\mathbb{P}_\beta(0\leftrightarrow x) \propto \int e^{t_x}\, e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)} \left(\det{}^0(-\Delta_{\beta(t)})\right)^{3/2}\prod_{i\neq 0} e^{-3t_i}\,dt_i .$$

Three sources for the magic.

$\mathbb{H}^{0|2} \longrightarrow \mathbb{H}^{2|4}$

$\mathbb{H}^{2|4}$ consists of supervectors $u = (\underbrace{\phi^1, \phi^2, z}_{\text{even}}, \underbrace{\xi^1, \eta^1, \xi^2, \eta^2}_{\text{odd}})$

$$u_i \cdot u_i = -1$$

$$\left.\begin{aligned} u_i \cdot u_j = {} & \phi_i^1 \phi_j^1 + \phi_i^2 \phi_j^2 - z_i z_j \\ & + \eta_i^1 \xi_j^1 + \eta_j^1 \xi_i^1 + \eta_i^2 \xi_j^2 + \eta_j^2 \xi_i^2 \end{aligned}\right\} \quad OSp(2,1|2)$$

Localisation:

$$\left\langle f((u_i \cdot u_j), (z_i)) \right\rangle_{\beta,h}^{\mathbb{H}^{0|2}} = \left\langle f((u_i \cdot u_j), (z_i)) \right\rangle_{\beta,h}^{\mathbb{H}^{2|4}}$$

Horospherical coordinates for $\mathbb{H}^{2|4}$

$$\begin{cases} z = \cosh t + \frac{1}{2} e^t s^2 \\ \phi^1 = \sinh t - \frac{1}{2} e^t s^2 \\ \phi^2 = e^t s \\ \xi^i = e^t \bar{\psi}^i \\ \eta^i = e^t \psi^i \end{cases}$$

$\left(\begin{array}{l} t \in \mathbb{R},\ s \in \mathbb{R} \\ \bar{\psi}^1, \psi^1, \bar{\psi}^2, \psi^2 \text{ Grassmann} \end{array} \right)$

quadratic in $s, \bar{\psi}^1, \psi^1, \bar{\psi}^2, \psi^2$
exponential in t

$$\frac{\beta}{2}(u, -\Delta u) + h \sum_i z_i \longrightarrow \beta \sum_{ij} \cosh(t_i - t_j) + \frac{1}{2} \sum_{ij} e^{t_i + t_j} (s_i - s_j)^2$$

$$+ (\cdots)$$

Hyperbolic σ-model

$$\propto e^{-\frac{\beta}{2}\sum_{ij\in E}(u_i-u_j)\cdot(u_i-u_j)} \prod du_i$$

$$= e^{-\beta\sum_{ij\in E}\left(\cosh(t_i-t_j)+\frac{1}{2}e^{t_i+t_j}(s_i-s_j)^2\right)} \prod_i e^{-(n-1)t_i} dt_i\, ds_i$$

Gaussian in s!

Hyperbolic σ-model

$$\propto e^{-\frac{\beta}{2}\sum_{ij\in E}(u_i-u_j)\cdot(u_i-u_j)} \prod du_i$$

$$= e^{-\beta\sum_{ij\in E}\left(\cosh(t_i-t_j)+\frac{1}{2}e^{t_i+t_j}(s_i-s_j)^2\right)} \prod_i e^{-(n-1)t_i} dt_i\, ds_i$$

Gaussian in s!

Thus t-marginal is

$$e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^0(-\Delta_{\beta(t)})\right)^{-(n-1)/2}\prod_{i\neq 0} e^{-(n-1)t_i} dt_i$$

Thm (Sabot–Tarrès). Mixing measure for VRJP:

$$d\nu_\beta(t) \propto e^{-\sum_{i\sim j}\beta\cosh(t_i - t_j)} \left(\det{}^0(-\Delta_{\beta(t)})\right)^{1/2} \prod_{i\neq 0} e^{-t_i}\, dt_i$$

n=0

$$(\Delta_{\beta(t)} f)_i = \sum_{j\sim i} \beta e^{t_i+t_j}(f_j - f_i)$$

Thm (BCHS). For the arboreal gas,

n=−2

$$\mathbb{P}_\beta(0 \leftrightarrow x) \propto \int e^{t_x}\, e^{-\sum_{i\sim j}\beta\cosh(t_i - t_j)} \left(\det{}^0(-\Delta_{\beta(t)})\right)^{3/2} \prod_{i\neq 0} e^{-3t_i}\, dt_i .$$

What is $\mathbb{H}^n$ if $n \leq 0$?

Thm (Sabot-Tarrès). Mixing measure for VRJP:

$$d\nu_\beta(t) \propto e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^{0}(-\Delta_{\beta(t)})\right)^{1/2}\prod_{i\neq 0} e^{-t_i}\,dt_i$$

n=0

$$(\Delta_{\beta(t)}f)_i = \sum_{j\sim i}\beta e^{t_i+t_j}(f_j-f_i)$$

Thm (BCHS). For the arboreal gas,

n=-2

$$\mathbb{P}_\beta(0\leftrightarrow x) \propto \int e^{t_x}\, e^{-\sum_{i\sim j}\beta\cosh(t_i-t_j)}\left(\det{}^{0}(-\Delta_{\beta(t)})\right)^{3/2}\prod_{i\neq 0} e^{-3t_i}\,dt_i\,.$$

What is $\mathbb{H}^n$ if $n\leq 0$? $\mathbb{H}^{p|2q}$ has dim. $n=p-2q$.