Uniqueness of BP fixed point for Ising models

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with: Qian Yu (Princeton) – *on the job market!*

What is “BP” in the title?

Belief propagation (BP) operator $\mathcal{Q}$ is a map of probability measures:

- Fix a probability measure μ on $\mathbb{R}$ and let $\mathcal{Q}\mu$ be the law of

$$R = \sum_{i=1}^{d} (-1)^{X_i} F_\delta(R_i)\,,$$

where $R_i \stackrel{iid}{\sim} \mu$ independent of $X_i \stackrel{iid}{\sim} \mathrm{Bern}(\delta)$ and

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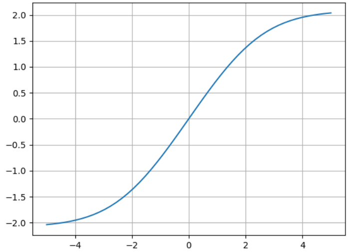

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- Variations:
  - d itself could be random (most often: $d \sim \mathrm{Poi}(\bar{d})$)
  - we can have "side information" or "survey" $S_0 \sim \mu_0$:

$$R = \sum_{i=1}^{d} (-1)^{X_i} F_\delta(R_i) + S_0$$

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Main Question: Characterize fixed points

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There exists at most one non-trivial fixed point μ^* and $\mathcal{Q}^k\mu \to \mu^*$ as $k \to \infty$ for any $\mu \neq \delta_0$.

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- [Kanade-Mossel-Schramm'2014]: labeled 2-SBM
- [Mossel-Xu'2015]: optimality of local algorithms for 2-SBM
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This talk: What is this relevant for?
How information theory helped us prove it?

Resc

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Three application domains

- Application 1: Statistical Physics
- Application 2: Machine Learning
- Application 3: Information propagation

Application 1: Ising Model on Sparse Graphs

- Model of correlated phenomena: Ising model on a graph $G = (V, E)$

$$\mathbb{P}[\mathbf{X} = \mathbf{x}] = \frac{1}{Z} e^{\beta H(\mathbf{x})}$$

with Hamiltonian

$$H(\mathbf{x}) = \sum_{u \sim v} x_u x_v$$

and $x_u \in \{\pm 1\}$ – binary spin variables.

- Many graphs are "complex" but "sparse". So model them as a random d-regular graph.

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- Locally such graphs are just trees.
- So let us understand Ising model on trees.

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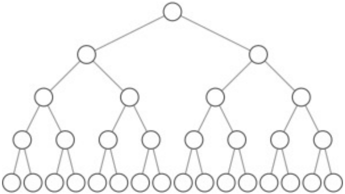

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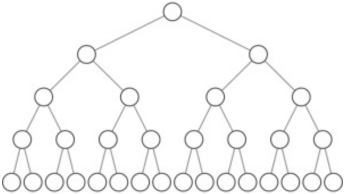

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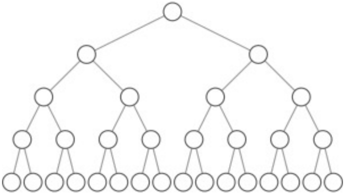

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- Similar issue on finite locally-tree-like graphs with correlated boundary.

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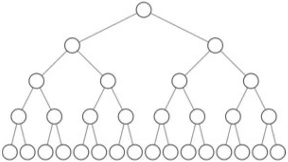

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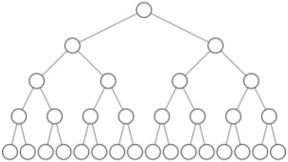

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- As usual there is a phase transition:
  - high temp $\beta \le \frac{1}{\operatorname{atanh}(d)}$: there is a unique Gibbs measure on infinite tree. The choice of boundary condition $\mathbf{y}_L$ is irrelevant.
  - medium temp $\beta > \frac{1}{\operatorname{atanh}(d)}$: depending on $\mathbf{y}_L$ we can get (uncountably many) Gibbs measures on infinite tree.

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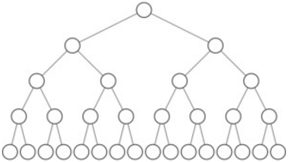

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> Problem: How to classify these Gibbs measures?

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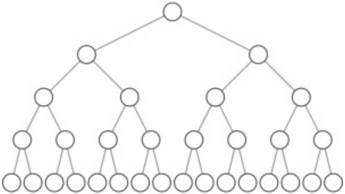

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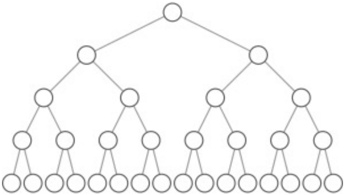

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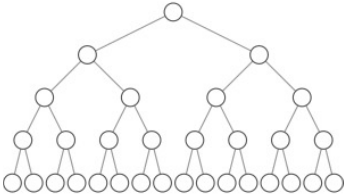

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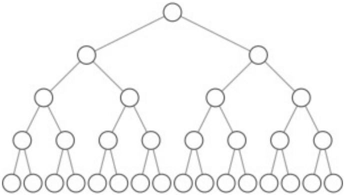

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- Question: How to "count" $\mathbb{P}_\alpha$'s?

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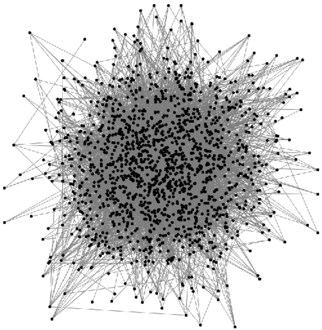
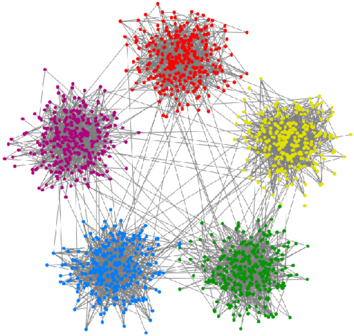
- Our contribution. We prove this ansatz: Iterations $\mathcal{Q} \circ \mathcal{Q} \cdots \mathcal{Q}\mu_0$ converge to a unique fixed point regardless of μ_0.





- Unsupervised clustering problem
- Input: graph
- Want: Label clusters

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- Key fact: locally the recovery problem looks like BOT (next):

$$d \triangleq \deg(u) \sim \mathrm{Poi}(a+b), \qquad \delta \triangleq \mathbb{P}[X_u \neq X_v | u \sim v] = \frac{b}{a+b}$$

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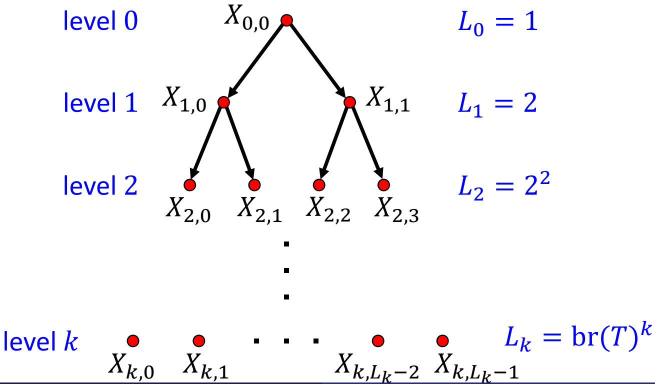
- Fix infinite oriented tree T with branching number $\mathrm{br}(T) := d$.



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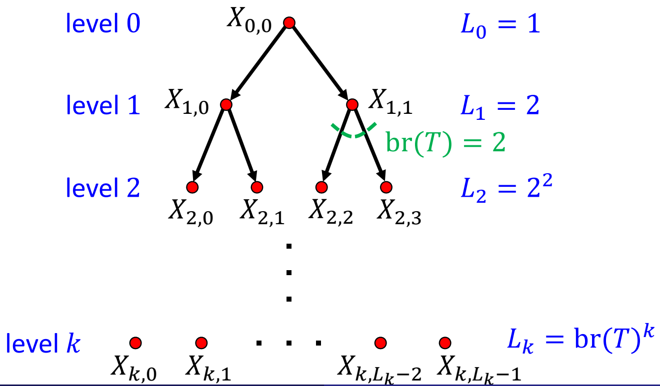

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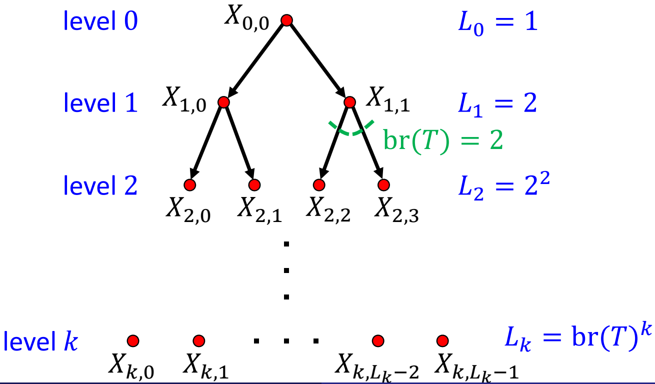

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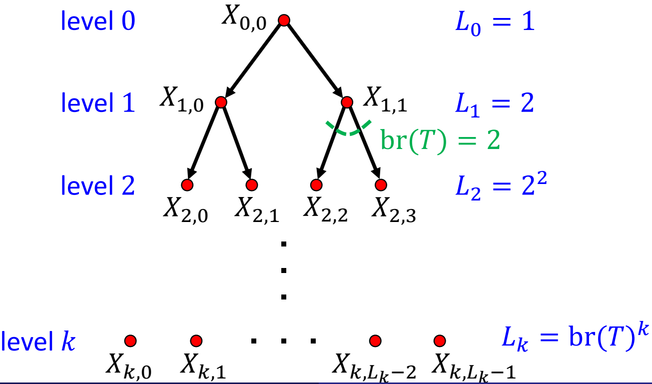

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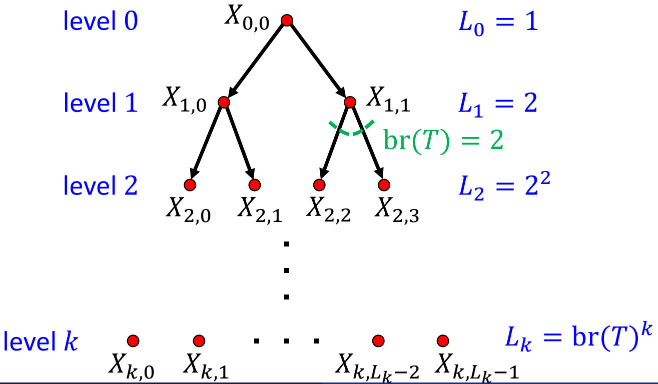
- Goal: Reconstruct $X_{0,0}$ from $X_k = \left(X_{k,0}, \ldots, X_{k,d^k-1}\right)$.



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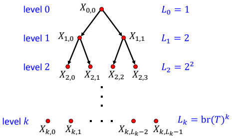

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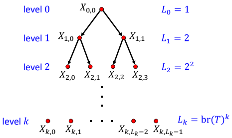

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$$R^{(k)} \triangleq \log \frac{\mathbb{P}[X_{0,0} = +1 | X_k]}{\mathbb{P}[X_{0,0} = -1 | X_k]}$$

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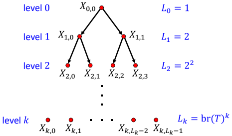

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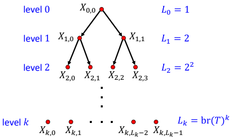
Broadcasting on Trees and BP fixed point



- Root variable $X_{0,0}$ is the information source
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Can show that $R^{(k)} \xrightarrow{d} \mu^*$ – BP fixed point.

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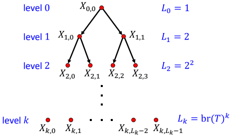

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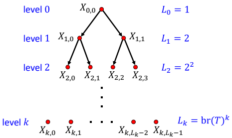

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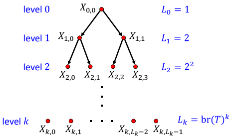

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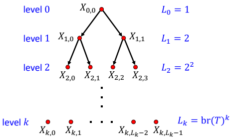

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- Our main result: $\mathcal{Q}^k \tilde{\mu}_0 \to \tilde{\mu}_\infty$ and $\tilde{\mu}_\infty$ does not depend on τ (!!)

Proof ideas

Binary symmetric (BMS) channels

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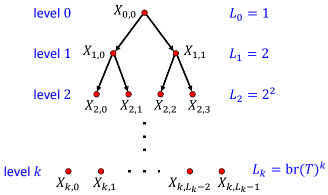

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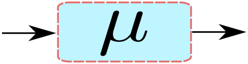
- Denote a BMS corresp. to μ as 

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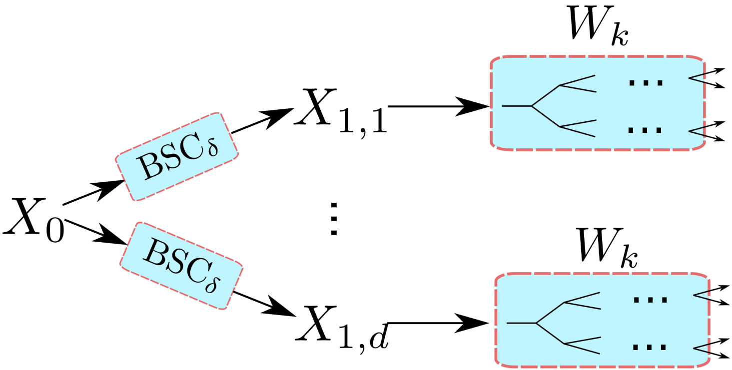

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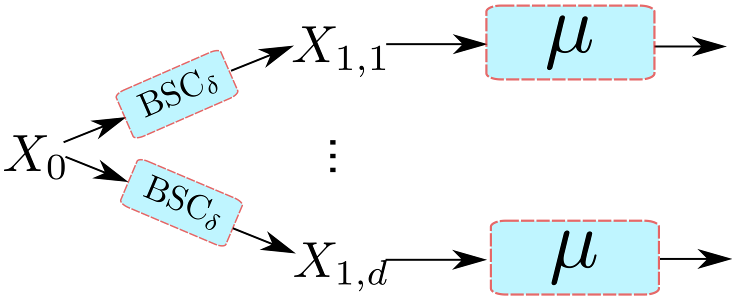

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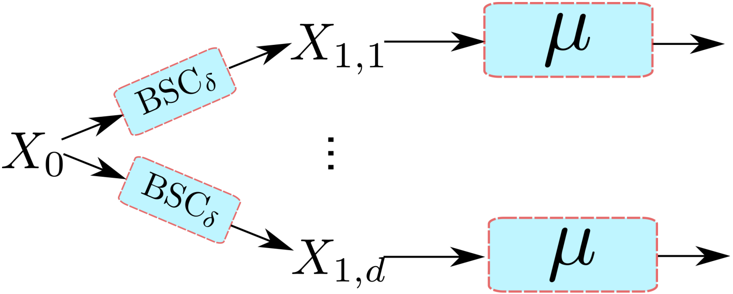

Infinite divisibility: a fixed point channel has property of not changing upon d-fold copying of its δ-noisy version.

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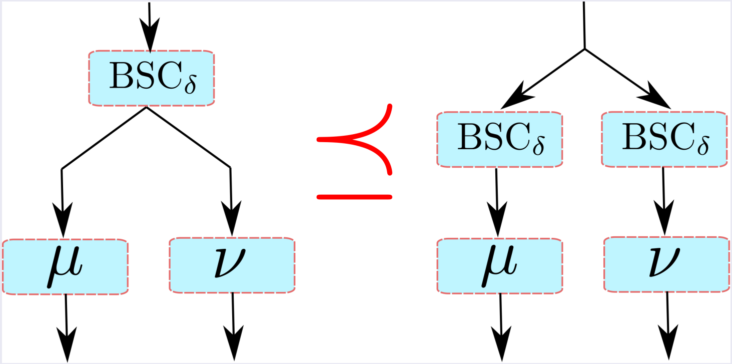

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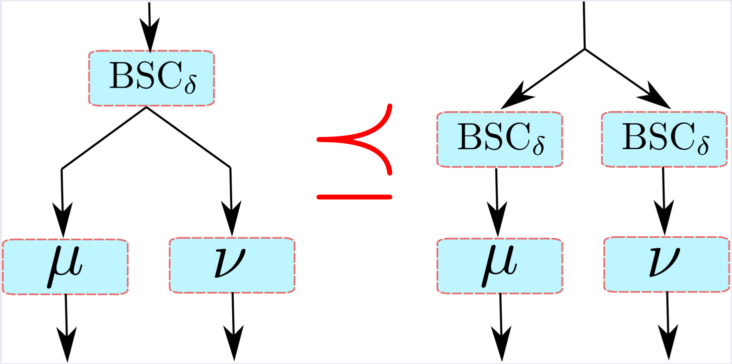

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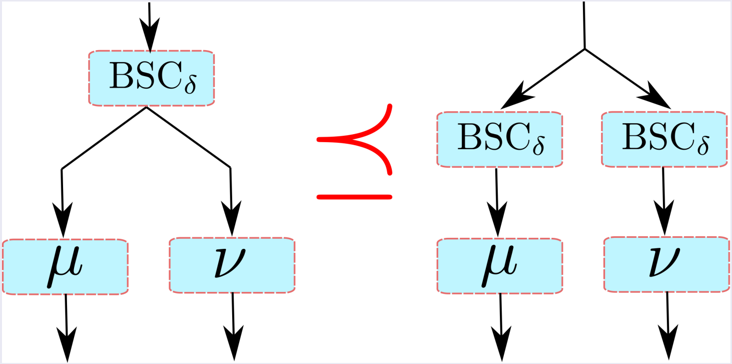

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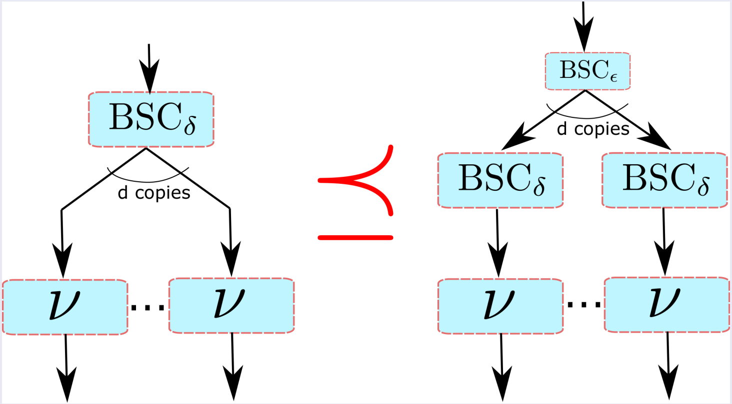
Applying iteratively $1 - 2\delta_k = (1 - 2\delta)^k$ [EKPS'2000] bound W_k and showed $\mathcal{Q}^k \delta_\infty \to \delta_0$ if $(1 - 2\delta)^2 d < 1$.

Theorem (Main technical discovery of Yu-P.'2022)

For every ν, $\delta \in (0,1)$ *and* $d \geq 3$ *there exists* $\epsilon > 0$ *such that:*



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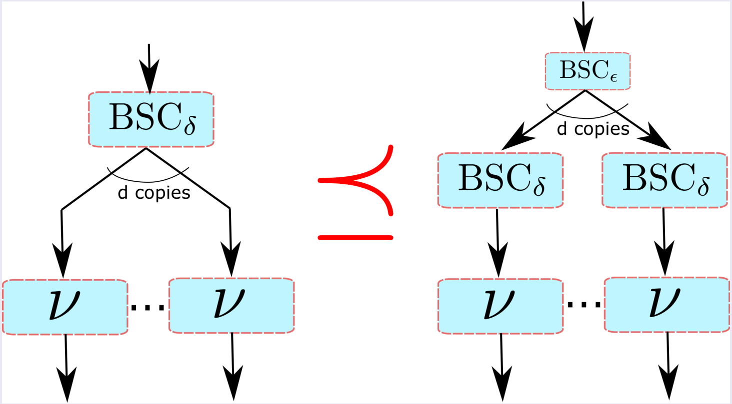

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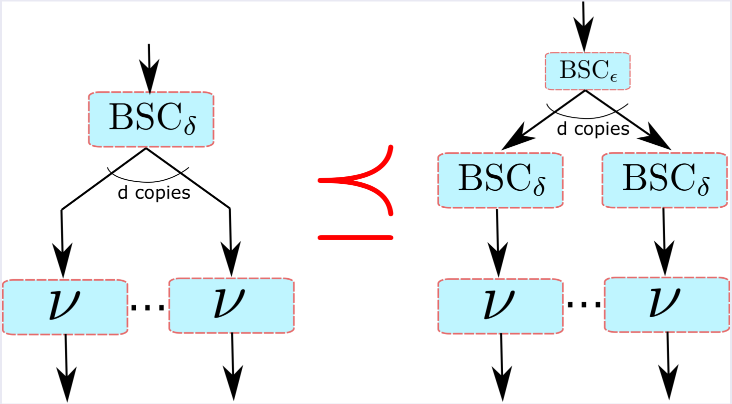

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Let us see how this result implies unicity of BP-fixed point.

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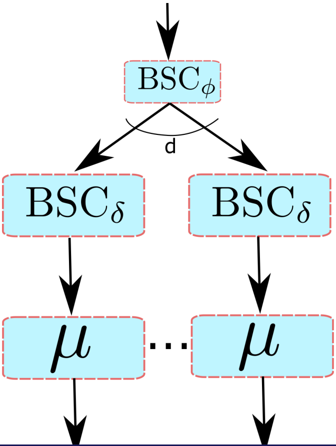

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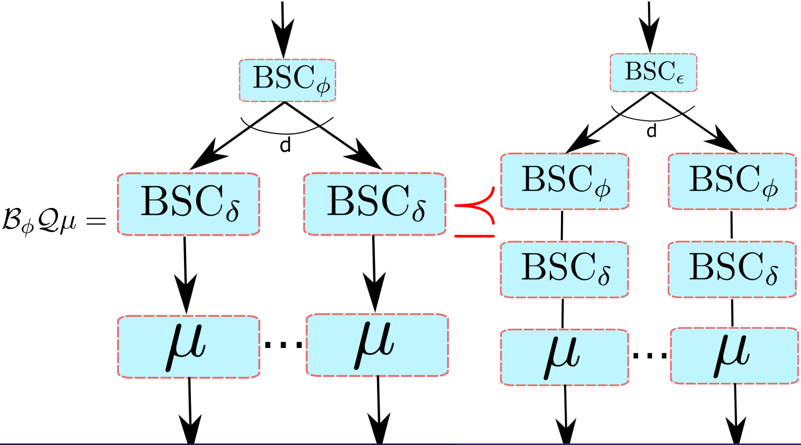

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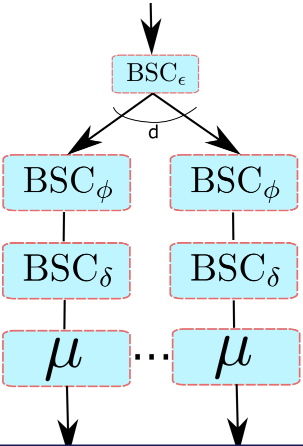

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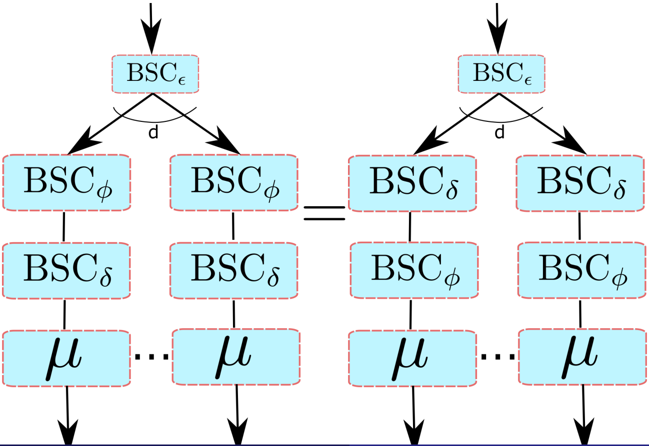

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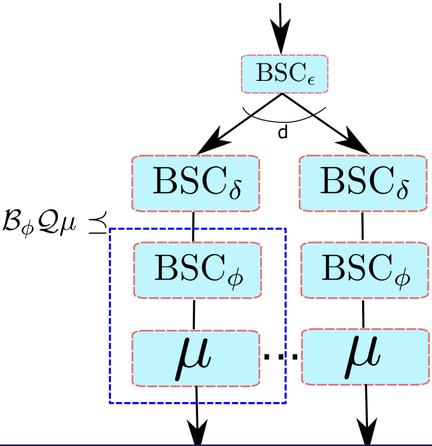

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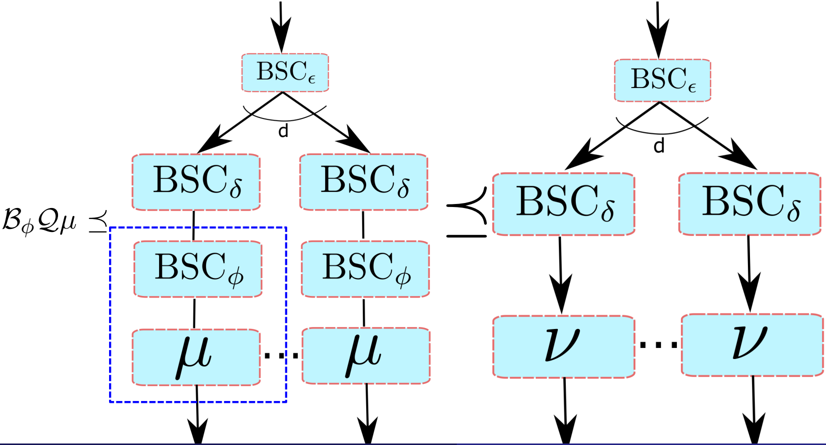

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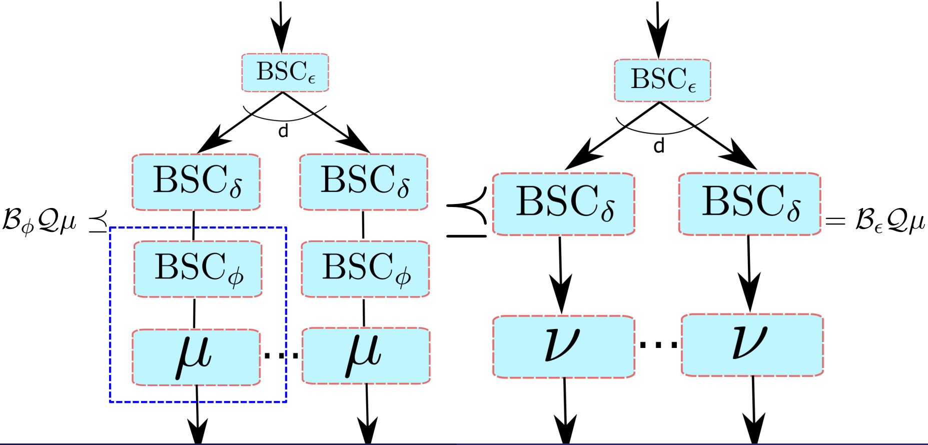
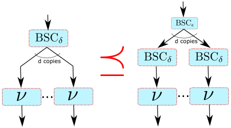
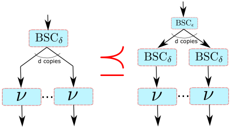


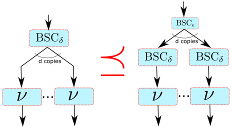


- One word about the proof of improved STL



- One word about the proof of improved STL
- Blackwell preorder: $P_{Y|X} \preceq P_{Z|X}$ if there is $P_{Y|Z}$ s.t.

$$P_{Y|X}(y|x) = \sum_z P_{Z|X}(z|x) P_{Y|Z}(y|z) \quad \iff \quad X \xrightarrow{P_{Y|X}} Z \xrightarrow{P_{Y|Z}} Y$$



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- Idea 1: Our trick was to avoid the (hard) job of constructing $P_{Y|Z}$:

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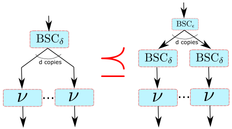

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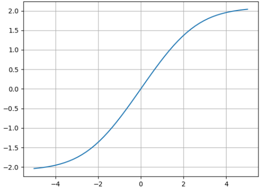
- Idea 2: Any μ is a mixture of elementary ones:

$$\mathcal{B}_\tau = (1-\tau)\delta_{-c} + \tau\delta_c, \qquad c = \log\frac{\tau}{1-\tau}\,.$$

So key inequalities are checked for BSC$_\tau$'s in place of generic ν.

Conclusion

$$F_\delta(x) =$$



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Theorem (Main result)

There exists at most one non-trivial fixed point μ^* of $\mathcal{Q}$.

- Unusual method based on (new) properties of channel compositions.

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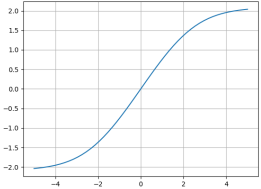
$F_\delta(x) =$ 

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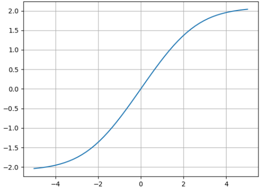

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- **Future work:** Extending to Potts (q-ary) models.

Thank You!

The draft is available here:
https://www.mit.edu/~ypol