Rigorous results about entropies in Quantum Field Theory

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Motivation

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In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; However, some very basic mathematical questions remain open. For example, most of the entropies computed in the physics literature are infinite, so the singularity structures, and sometimes the cut off independent quantities, are of most interest. Often, the mutual information is argued to be finite based on heuristic physical arguments, and one can derive the singularities of the entropies from the mutual information by taking singular limits. But it is not even clear that such mutual information, which is well defined as a special case of Araki's relative entropy, is indeed finite.
We begin to address some of these fundamental mathematical questions motivated by the physicists' work on entropy.

Motivation and Main Results

Main Results

Unlike the main focus in recent work such as by Hollands and Sanders, the relative entropy, in particular mutual information considered in our paper can be computed explicitly in many cases and satisfies many conditions, but not all, proposed by physicists such as those considered by Casini and Huerta. Our work is strongly motivated by Edward Witten's questions, in particular his question to make physicists' entropy computations rigorous. In this talk we focus on the Chiral CFT in two dimensions, where the results we have obtained are most explicit and have interesting connections to subfactor theory, even though some of our results do not depend on conformal symmetries and apply to more general QFT. The main results are:

1) Exact computation of the mutual information (through the relative entropy as defined by Araki for general states on von Neumann algebras) for free fermions.

Motivation and Main Results

Main Results

Note that this was not even known to be finite, for example the main quantity defined by Hollands and Sanders is smaller. Our proof uses Lieb's convexity and the theory of singular integrals; to the best of our knowledge, this and related cases are the first time that such relative entropies are computed in a mathematical rigorous way. The results verify earlier computations by physicists based on (cut off dependent) In particular, for the free chiral net $\mathcal{A}_r$ associated with r fermions, and two intervals $A = (a_1, b_1)$, $B = (a_2, b_2)$ of the real line, where $b_1 < a_2$, the mutual information associated with A, B is

$$F(A, B) = -\frac{r}{6} \log \eta \ ,$$

where $\eta = \frac{(b_1 - a_2)(b_2 - a_1)}{(b_1 - a_1)(b_2 - a_2)}$ is the cross ratio of A, B, $0 < \eta < 1$.

Motivation and Main Results

Main Results

2) It follows from 1) and the monotonicity of the relative entropy that any chiral CFT in two dimensions that embeds into free fermions, and their finite index extensions, verify most of the conditions (not all) discussed for example by Casini and Huerta. This includes a large family (in fact all known examples) of chiral CFTs. Much more can be obtained if the embedding has finite index. In this case, we also verify a proposal of Casini and Huerta about an entropy formula related to a derivation of the c theorem. Our theorem also connects relative entropy and index of subfactors in an interesting and unexpected way. This connection is different but related to Pimsner-Popa result that connects Connes-Stormer entropy to index. There is also one bit of surprise: it is usually postulated that the mutual information of a pure state such as vacuum state for complementary regions should be the same. But in the Chiral case this is not true, and the violation is measured by global dimension.

Main Results

The violation, which is in some sense proportional to the logarithm of global index, also turns out to be what is called topological entanglement entropy. (In fact there is a precise formula relating such violation to relative entropy defined by conditional expectation to disjoint interval nets.)

OUTLINE

Entropy and relative entropy

von Neumann entropy is the quantity associated with a density matrix ρ on a Hilbert space $\mathcal{H}$ by

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \ .$$

von Neumann entropy can be viewed as a measure of the lack of information about a system to which one has ascribed the state. This interpretation is in accord for instance with the facts that $S(\rho) \geq 0$ and that a pure state $\rho = |\Psi\rangle\langle\Psi|$ has vanishing von Neumann entropy. Note that von Neumann entropy is non-linear in the state and in general not easy to compute even in finite dimensional cases such as those in the statistical mechanical model on finite lattices.

A related notion is that of the relative entropy. It is defined for two density matrices ρ, ρ' by

$$S(\rho, \rho') = \mathrm{Tr}(\rho \log \rho - \rho \log \rho') \ . \quad (1)$$

Like $S(\rho)$, $S(\rho, \rho')$ is non-negative, and can be infinite.

Entropy and relative entropy

Entropy and relative entropy

A generalization of the relative entropy in the context of von Neumann algebras of arbitrary type was found by Araki and is formulated using modular theory. Given two faithful, normal states ω, ω' on a von Neumann algebra $\mathcal{A}$ in standard form, we choose the vector representatives in the natural cone $\mathcal{P}^\sharp$, called $|\Omega\rangle, |\Omega'\rangle$. The anti-linear operator $S_{\omega,\omega'} a|\Omega'\rangle = a^*|\Omega\rangle$, $a \in \mathcal{A}$, is closable and one considers again the polar decomposition of its closure $\bar{S}_{\omega,\omega'} = J\Delta^{1/2}_{\omega,\omega'}$. Here J is the modular conjugation of $\mathcal{A}$ associated with $\mathcal{P}^\sharp$ and $\Delta_{\omega,\omega'} = S^*_{\omega,\omega'}\bar{S}_{\omega,\omega'}$ is the relative modular operator w.r.t. $|\Omega\rangle, |\Omega'\rangle$. Of course, if $\omega = \omega'$ then $\Delta_\omega = \Delta_{\omega,\omega'}$ is the usual modular operator or modular Hamiltonian in physics literature.

The relative entropy w.r.t. ω and ω' is defined by

$$S(\omega, \omega') = \langle \Omega | \log \Delta_{\omega,\omega'} \ \Omega \rangle = \lim_{t \to 0} \frac{\omega([D\omega : D\omega']_t - 1)}{it} \ ,$$

S is extended to positive linear functionals that are not necessarily normalized by the formula $S(\lambda\omega, \lambda'\omega') = \lambda S(\omega, \omega') + \lambda \log(\lambda/\lambda')$, where $\lambda, \lambda' > 0$ and ω, ω' are normalized. If ω' is not normal, then one sets $S(\omega, \omega') = \infty$.

For a type I algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, states ω, ω' correspond to density matrices ρ, ρ'. The square root of the relative modular operator $\Delta^{1/2}_{\omega,\omega'}$ corresponds to $\rho^{1/2} \otimes \rho'^{-1/2}$ in the standard representation of $\mathcal{A}$ on $\mathcal{H} \otimes \bar{\mathcal{H}}$; namely $\mathcal{H} \otimes \bar{\mathcal{H}}$ is identified with the Hilbert-Schmidt operators $HS(\mathcal{H})$ with the left/right multiplication of $\mathcal{A}/\mathcal{A}'$. In this representation, ω corresponds to the vector state $|\Omega\rangle = \rho^{1/2} \in \mathcal{H} \otimes \bar{\mathcal{H}}$, and the abstract definition of the relative entropy becomes

$$\langle\Omega| \log \Delta_{\omega,\omega'} \Omega\rangle = \mathrm{Tr}_{\mathcal{H}} \rho^{\frac{1}{2}} \left(\log \rho \otimes 1 - 1 \otimes \log \rho'\right) \rho^{\frac{1}{2}} = \mathrm{Tr}_{\mathcal{H}}(\rho \log \rho - \rho \log \rho') \, . \tag{2}$$

As another example, let us consider a bi-partite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and observable algebra $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. A normal state ω_{AB} on $\mathcal{A}$ corresponds to a density matrix ρ_{AB}. One calls $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho_{AB}$ the "reduced density matrix", which defines a state ω_A on $\mathcal{B}(\mathcal{H}_A)$ (and similarly for system B). The mutual information is given in our example system by

$$S(\rho_{AB}, \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \ . \tag{3}$$

For tri-partite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and observable algebra $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \otimes \mathcal{B}(\mathcal{H}_C)$, we have the following strong subadditivity :

$$S(\rho_{AB}) + S(\rho_{AC}) - S(\rho_A) - S(\rho_{ABC}) \geq 0 \ . \tag{4}$$

This was originally proved by Lieb and Ruskai but follows rather easily from monotonicity of relative entropy.

Kosaki's formula

In general it is desirable to have a formula for $S(\omega, \omega')$ directly in terms of states. This is provided by Kosaki:

$$S(\omega, \omega') = \sup_{m \in \mathbb{N}} \sup_{x_t + y_t = 1} \left(lnm - \int_{m^{-1}}^{\infty} \left(\omega(x_t^* x_t) \frac{1}{t} + \omega'(y_t y_t^*) \frac{1}{t^2} \right) dt \right) \, ,$$

where x_t is a step function valued in M which is equal to 0 when t is sufficiently large. Many properties of relative entropies follow easily from Kosaki's formula. For an example: Let ω and ϕ be two normal states on a von Neumann algebra M, and denote by ω_1 and ϕ_1 the restrictions of ω and ϕ to a von Neumann subalgebra $M_1 \subset M$ respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$. As another example: Let be M_i an increasing net of von Neumann subalgebras of M with the property $(\bigcup_i M_i)'' = M$. Then $S(\omega_1 \restriction M_i, \omega_2 \restriction M_i)$ converges to $S(\omega_1, \omega_2)$ where ω_1, ω_2 are two normal states on M;

Finally Let ω and ω_1 be two normal states on a von Neumann algebra M. If $\omega_1 \geq \mu\omega$, then $S(\omega, \omega_1) \leq ln\mu^{-1}$; Here is a property of relative entropies that does not follow directly from Kosaki's formula: Let M be a von Neumann algebra and M_1 a von Neumann subalgebra of M. Assume that there exists a faithful normal conditional expectation E of Monto M_1. If ψ and ω are states of M_1 and M, respectively, then $S(\omega, \psi \cdot E) = S(\omega \restriction M_1, \psi) + S(\omega, \omega \cdot E)$; For type III factors, the von Neumann entropy is always infinite, but we shall see that in many cases mutual information is finite. By taking singular limits, we can also explore the singularities of von Neumann entropy from mutual information which is important from physicists' point of view. The formal properties of von Neumann entropies are useful in proving properties of mutual information.

OUTLINE

Graded nets and subnets

We shall denote by Möb the Möbius group, which is isomorphic to $SL(2,\mathbb{R})/\mathbb{Z}_2$ and acts naturally and faithfully on the circle S^1.
By an interval of S^1 we mean, as usual, a non-empty, non-dense, open, connected subset of S^1 and we denote by $\mathcal{I}$ the set of all intervals. If $I \in \mathcal{I}$, then also $I' \in \mathcal{I}$ where I' is the interior of the complement of I. Intervals are disjoint if their closure are disjoint. We will denote by $\mathcal{PI}$ the set which consists of disjoint union of intervals.

Möbius covariant net

This is an adaption of DHR analysis to chiral CFT which is most suitable for our purposes.
By an *interval* we shall always mean an open connected subset I of S^1 such that I and the interior I' of its complement are non-empty. We shall denote by $\mathcal{I}$ the set of intervals in S^1.
A *Möbius covariant* net $\mathcal{A}$ of von Neumann algebras on the intervals of S^1 is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$ to the von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the following:

Möbius covariant

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G. Conformal covariance.

A. ISOTONY

If I_1, I_2 are intervals and $I_1 \subset I_2$, then

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B. Möbius covariance

There is a nontrivial unitary representation U of $\mathbf{G}$ (the universal covering group of $PSL(2, \mathbf{R})$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)\,, \qquad g \in \mathbf{G}, \quad I \in \mathcal{I}\,.$$

The group $PSL(2, \mathbf{R})$ is identified with the Möbius group of S^1, i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore $\mathbf{G}$ has a natural action on S^1.

C. Positivity of the energy

The generator of the rotation subgroup $U(R)(\cdot)$ is positive. Here $R(\vartheta)$ denotes the (lifting to **G** of the) rotation by an angle ϑ.

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D. Graded Locality

There exists a grading automorphism **g** of $\mathcal{A}$ such that, if I_1 and I_2 are disjoint intervals,

$$[x, y] = 0, \quad x \in \mathcal{A}(I_1), y \in \mathcal{A}(I_2) \ .$$

Here $[x, y]$ is the graded commutator with respect to the grading automorphism **g**.

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There exists a unit vector Ω (vacuum vector) which is $U(\mathbf{G})$-invariant and cyclic for $\vee_{I \in \mathcal{I}} \mathcal{A}(I)$.

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F. Uniqueness of the vacuum (or irreducibility)

By a *conformal net* (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:
G. Conformal covariance There exists a projective unitary representation U of $Diff(S^1)$ on $\mathcal{H}$ extending the unitary representation of **G** such that for all $I \in \mathcal{I}$ we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in Diff(S^1), \\ U(g)xU(g)^* &= x, \quad x \in \mathcal{A}(I),\ g \in Diff(I'), \end{aligned}$$

where $Diff(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of S^1 and $Diff(I)$ the subgroup of diffeomorphisms g such that $g(z) = z$ for all $z \in I'$.

Moreover, setting

$$Z \equiv \frac{1 - i\Gamma}{1 - i} \ ,$$

we have that the unitary Z fixes Ω and

$$\mathcal{A}(I') \subset Z\mathcal{A}(I)'Z^*$$

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THEOREM 1

Let $\mathcal{A}$ be a Möbius covariant Fermi net on S^1. Then Ω is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.

If $I \in \mathcal{I}$, we shall denote by Λ_I the one parameter subgroup of Möb of "dilation associated with I".

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Theorem 2

Let $I \in \mathcal{I}$ and Δ_I, J_I be the modular operator and the modular conjugation of $(\mathcal{A}(I), \Omega)$. Then we have:
(i):

$$\Delta_I^{it} = U(\Lambda_I(-2\pi t)),\ t \in \mathbb{R}, \tag{5}$$

(ii): U extends to an (anti-)unitary representation of Möb $\ltimes \mathbb{Z}_2$ *determined by*

$$U(r_I) = ZJ_I,\ I \in \mathcal{I},$$

acting covariantly on $\mathcal{A}$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(\dot{g}I) \quad g \in \text{Möb} \ltimes \mathbb{Z}_2\ I \in \mathcal{I}\ .$$

Here $r_I : S^1 \to S^1$ is the reflection mapping I onto I'.

Part (1) of the above theorem says that the modular Hamiltonian is the boost generator, or as mathematicians would say that the modular automorphism group is geometric, and plays an important role in recent work on entropies in physics literature.

COROLLARY 3

(Additivity) *Let I and I_i be intervals with $I \subset \cup_i I_i$. Then $\mathcal{A}(I) \subset \vee_i \mathcal{A}(I_i)$.*

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Theorem 4

For every $I \in \mathcal{I}$, we have:

$$\mathcal{A}(I') = Z\mathcal{A}(I)'Z^* \ .$$

Let now G be a simply connected compact Lie group. Then the vacuum positive energy representation of the loop group LG at level k gives rise to an irreducible local net denoted by $\mathcal{A}_{G_k}$. Every irreducible positive energy representation of the loop group LG at level k gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$. When no confusion arises we will write $\mathcal{A}_{G_k}$ simply as G_k . These CFT are what is also called Wess-Zumino-Witten CFT with gauge group G and are important building blocks of rational CFT.

Möbius covariant *representation*

Assume $\mathcal{A}$ is a Möbius covariant net. A Möbius covariant *representation* π of $\mathcal{A}$ is a family of representations π_I of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation U_π of the covering group **G** of $PSL(2, \mathbf{R})$, with *positive energy*, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$I \supset \bar{I} \Rightarrow \pi_{\bar{I}} \mid_{\mathcal{A}(I)} = \pi_I \quad \text{(isotony)}$$
$$\mathrm{ad}\, U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \mathrm{ad}\, U(g) \text{(covariance)} .$$

A unitary equivalence class of Möbius covariant representations of $\mathcal{A}$ is called *superselection sector*.

Connes's fusion

The composition of two superselection sectors are known as Connes's fusion . The composition is manifestly unitary and associative, and this is one of the most important virtues of the above formulation. The main question is to study all superselection sectors of $\mathcal{A}$ and their compositions. Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let G be a group. Let $V : G \to U(\mathcal{H})$ be a faithful unitary representation of G on $\mathcal{H}$. If $V : G \to U(\mathcal{H})$ is not faithful, we can take $G' := G/kerV$ and consider G' instead.

Proper Action

We say that G acts properly on $\mathcal{A}$ if the following conditions are satisfied:
(1) For each fixed interval I and each $s \in G$,
$\alpha_s(a) := V(s)aV(s^*) \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;
(2) For each $s \in G$, $V(s)\Omega = \Omega, \forall s \in G$. We will denote by $\mathrm{Aut}(\mathcal{A})$ all automorphisms of $\mathcal{A}$ which are implemented by proper actions.
Define $\mathcal{A}^G(I) := \mathcal{B}(I)P_0$ on $\mathcal{H}_0$, where $\mathcal{H}_0$ is the space of G invariant vectors and P_0 is the projection onto $\mathcal{H}_0$. The unitary representation U of **G** on $\mathcal{H}$ restricts to a unitary representation (still denoted by U) of **G** on $\mathcal{H}_0$. Then :

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Proposition

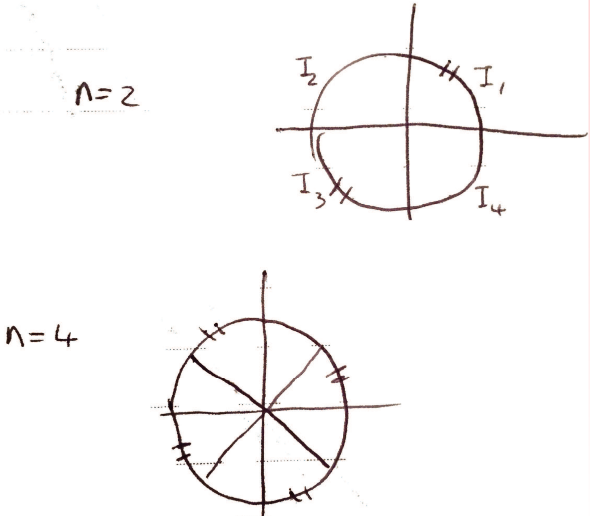
The map $I \in \mathcal{I} \rightarrow \mathcal{A}^G(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by U) of **G** on $\mathcal{H}_0$ is an irreducible conformal net. We say that $\mathcal{A}^G$ is obtained by *orbifold* construction from $\mathcal{A}$.

Complete rationality

By an interval of the circle we mean an open connected proper subset of the circle. If I is such an interval then I' will denote the interior of the complement of I in the circle. We will denote by $\mathcal{I}$ the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that I_1, I_2 are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of I in S^1.. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible conformal net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in S^1 where I_3, I_4 are disjoint intervals. Let

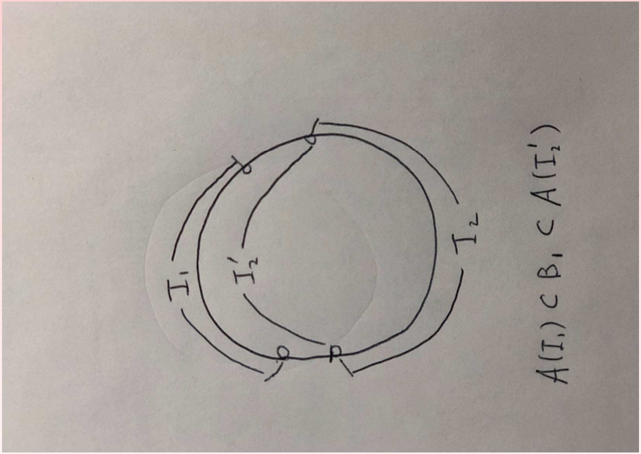
$$\mathcal{A}(E) := \mathcal{A}(I_1) \vee \mathcal{A}(I_3), \hat{\mathcal{A}}(E) := (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is *strongly additive* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from I. We note that a conformal net is automatically split


n=2
I2
I1
I3
I4
n=4

A REMARK ABOUT SPLIT PROPERTY

Recall that a net $\mathcal{A}$ is *split* if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. Any conformal net $\mathcal{A}$ is split. Since $\mathcal{A}(I_1)$ are type *III* factors, there exists a unitary operator $U_1 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ such that:
$U_1 A B U_1^* = A \otimes B \quad \text{for every} \quad A \in \mathcal{A}(I_1), \ B \in \mathcal{A}(I_2)$. Therefore $B_1 = U_1(B(\mathcal{H}) \otimes 1)U_1^*$ is a type *I* factor such that $\mathcal{A}(I_1) \subset B_1 \subset \mathcal{A}(I_2)' = \mathcal{A}(I_2')$.


I_1
I_2'
I_2
$A(I_1) \subset B_1 \subset A(I_2')$

A remark about split property

By choosing increasing sequence of $I_n \subset I_2'$ such that $\cup_n I_n = I_2'$, we can get an increasing sequence type I factors B_n such that $\cup_n B_n$ is strongly dense in $\mathcal{A}(I_2')$. This allows us to approximate relative entropy by restricting to these finite dimensional approximations. This is one reason that many formal calculations in physics literature, which only applies to type I factors, under certain conditions turn out to be true also for local algebras in QFT which is always III. These conditions are such that the formal computations in type I case (also in the case of suitable cut off such as in the finite lattice approximation) converge to Araki's relative entropy, and in general are quite nontrivial to prove (free of ultra violet divergences).

Definition

$\mathcal{A}$ is said to be completely rational, or μ-rational, if the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$is denoted by $\mu_{\mathcal{A}}$ and is called the μ-index of $\mathcal{A}$. $\mathcal{A}$ is holomorphic if $\mu_{\mathcal{A}} = 1$. $\log \mu_{\mathcal{A}}$ is also known as *Topological Entanglement Entropy* by Kitaev and Preskill.

Definition

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Theorem

Let $\mathcal{A}$ be an irreducible conformal net and let G be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational. Then:
(1): $\mathcal{A}^G$ is completely rational and $\mu_{\mathcal{A}^G} = |G|^2 \mu_{\mathcal{A}}$;
(2): There are only a finite number of irreducible covariant representations of $\mathcal{A}^G$ and they give rise to a unitary modular category.

An applications to twisted representations

First by KLM $\mu_{\mathcal{A}} = \sum_i d_i^2$ while the sum is over all irreducible reps i of $\mathcal{A}$, and d_i^2 is the Jones index or square of quantum dimension. The formula is similar to $|G| = \sum_i (\dim i)^2$ which is classical Frobenius formula. From the theorem about orbifold we get that $\mu_{\mathcal{A}^G} = \mu_{\mathcal{A}}|G|^2 = \sum_i d_i^2$, where the sum is now over irreducible reps of $\mathcal{A}^G$, but if we restrict the sum to be over the set of *non-twisted* representations of G, we get that such sum is bounded by $\mu_{\mathcal{A}}|G|$, and since $\mu_{\mathcal{A}}|G|^2 > \mu_{\mathcal{A}}|G|$ if G is nontrivial, we have proved that twisted representation always exists.

Let $\mathcal{A}$ be a graded Möbius net. By a *Möbius subnet* we shall mean a map

$$I \in \mathcal{I} \to \mathcal{B}(I) \subset \mathcal{A}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ of $\mathcal{A}(I)$, which is isotonic

$$\mathcal{B}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and Möbius covariant with respect to the representation U, namely

$$U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$$

for all $g \in$ Möb and $I \in \mathcal{I}$, and we also require that $\mathrm{Ad}\Gamma$ preserves $\mathcal{B}$ as a set.

The case when $\mathcal{B} \subset \mathcal{A}$ has finite index will be most interesting. For an example we have

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Lemma 5

If $\mathcal{B} \subset \mathcal{A}$ is a Möbius subnet such that $\mu_{\mathcal{A}}$ is finite and $[\mathcal{A} : \mathcal{B}] < \infty$. Then $\mu_{\mathcal{B}} = \mu_{\mathcal{A}}[\mathcal{A} : \mathcal{B}]^2$.

OUTLINE

Mutual information in the case of free fermions

Let H denote the Hilbert space $L^2(S^1; \mathbb{C}^r)$ of square-summable $\mathbb{C}^r$-valued functions on the circle. The group LU_r of smooth maps $S^1 \to U_r$, with U_r the unitary group on $\mathbb{C}^r$, acts on H multiplication operators.
Let us decompose $H = H_+ \oplus H_-$, where

$$H_+ = \{\text{functions whose negative Fourier coeffients vanish}\}.$$

We denote by p the Hardy projection from H onto H_+.
Denote by $U_{\mathrm{res}}(H)$ the group consisting of unitary operator A on H such that the commutator $[p, A]$ is a Hilbert-Schmidt operator. Denote by $\mathrm{Diff}^+(S^1)$ the group of orientation preserving diffeomorphism of the circle. It follows that LU_r and $\mathrm{Diff}^+(S^1)$ are subgroups of $U_{\mathrm{res}}(H)$. The basic representation of LU_r is the representation on Fermionic Fock space $F_p = \Lambda(pH) \otimes \Lambda((1-p)H)^*$.

Such a representation gives rise to a graded net as follows. Denote by $\mathcal{A}_r(I)$ the von Neumann algebra generated by $c(\xi)'s$, with $\xi \in L^2(I, \mathbb{C}^r)$. Here $c(\xi) = a(\xi) + a(\xi)^*$ and $a(\xi)$ is the creation operator. Let $Z : F_p \to F_p$ be the Klein transformation given by multiplication by 1 on even forms and by i on odd forms. $\mathcal{A}_r$ is a graded Möbius covariant net, and $\mathcal{A}_r$ will be called the *net of r free fermions*. $\mathcal{A}_r$ is strongly additive and $\mu_{\mathcal{A}_r} = 1$.

Fix $I_i \in \mathcal{PI}, i = 1, 2$, and I_1, I_2 disjoint, that is $\bar{I_1} \cap \bar{I_2} = \emptyset$, and $I = I_1 \cup I_2$. The mutual information we will compute is $S(\omega, \omega_1 \otimes_2 \omega_2)$. Here $\omega_1 \otimes_2$ denotes the restriction of the vacuum state to $\mathcal{A}_r(I_1) \otimes_2 \mathcal{A}_r(I_2)$ which is a graded tensor product. ω on $\mathcal{A}_r(I)$ is quasi-free state as studied by Araki. To describe this state, it is convenient to use Cayley transform $V(x) = (x - i)/(x + i)$, which carries the (one point compactification of the) real line onto the circle and the upper half plane onto the unit disk. It induces a unitary map

$$Uf(x) = \pi^{-\frac{1}{2}}(x + i)^{-1} f(V(x))$$

of $L^2(S^1, \mathbb{C}^r)$ onto $L^2(\mathbb{R}, \mathbb{C}^r)$. The operator U carries the Hardy space on the circle onto the Hardy space on the real line . We will use the Cayley transform to identify intervals on the circle with one point removed to intervals on the real line.

Under the unitary transformation above, the Hardy projection on $L^2(S^1, \mathbb{C}^r)$ is transformed to the Hardy projection on $L^2(\mathbb{R}, \mathbb{C}^r)$ given by :

$$Pf(x) = \frac{1}{2}f(x) + \int \frac{i}{2\pi}\frac{1}{(x-y)}f(y)dy \ ,$$

where the singular integral is (proportional to) the Hilbert transform. We write the kernel of the above integral transformation as C:

$$C(x,y) = \frac{1}{2}\delta(x-y) - \frac{i}{2\pi}\frac{1}{(x-y)} \ . \tag{6}$$

The quasi free state ω is determined by

$$\omega\big(a(f)^* a(g)\big) = \langle g, Pf \rangle.$$

Slightly abusing our notations, we will identify P with its kernel C and simply write

$$\omega\big(a(f)^* a(g)\big) = \langle g, Cf \rangle.$$

Computation of mutual information in finite dimensional case

Choose finite dimensional subspaces H_i of $L^2(I_i, \mathbb{C}_r), i = 1, 2$, and denote by $\mathrm{CAR}(H_i) \subset \mathcal{A}(I_i)$ the corresponding finite dimensional factors of dimensions $2^{2\dim H_i}$ generated by $a(f), f \in H_i$. Let ρ_{12}, ρ_1, ρ_2 be the density matrices of the restriction of ω to $\mathrm{CAR}(H_1) \otimes_2 \mathrm{CAR}(H_2)$, $\mathrm{CAR}(H_1)$, $\mathrm{CAR}(H_2)$ respectively, and $\rho_1 \otimes_2 \rho_2$ of the restriction of $\omega_1 \otimes_2 \omega_2$ to $\mathrm{CAR}(H_1) \otimes_2 \mathrm{CAR}(H_2)$. When working carefully with graded tensor product, we have the analog of (3) in this graded local context:

$$S(\rho_{12}, \rho_1 \otimes_2 \rho_2) = S(\rho_1) + S(\rho_2) - S(\rho_{12}) \ .$$

This is the formula for mutual information in type I factor case.

Now we turn to the computation of von Neumann entropy $S(\rho_1)$. Let p_1 be the projection onto the finite dimensional subspace H_1 of $L^2(I_1, \mathbb{C}_r)$. ρ_1 on $\mathrm{CAR}(H_1)$ is quasi free state given by covariance operator $C_{p_1} = p_1 C p_1$. According to Araki

$$S(\rho_1) = \mathrm{Tr}\big((1 - C_{p_1}) \log(1 - C_{p_1}) + C_{p_1} \log C_{p_1}\big)$$

Let $\mathbf{P}_i$ be projections from $L^2(I, \mathbb{C}^r)$ onto $L^2(I_i, \mathbb{C}^r)$, and $C_i = \mathbf{P}_i C \mathbf{P}_i, i = 1, 2$.
Let

$$\sigma_C = \mathbf{P_1}\big(C \log C + (1-C) \log(1-C)\big)\mathbf{P_1} - \big(C_1 \log C_1 + (\mathbf{P_1} - C_1) \log(\mathbf{P_1} - C_1)\big)$$
$$\mathbf{P_2}\big(C \log C + (1-C) \log(1-C)\big)\mathbf{P_2} - \big(C_2 \log C_2 + (\mathbf{P_2} - C_2) \log(\mathbf{P_2} - C_2)\big)$$

and σ_{C_p} be the same as in the definition of σ_C with C replaced by $C_p = pCp$, if p is a projection commuting with $\mathbf{P}_1$.

Denote by p the projection from $L^2(I, \mathbb{C}^r)$ onto $H_1 \oplus H_2$. We have proved the following

$$S(\rho_{12}, \rho_1 \otimes_2 \rho_2) = \mathrm{Tr}(\sigma_{C_p}) \ .$$

It is clear that σ_{C_p} converges strongly to σ_C as P converges to identity. To compute our mutual information, we like to show that this convergence is actually in trace. Unfortunately this is much harder. Instead we explore additional subtle properties of such operators.

Inequality from operator convexity

Inequality from operator convexity

Theorem 6

(1) For all operator convex functions f on $\mathbb{R}$, and all orthogonal projections p, we have $pf(pAp)p \leq pf(A)p$ for every selfadjoint operator A; (2) $f(t) = t\log(t)$ is operator convex.

(1) of the above Theorem is known as Sherman-Davis Inequality. It in instructive to review the idea of the proof of (1) which is also used later: Consider the selfadjoint unitary operator $U^p = 2p - I$; by operator convexity we have

$$f\big(\frac{1}{2}A + \frac{1}{2}U^pAU^p\big) \leq \frac{1}{2}f(A) + \frac{1}{2}f(U^pAU^p) \ .$$

Now notice that

$$\frac{1}{2}A + \frac{1}{2}U^pAU^p = A_p + A_{1-p}, \quad f(U^pAU^p) = U^pf(A)U^p \ ,$$

where $A_p = pAp$, and the inequality follows.

$S(\omega, \omega_1 \otimes_2 \omega_2) = \lim_{p \to 1} \mathrm{Tr}(\sigma_{C_p}) \geq \mathrm{Tr}(\sigma_C)$ where $p \to 1$ strongly. The first identity follows from Martingale property of relative entropy. To prove the inequality, we use the fact that $x \log x$ is operator convex, and so $\mathbf{P_1} C \log C \mathbf{P_1} \geq C_1 \log C_1$, and similarly with C replaced by $1 - C$. It follows that $\sigma \geq 0, \sigma_p \geq 0$. Since σ_p goes to σ strongly as $p \to 1$ strongly, the inequality follows.

We shall prove later that the inequality in the above Lemma is actually an equality. It would follow if one can show that σ_{C_p} goes to σ_C in tracial norm. This is not so easy, and we note that
$\mathbf{P_1}\big(C\log C + (1-C)\log(1-C)\big)\mathbf{P_1}$ is not trace class. To overcome this difficulty and to compute the mutual information we prove the reverse inequality by applying Lieb's joint convexity and do explicit computation by using solutions to a Riemann-Hilbert problem as in the next two sections.

Reversed inequality from Lieb's joint convexity

We begin with the following Lieb's Concavity Theorem:

Reversed inequality from Lieb's joint convexity

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Theorem 7

(1) For all $m \times n$ matrices K, and all $0 \leq t \leq 1$, the real valued map given by $(A, B) \to \mathrm{Tr}(K^* A^{1-t} K B)$ is concave where A, B are non-negative $m \times m$ and $n \times n$ matrices respectively;
(2) If $A \geq 0, B \geq 0$ and K is trace class, then

$$(A, B) \to \mathrm{Tr}(K^* A^{1-t} K B), \quad 0 \leq t \leq 1,$$

is jointly concave;
(3) If $A \geq \epsilon I, B \geq \epsilon I, \epsilon > 0$ and K is trace class, then

$$(A, B) \to \mathrm{Tr}(K^* A \log AK - K^* AK \log B)$$

is jointly convex;

To prove (3), we note that

$$\mathrm{Tr}(K^* A \log AK - K^* AK \log B) = \lim_{t \to 0} \frac{\mathrm{Tr}(K^* A^{1-t} KB) - \mathrm{Tr}(K^* AK)}{t-1}$$

and (3) follows from (2).

Theorem 8

Let $A \geq 0, B := \mathbf{P_1} A \mathbf{P_1} + \mathbf{P_2} A \mathbf{P_2}$, *where* $\mathbf{P_1}$ *is a projection,* $\mathbf{P_1} + \mathbf{P_2} = 1$, *and* p *is a finite rank projection commuting with* $\mathbf{P_1}$. *Assume that* $A - B$ *is trace class. Then*

$$\mathrm{Tr}\big(A(\log A - \log B)\big) \geq \mathrm{Tr}\big(A_p(\log A_p - \log B_p)\big) \ .$$

The idea of the proof is to apply Lieb's joint convexity to A, B and unitary $U^p = 2P - I$, with $f(A, B, K) = \mathrm{Tr}(K^* A \log AK - K^* AK \log B)$, K is a finite rank projection, and then let K goes to identity strongly. This works well when A is strictly positive, and general case follows by a careful approximation argument.

Resolvent for one free fermion case

Apply the above theorem directly to the covariance operator C we get the reversed inequality and hence equality $S(\omega, \omega_1 \otimes_2 \omega_2) = \lim_{p\to 1} \mathrm{Tr}(\sigma_{C_p}) = \mathrm{Tr}(\sigma_C)$ To compute trace we need explicit formula for the kernel of the resolvent of C. This is related to Riemann-Hilbert problem.

Riemann-Hilbert problem

Recall $I_i \in \mathcal{PI}, i = 1, 2$, and I_1, I_2 are disjoint, that is $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, and $I = I_1 \cup I_2$. We assume that $I = (a_1, b_1) \cup (a_1, b_1) \cup ... \cup (a_n, b_n)$ in increasing order. Close up I to a simple contour, and if a function ϕ is defined on I, extend its definition on the contour by simply defining it to be zero on the complement of I. We will still denote by I the simple contour containing I. Consider the following equation:

$$a\phi(t) + \frac{b}{\pi i} \int_I \frac{\phi(\xi)}{\xi - t} d\xi = f(t)$$

For simplicity it will be assumed that a, b are constants and that $a^2 - b^2 \neq 0$. As a new unknown let us introduce the Cauchy type integral with density ϕ:

$$F(z) = \frac{1}{2\pi i} \int_I \frac{\phi(\xi)}{\xi - z} d\xi$$

We have:

$$F_i(t) - F_o(t) = \phi(t), F_i(t) + F_o(t) = \frac{1}{2\pi i} \int_l \frac{\phi(\xi)}{\xi - t} d\xi$$

Substituting this , we obtain the following equation:

$$(a+b)F_i(t) - (a-b)F_o(t) = f(t)$$

. We arrive in this manner at the Riemann-Hilbert problem: to find a function $F(z)$ from a given linear relation between its limiting values on the inside and on the outside of a contour. The solution is known.

Using the known solution to the Riemann-Hilbert problem, the resolvent of C as restriction of an operator on $L^2(I, \mathbb{C})$

$$R^0(\beta) = (C - 1/2 + \beta)^{-1} \equiv \left(-\frac{i}{2\pi} \frac{1}{x - y} + \beta\, \delta(x - y) \right)^{-1} \quad (7)$$

has the following expression :

$$R^0(\beta) = \left(\beta^2 - 1/4\right)^{-1} \left(\beta\, \delta(x - y) + \frac{i}{2\pi} \frac{e^{-\frac{i}{2\pi} \log\left(\frac{\beta - 1/2}{\beta + 1/2}\right)(Z(x) - Z(y))}}{x - y} \right), \quad (8)$$

where

$$Z(x) = \log\left(-\frac{\prod_{i=1}^{n}(x - a_i)}{\prod_{i=1}^{n}(x - b_i)} \right). \quad (9)$$

We shall denote by $Z_{I,I_1}(x) = Z_I(x) - Z_{I_1}(x)$. Even though both $Z_I(x)$ and $Z_{I_1}(x)$ are singular when x is close to the boundary of its domain, it is crucial that $Z_{I,I_1}(x)$ is a smooth function on the closure of $\bar{I}_1$.

Let

$$G(t,x,y) = \frac{\sin\left(\frac{1}{2\pi}\log\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)(Z_I(x)-Z_I(y))\right) - \sin\left(\frac{1}{2\pi}\log\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)(Z_{I_1}(x)\right.}{x-y}$$

if $x \neq y$ and $G(t,x,x) = \frac{1}{2\pi}\log\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)(Z_I'(x) - Z_{I_1}'(x))$, $t > \frac{1}{2}$.
Then $G(t,x,y)$ is continuous on $(\frac{1}{2},\infty)\times I_1 \times I_1$ and

$$|G(t,x,y)| \leq \left|\frac{1}{2\pi}\log\left(\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right)\right| M, \quad (t,x,y) \in (\frac{1}{2},\infty)\times I_1 \times I_1 \ ,$$

where M is a constant.

The kernel for the computation of mutual information is given by

$$K_1(x, y) = \frac{-1}{\pi} \int_{\frac{1}{2}}^{\infty} \frac{1}{t + 1/2} G(t, x, y) dt \ .$$

Lemma 9

(1) $K_1(x,y)$ is continuous, uniformly bounded on $I_1 \times I_1$;
(2) The kernel of

$$\mathbf{P}_1 C \log C + (1-C)\log(1-C)\mathbf{P}_1 - C_1 \log C_1 - (\mathbf{P}_1 - C_1)\log(\mathbf{P}_1 - C_1)$$

is given by the bounded continuous function $K_1(x,y)$, and moreover its trace is given by

$$\int_{I_1} K_1^0(x,x)dx = \frac{1}{12} \sum_{(a_i,b_i)\in I_2,(a_j,b_j)\in I_1} \log\left(\frac{(a_j-a_i)(b_j-b_i)}{(b_j-a_i)(a_j-b_i)}\right) .$$

The reader is invited to do the computation in (3) of the above Lemma. The number $\frac{1}{12}$ comes from Euler's famous solution to Basel's problem that $\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$.

The reader is invited to do the computation in (2) of the above Lemma. The number $\frac{1}{12}$ comes from Euler's famous solution to Basel's problem that $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.
If $I = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_n, b_n)$ is in increasing order, define

$$G(I) := \frac{1}{6}\left(\sum_{i,j} \log|b_i - a_j| - \sum_{i<j} \log|a_i - a_j| - \sum_{i<j} \log|b_i - b_j|\right) .$$

Combined all the ingredients we have proved the following:

The reader is invited to do the computation in (2) of the above Lemma. The number $\frac{1}{12}$ comes from Euler's famous solution to Basel's problem that $\sum_{n\geq 1}\frac{1}{n^2}=\frac{\pi^2}{6}$.
If $I=(a_1,b_1)\cup(a_2,b_2)\cup\ldots\cup(a_n,b_n)$ is in increasing order, define

$$G(I):=\frac{1}{6}\left(\sum_{i,j}\log|b_i-a_j|-\sum_{i<j}\log|a_i-a_j|-\sum_{i<j}\log|b_i-b_j|\right)\ .$$

Combined all the ingredients we have proved the following:

Theorem 10

Let $I=(a_1,b_1)\cup(a_2,b_2)\cup\ldots\cup(a_n,b_n)\in\mathcal{PI}$ and $I_1\cup I_2=I,\bar{I}_1\cap\bar{I}_2=\emptyset$. Then

$$S_{\mathcal{A}_r}(\omega,\omega_1\otimes_2\omega_2)=r\big(G(I_1)+G(I_2)-G(I_1\cup I_2)\big)\ .$$

OUTLINE

Formal properties of entropy for free fermion nets and their subnets

In the previous section we use Cayley transformation to identify punctured circle with real line as a tool to compute relative entropy. Now we return to general discussion about formal properties of entropy, and it is now convenient to be back to intervals on the circle. Let $I \in \mathcal{PI}$ be disjoint union of intervals on the circle. Explicitly we write $I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ in anti-clockwise order on the unit circle. We note that relative entropies are invariant under Möb transformations on the circle.

By Theorem 10, we have $F_{\mathcal{A}_r}(A, B) := S(\omega, \omega_A \otimes_2 \omega_B) < \infty$ where A, B are union of disjoint intervals . When no confusion arises, we will simply write $F_{\mathcal{A}_r}(A, B)$ as $F(A, B)$.

We can extend the definition mutual information to more general union of disjoint intervals by the following

$F(A \cup B, A \cup C) = F(A, B \cup C) + F(B, C) - F(A, C) - F(A, B).$

Theorem 11

(1) $F(A\cup B, A\cup C)\geq 0$; $F(A\cup B, A\cup C)$ *is continuous from inside;*
(2)$F(A,B)+F(A,C)+F(A\cup B, A\cup C)+F(A\cap C, A\cap B)=$
$F(B,C)+F(A,B\cup C)+F(A,B\cap C)$;
(3) There exists function $G:\mathcal{PI}\rightarrow\mathbb{R}$ *such that*
$F(A,B)=G(A)+G(B)-G(A\cup B)-G(A\cap B)$. *Such* G *is uniquely determined by its value on connected open intervals;*

(4) One can choose $G(a,b) = \frac{r}{6}\log|b-a|$ in (3) for the r free fermion net $\mathcal{A}_r$, and such a choice determines
$G(I) = \frac{r}{6}\left(\sum_{i,j}\log|b_i - a_j| - \sum_{i<j}\log|a_i - a_j| - \sum_{i<j}\log|b_i - b_j|\right)$ for
$I = (a_1, b_1) \cup (a_2, b_2) \cup ... \cup (a_n, b_n)$ on unit circle with anti-clockwise order;
(5) $F(A \cup B, A \cup C) = F(A \cup B, C) - F(A, C) = F(B, A \cup C) - F(B, A)$;
In particular $F(A \cup B, A \cup C)$ increases with B, C;
(6) If $\mathcal{B} \subset \mathcal{A}$ is a graded subnet, then (1), (2), (3) is also true for the system of mutual information associated with $\mathcal{B}$.

(1) and (5) for free fermions can be checked by using explicit formulas in Th. 10, but here we present general arguments which will also works for other cases such as subnets of free fermions, and also explains the origins of such formulas which are formal manipulations of von Neumann entropies.
Choose increasing sequence of finite dimensional factors I_{A_n}, I_{B_n}, invariant under the conjugate action of Γ such that $(\bigcup_n I_{A_n})'' = \mathcal{A}_r(A)$, $(\bigcup_n I_{B_n})'' = \mathcal{A}_r(B)$, and denote by $\rho_{A_n B_n}, \rho_{A_n} \otimes_2 \rho_{B_n}$ the restrictions of ω and $\omega_1 \otimes_2 \omega_2$ to $I_{A_n} \vee I_{B_n}$ respectively. Let ρ_{A_n} and ρ_{B_n} be the restrictions of ω to $I_{A_n}$and I_{B_n} respectively.

Then we have

$$S(\rho_{A_nB_n}, \rho_{A_n} \otimes_2 \rho_{B_n}) = S(\rho_{A_n}) + S(\rho_{B_n}) - S(\rho_{A_nB_n}) \ .$$

To simplify notations, let us write
$S(A_n) := S(\rho_{A_n}), S(A_n \cup B_n) := S(\rho_{A_nB_n})$. Then we have

$$F(A, B) = \lim_{n\to\infty} S(A_n) + S(B_n) - S(A_n \cup B_n) \ .$$

It follows that

$$F(A\cup B, A\cup C) = \lim_{n\to\infty} \left(S(A_n\cup B_n) + S(A_n\cup C_n) - S(A_n) - S(A_n\cup B_n\cup C_n)\right).$$

Note that

$$S(A_n \cup B_n) + S(A_n \cup C_n) - S(A_n) - S(A_n \cup B_n \cup C_n) \geq 0$$

by strong subadditivity of von Neumann entropy, (1) follows and (2) also follows from the limit formula and the fact that $F(A, B)$ is finite by

OUTLINE

Structure of singularities

G from (3) in Th. 11 can be thought as "regularized" version of von Neumann entropy which is always infinite in our case . From (3) of the above Theorem we see that if we only allow G to be defined on $\mathcal{PI}$ then G is highly non unique. Due to the continuity properties of $F(A, B)$, we require that $G(A)$ depends continuously only on the length r_A of interval A. In addition we require that $G(A) = G(A^c)$ for a connected interval, and we set $G(\emptyset) = 0$. Still such G is highly non unique. However, we shall impose further conditions coming from studying the singularities of relative entropy when we allow intervals to approach each other. Let $B_\epsilon = (a_1, a_{2\epsilon}) \cup C = (a_2, b_2) \in \mathcal{PI}$, with $|a_{2\epsilon} - a_2| = \epsilon > 0$. We shall consider the singular limit when ϵ goes to zero while fixing a_1 and C. Let $B_0 = (a_1, a_2)$. We will denote by $B_0 \bar{\cup} C = (a_1, b_2)$, i.e., $B_0 \bar{\cup} C$ is obtained from $B_0 \cup C$ by adding the point a_2: notice in the process the number of components decrease by 1. The nature condition turns out to be:

$$\lim_{\epsilon \to 0} G(B_\epsilon \cup C) - P(\epsilon) = G(B_0 \bar{\cup} C) \tag{10}$$

In general we may take multiple singular limits. Equation (10) allows us to evaluate such limits. Let us consider such an example in details. Let $A = (a_2, b_2)$, $B_{\epsilon_1} = (a_1, a_{2\epsilon_1})$, $C_{\epsilon_2} = (b_{2\epsilon_2}, b_3)$, $|a_{2\epsilon_1} - a_2| = \epsilon_1 > 0$, $|b_{2\epsilon_2} - b_2| = \epsilon_2 > 0$. Let ϵ_1 goes to 0 first, we find

$$F(A\bar{\cup}B_0, A \cup C_{\epsilon_2}) = G(A\bar{\cup}B_0) + G(A \cup C_{\epsilon_2}) - G(A\bar{\cup}B_0 \cup C_{\epsilon_2}) - G(A)$$

since the same function $P(\epsilon_1)$ appears in both $G(A \cup B_{\epsilon_1})$ and $G(A \cup B_{\epsilon_1} \cup C_{\epsilon_2})$ with opposite signs. Then let ϵ_2 goes to 0 we get by the same argument

$$F(A\bar{\cup}B_0, A\bar{\cup}C_0) = G(A\bar{\cup}B_0) + G(A\bar{\cup}C_0) - G(A\bar{\cup}B_0\bar{\cup}C_0) - G(A) .$$

It is easy to see that the result is independent of the order of taking limits, and this way we can extend the definition of $F(A, B)$ to any $F(A, B)$ with $A \in \mathcal{PI}, B \in \mathcal{PI}$.

In the case of free fermions, by Th. 10 we have that $P(\epsilon) = r/6 \log \epsilon + o(\epsilon)$, and we have

$$F(A\bar{\cup}B_0, A\bar{\cup}C_0) = -\frac{r}{6}\log\left|\frac{(b_2-a_2)(b_3-a_1)}{(b_3-a_2)(b_2-a_1)}\right| \ .$$

Theorem 12

Assume that a subnet $\mathcal{B} \subset \mathcal{A}_r$ *has finite index, then:*
(1): $G_{\mathcal{B}}((a,b)) = \frac{r}{6}\log|b-a|$ *and verifies equation (3) of Th. 11, and*

$$F_{\mathcal{B}}(A,B) = \frac{r}{6}|\log \eta_{AB}| \ ,$$

where A, B *are two overlapping intervals with cross ratio* $0 < \eta_{AB} < 1$*;*
(2) Let $B = (a_1, a_{2\epsilon})$*,* $C = (a_2, b_2)$*,* $|a_{2\epsilon} - a_2| = \epsilon > 0$*. Then:*

$$F_{\mathcal{B}}(B,C) = \frac{r}{6}\big(\log|a_2-a_1| + \log|b_2-a_2| - \log|b_2-a_1| - \log(\epsilon)\big) - \frac{1}{2}\log\mu_{\mathcal{B}} + o(\epsilon)$$

as ϵ *goes to* 0*.*

(1) in the above theorem agrees with postulates of Casini and Huerta in their discussion of c theorem using relative entropies.
It is interesting to note that the constant term in (2) of above Th. seems to be related to the topological entropy discussed for example by Kitaev and Preskill. al even with the right factor: in our case we have additional factor $1/2$ since we are discussing chiral half of CFT.

OUTLINE

Failure of duality is related to nontrivial global dimension or topological entanglement entropy

By our theorem for the free fermion net $\mathcal{A}_r$, and two intervals $A = (a_1, b_1)$, $B = (a_2, b_2)$, where $b_1 < a_2$, we have

$$F_{\mathcal{A}}(A, B) = \frac{-r}{6} \log \eta \ ,$$

where $\eta = \frac{(b_1 - a_2)(b_2 - a_1)}{(b_1 - a_1)(b_2 - a_2)}$ is the cross ratio, $0 < \eta < 1$. For simplicity we denote by $F_{\mathcal{A}_r}(\eta) = F_{\mathcal{A}}(A, B)$.
One checks that $F_{\mathcal{A}_r}(A, B) = F_{\mathcal{A}_r}(A^c, B^c)$, which is in fact equivalent to

$$F_{\mathcal{A}_r}(\eta) - F_{\mathcal{A}_r}(1 - \eta) = \frac{-r}{6} \log \left(\frac{\eta}{1 - \eta} \right) \ .$$

Similarly for $\mathcal{B} \subset \mathcal{A}_r$ with finite index, by Th. 12 $F_{\mathcal{B}}(A, B) = F_{\mathcal{B}}(A^c, B^c)$ is equivalent to

$$F_{\mathcal{B}}(\eta) - F_{\mathcal{B}}(1 - \eta) = \frac{-r}{6} \log \left(\frac{\eta}{1 - \eta} \right) \ .$$

We note that $F_{\mathcal{A}_r}(A, B) = F_{\mathcal{A}_r}(A^c, B^c)$ for the free fermion net $\mathcal{A}_r$. However here we show that $F_{\mathcal{B}}(A, B) \neq F_{\mathcal{B}}(A^c, B^c)$ with $\mathcal{B} \subset \mathcal{A}_r$ has finite index $[\mathcal{A}_r : \mathcal{B}] = \lambda^{-1} > 1$. By Lemma 5 $\mu_{\mathcal{B}} = [\mathcal{A}_r : \mathcal{B}]^2$.
We note that, $S(\omega, \omega \cdot E) = F_1(\eta) = F_{\mathcal{A}}(\eta) - F_{\mathcal{B}}(\eta)$ is a decreasing function of η, and $0 \leq F_1(\eta) \leq F_{\mathcal{A}}(\eta)$. So we have

$$\lim_{\eta \to 1} F_1(\eta) = 0 \ .$$

On the other hand, by Th. **??**

$$\lim_{\eta \to 0} F_1(\eta) = \log[\mathcal{A}_r : \mathcal{B}] = \frac{1}{2} \log \mu_{\mathcal{B}} \ .$$

It follows that $F_{\mathcal{B}}(A, B) \neq F_{\mathcal{B}}(A^c, B^c)$ due to the fact that $\mu_{\mathcal{B}} > 1$.

OUTLINE

Formally one has $F(A,B) = S(A) + S(B) - S(A \cap B) - S(A \cup B)$, and for pure states we have $S(A) = S(A^c)$, and it follows that $F(A,B) = F(A^c, B^c)$, but the results of the previous section shows that this is not true (In fact initially we tried very hard to prove it is true). The reason is because that our algebras are not type I, therefore even though the two algebras associated with A and A^c commuting with each other and generate the algebra of all bounded operators on the underlying Hilbert space, these two algebras are in general not each others commutant. Moreover the formula $F(A,B) = S(A) + S(B) - S(A \cap B) - S(A \cup B)$, is only true in the sense that $F(A,B) = \lim_n (S(A_n) + S(B_n) - S(A_n \cap B_n) - S(A_n \cup B_n))$, where A_n is an increasing sequence of type I factors approximating our net localized on A. Even though $S(A_n) = S(A_n')$ for pure states, we only have $A_n^c \subset A_n'$, and we can't conclude that $S(A_n) = S(A_n^c)$, and there is no continuity that can help because both $S(A_n)$ and $S(A_n^c)$ go to infinity as n goes to ∞.

OUTLINE

More on duality

To give the reader an idea the results in our paper "Relative entropy and Global index" let us consider a special case of a conformal net $\mathcal{A}$ that is chain related to free fermion net $\mathcal{A}_r$. Then for $I = I_1 \cup I_2, I' = J_1 \cup J_2$ we have

$$S(\omega, \omega_{J_1} \otimes \omega_{J_2}) - S(\omega, \omega_{I_1} \otimes \omega_{I_2}) - \frac{c}{6} \log \eta = S(\omega, \omega F_I) - \frac{1}{2} \log \mu_{\mathcal{A}}$$

where S is the relative entropy, ω is the vacuum state, c is the central charge, $\mu_{\mathcal{A}}$ is the global index of $\mathcal{A}$, $\eta = \frac{r_{J_1} r_{J_2}}{r_{I_1} r_{I_2}}$ is a cross ratio, and $F_I : \mathcal{A}(J_1 \cup J_2)' \to \mathcal{A}(I_1 \cup I_2)$ is the conditional expectation. Previously relations among relative entropies, central charge and global index are given in asymptotic form . The above relation is an identity. The duality condition as described above holds when the righthand side is 0.

More on duality

Since $\log \mu_{\mathcal{A}} \geq S(\omega, \omega F_I) \geq 0$ is monotonically increasing with respect to interval I, we have that

$$\frac{1}{2}\log \mu_{\mathcal{A}} \geq S(\omega, \omega_{J_1} \otimes \omega_{J_2}) - S(\omega, \omega_{I_1} \otimes \omega_{I_2}) - \frac{c}{6}\log \eta \geq -\frac{1}{2}\log \mu_{\mathcal{A}}$$

and is monotonically increasing in I, a result which is already quite nontrivial.

OUTLINE

A more refined invariant than index?

The proof of such relations are partially based on the following result from our paper "Relative entropy and global index":

A more refined invariant than index?

The proof of such relations are partially based on the following result from our paper "Relative entropy and global index":

Lemma 13

Let $N_3 \subset N_2 \subset N_1$ be factors on a Hilbert space H and ω is a vector state on $B(H)$ given by a vector $\Omega \in H$. Let $E_i, N_i \rightarrow N_{i+1}, i = 1, 2$ be conditional expectation with finite index. Assume that Ω is cyclic and separating for N_2. Then

$$S(\omega, \omega E_2 E_1) = S(\omega, \omega E_2) + S(\omega, \omega E_1)$$

A more refined invariant than index?

The above result suggests that $S(\omega, \omega E)$ behaves like $\log \mathrm{Ind} E$. Note that $S(\omega, \omega E) \leq \log \mathrm{Ind} E$, and in cases such as those from conformal nets where E depends on an interval I, one can show that as I approximate the whole circle,

$$\lim_{I \to S^1} S(\omega, \omega E_I) = \log \mathrm{Ind} E$$

. So $S(\omega, \omega E_I)$ recovers $\log \mathrm{I}nd E$ in a limit and seems to be a more refined invariant than $\log \mathrm{Ind} E$.