Analytic Langlands correspondence for complex curves

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April 5, 2022

Introduction

I will talk about a joint work with Pavel Etingof and David Kazhdan:

(1) *An analytic version of the Langlands correspondence for complex curves*, **arXiv:1908.09677**, in the Dubrovin Memorial Volume published by AMS;

(2) *Hecke operators and analytic Langlands correspondence for curves over local fields*, **arXiv:2103.01509**;

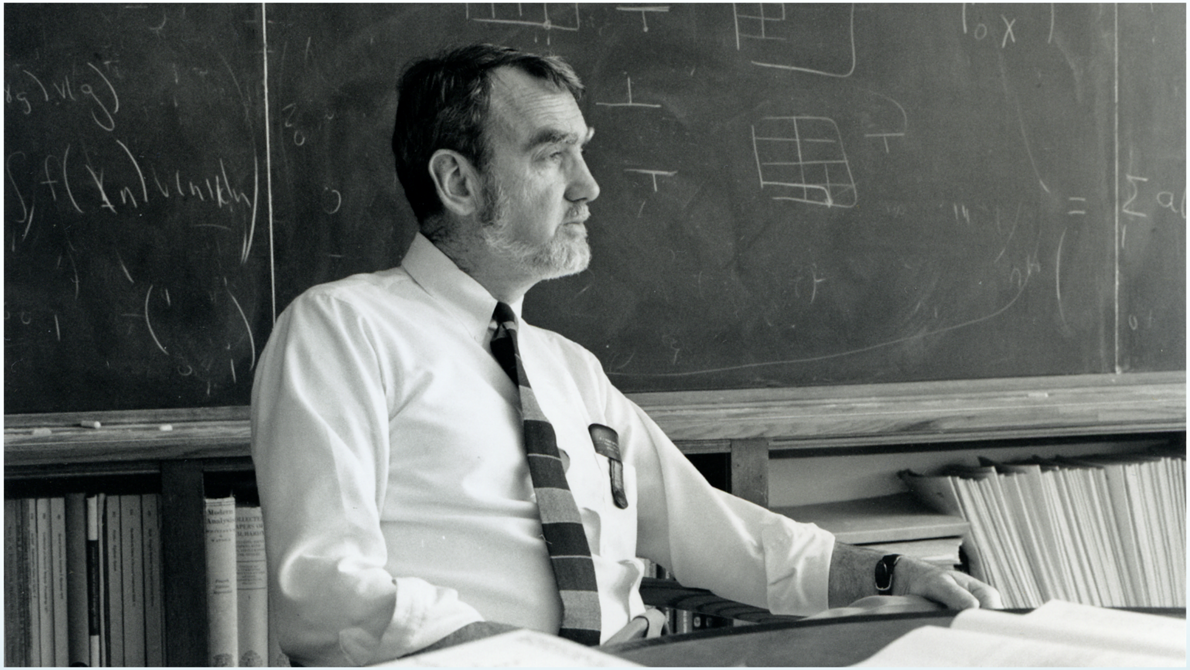
(3) *Analytic Langlands correspondence for PGL_2 on $\mathbb{P}^1$ with parabolic structures over local fields*, **arXiv:2106.05243**.

Motivated by a suggestion of R.P. Langlands and results of J. Teschner, we propose an analytic version of the Langlands correspondence for complex curves.

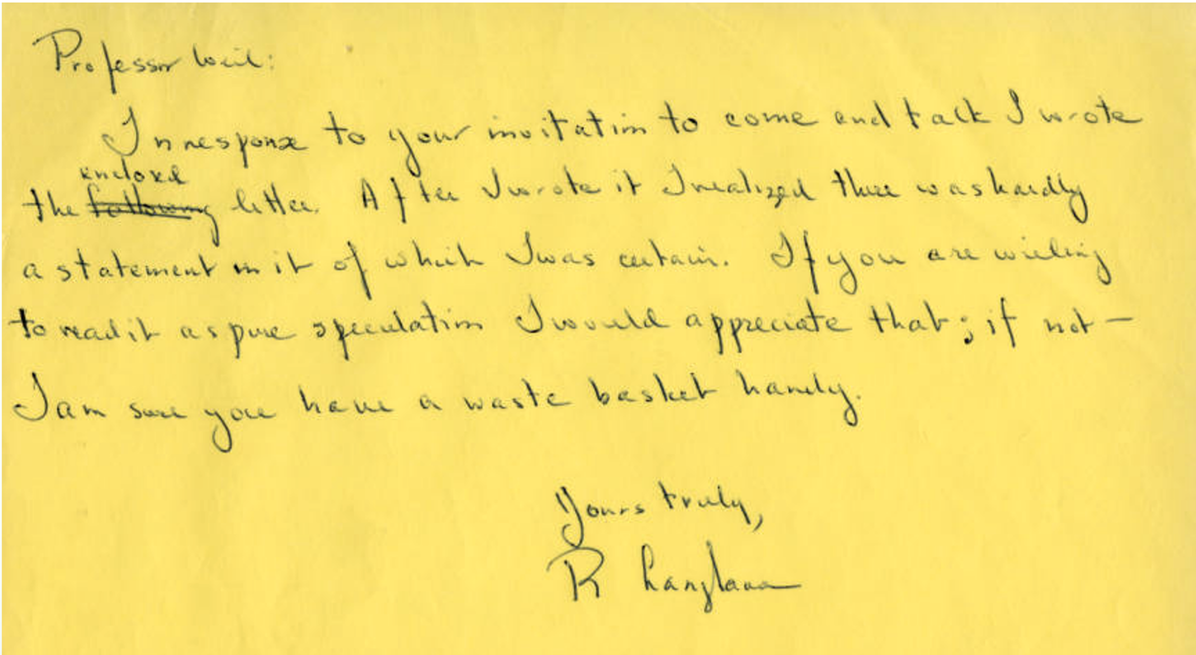
In the late 1960s Robert Langlands launched what has become known as the Langlands Program with the ambitious goal of relating deep questions in Number Theory to Harmonic Analysis.

In particular, Langlands conjectured that Galois representations and motives can be related to automorphic representations.

Moreover, using this Langlands correspondence, intricate number-theoretic data associated to the Galois representations can be obtained from more easily discernable analytic data associated to the automorphic representations.



Robert Langlands at his office at the Institute for Advanced Study, 1999



Professor Weil:

In response to your invitation to come and talk I wrote the ~~following~~ enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not — I am sure you have a waste basket handy.

Yours truly,

R. Langlands

Cover page of Langlands' letter to André Weil, 1967

(from the archive of the Institute for Advanced Study)

A special case of the Langlands correspondence is the Shimura–Taniyama–Weil conjecture.

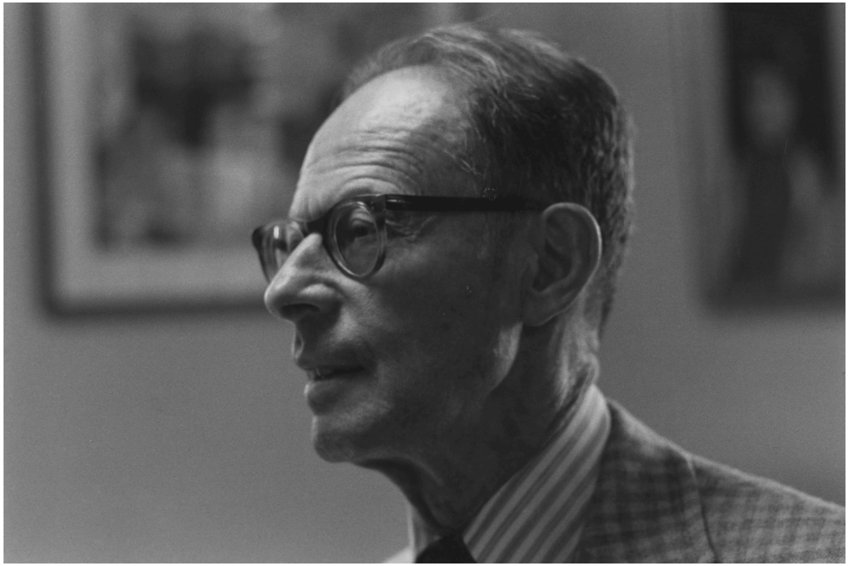
It states that information about 2-dim. Galois representations associated to a given elliptic curve over $\mathbb{Q}$ (which turns out to be equivalent to the knowledge of how many points this curve has over $\mathbb{F}_p$ for almost all primes p) can be found from the coefficients of the Fourier expansion of particular modular form on the upper-half plane. Finding those coefficients is a much easier task!

In 1995, Andrew Wiles and Richard Taylor proved this conjecture in the semistable case.

By a theorem of Ken Ribet, this implies Fermat's Last Theorem.

In fact, the Langlands correspondence can be formulated in 3 different scenarios (in the framework of André Weil's *Rosetta Stone*):

Number Fields *Curves over* $\mathbb{F}_q$ *Curves over* $\mathbb{C}$



André Weil

Exploration of S-duality in 4D supersymmetric gauge theory (and mirror symmetry in 2D QFT) may be viewed as the 4th column.

In 2006, A. Kapustin and E. Witten linked S-duality to the geometric/categorical Langlands correspondence for curves over $\mathbb{C}$. This has inspred a great deal of research in this area.

D. Gaiotto and E. Witten have recently given an S-duality and "brane quantization" interpretation of some of the results in the analytic Langlands correspondence that I am going to talk about today.

Number field $=$ finite extension of the field $\mathbb{Q}$ of rational numbers.

If X is a curve over $\mathbb{F}_q$, then the structure of its field of rational functions (called *function field*) is similar to that of a number field. For instance, if X is the projective line over $\mathbb{F}_q$, then F consists of all fractions $P(t)/Q(t)$, where P and Q are two relatively prime polynomials in one variable with coefficients in $\mathbb{F}_q$.

On the other hand, *function fields* of curves over $\mathbb{F}_q$ and $\mathbb{C}$ have their own similarities.

The formulations of the Langlands correspondence for number fields and for curves over $\mathbb{F}_q$ are *very close*. Let's consider the latter case.

Let F be the function field of a curve X over $\mathbb{F}_q$.

The unramified Langlands correspondence for GL_n, proved by Vladimir Drinfeld (for $n = 2$) and Laurent Lafforgue (for $n > 2$):

$$\boxed{\begin{array}{c}\text{unramified homomorphisms}\\ \mathrm{Gal}(\overline{F}/F) \to GL_n\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Hecke eigenfunctions on}\\ GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O}_F)\end{array}}$$

Here $\overline{F}$ is the algebraic closure of F and $\mathrm{Gal}(\overline{F}/F)$ the Galois group of automorphisms of $\overline{F}$ preserving F.

$\mathbb{A}_F = \prod'_{x \in |X|} F_x$ – the ring of adeles of F, where F_x is the completion of F corresponding to x.
$\mathcal{O}_F = \prod_{x \in |X|} \mathcal{O}_x$ – the subring of integer adeles, where $\mathcal{O}_x \subset F_x$ is the ring of integers.

If we replace GL_n on the RHS by a reductive group G, then GL_n on the LHS should be replaced by ${}^LG$, the *Langlands dual* group of G:

$$\boxed{\begin{array}{c}\text{unramified homomorphisms}\\ \mathrm{Gal}(\overline{F}/F) \to {}^LG\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Hecke eigenfunctions on}\\ G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_F)\end{array}}$$

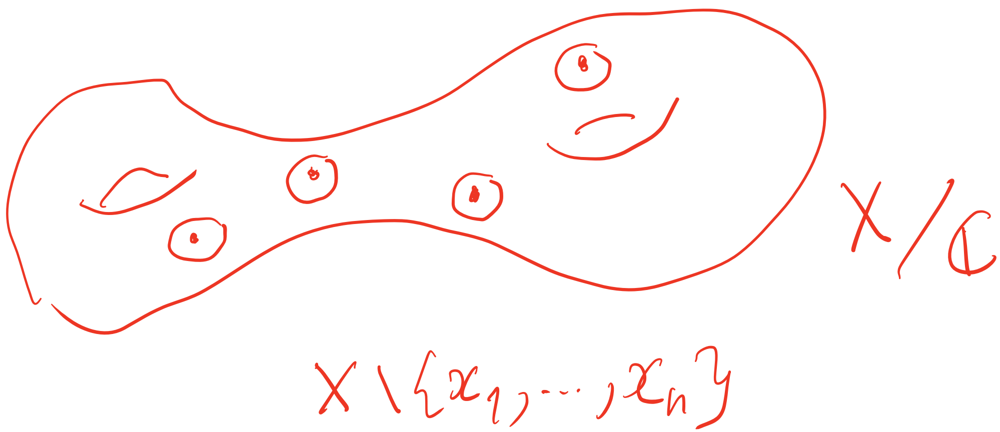
Galois side

${}^LG$	G
GL_n	GL_n
PGL_n	SL_n
SO_{2n+1}	Sp_{2n}
$SO_{2n}/\mathbb{Z}_2$	$Spin_{2n}$
E_8	E_8

automorphic side

Key point (André Weil): $G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}_F) \simeq \mathrm{Bun}_G(\mathbb{F}_q)$

where $\mathrm{Bun}_G(\mathbb{F}_q) := \{$equiv. classes of G-bundles on $X\}$



The same statement is also true if X is a curve over $\mathbb{C}$. However, the formulations of the Langlands correspondence for curves over $\mathbb{F}_q$ and $\mathbb{C}$ are *very different*.

If X is a curve over $\mathbb{F}_q$, then $\mathrm{Bun}_G(\mathbb{F}_q)$ is a *discrete countable set* with a natural measure assiging

$$\mathcal{P} \mapsto 1/|\operatorname{Aut}(\mathcal{P})|$$

to each equivalence class $\mathcal{P}$ of G-bundles.

This is well-defined because the groups $\operatorname{Aut}(\mathcal{P})$ are *finite*.

We can therefore define the Hilbert space $\mathcal{H}_G$ of square integrable $\mathbb{C}$-valued functions on $\mathrm{Bun}_G(\mathbb{F}_q)$with respect to this measure. It is equipped with the Hermitean inner product:

$$\langle f, g \rangle := \sum_{\mathcal{P} \in \mathrm{Bun}_G(\mathbb{F}_q)} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} f(\mathcal{P}) \, \overline{g(\mathcal{P})}.$$

Langlands Correspondence for curves over a finite field

Automorphic side: Joint spectrum of the commuting Hecke operators $H_{x,\lambda}$ acting on $\mathcal{H}_G$ (these are labeled by $x \in |X|$, $\lambda \in {}^L P^+$)

Galois side: unramified homomorphisms from the Galois group $\mathrm{Gal}(\overline{F}/F)$ to the Langlands dual group ${}^L G$ (here $F = \mathbb{F}_q(X)$)

(more precisely, from the Weil group of F; plus more data in general)

In other words, joint eigenvalues of the Hecke operators $H_{x,\lambda}$ on a specific eigenvector in $\mathcal{H}_G$ are encoded by a specific homomorphism $\sigma : \mathrm{Gal}(\overline{F}/F) \to {}^L G$ (namely, Hecke eigenvalues correspond to the images under σ of the Frobenius conjugacy classes associated to $x \in X$), and this sets up a one-to-one correspondence.

Now suppose that X is a curve over $\mathbb{C}$ (a compact Riemann surface).

In this case, we also have $\mathrm{Bun}_G(\mathbb{C})$, the set of isomorphism classes of principal G-bundles on X.

However, it is *not* possible to define integration measure on $\mathrm{Bun}_G(\mathbb{C})$ in the same way as over $\mathbb{F}_q$ because the groups $\mathrm{Aut}(\mathcal{P})$ of automorphisms of G-bundles can now be infinite.

For this reason, the Langlands correspondence for curves over $\mathbb{C}$ has been traditionally formulated in terms of sheaves rather than functions. It is usually referred to as *geometric* or *categorical*.

(V. Drinfeld, G. Laumon, A. Beilinson, ...)

P. Deligne

On the **automorphic side**:

Instead of functions on Bun_G, one considers the derived category of D-modules on Bun_G, and instead of Hecke operators one considers Hecke functors on this category.

Physics counterpart: category of A-branes on the *Hitchin moduli space* $\mathcal{M}_H(G, X)$ (with respect to ω_K) with 't Hooft line operators (Kapustin–Witten).

On the **Langlands dual side**:

Derived category (sutiably modified) of coherent sheaves on the moduli stack $\mathrm{Loc}_{{}^L G}$ of flat ${}^L G$-bundles on X (Arinkin–Gaitsgory).

Physics counterpart: category of B-branes on the *Hitchin moduli space* $\mathcal{M}_H({}^L G, X)$ (with respect to complex structure I) with Wilson line operators (Kapustin–Witten).

The *categorical* Langlands correspondence is a kind of non-abelian Fourier–Mukai transform (Belinson-Drinfeld).

This point of view has been the prevailing wisdom for how Langlands correspondence should be interpreted in the 3rd column of Weil's Rosetta stone ("curves over $\mathbb{C}$") for the past 30 years.

However, in our work with Etingof and Kazhdan we have shown that there is also a rich *analytic* Langlands correspondence for complex curves (i.e. *function-theoretic* instead of *sheaf-theoretic*).

Moreover, the two versions (categorical & analytic) complement each other. We can use each of them to gain new insights about the other.

Analogy: correlation functions in 2D conformal field theory are single-valued bilinear combinations of (multi-valued) conformal and anti-conformal blocks. $(X,(x_i))$ $\quad Z = \sum_{\alpha,\beta} a_{\alpha\beta} \varphi_\alpha \overline{\varphi_\beta}$

With Etingof and Kazhdan, we have associated to Bun_G of a curve $X/\mathbb{C}$ (and more generally X/F, where F is a local field) a natural Hilbert space $\mathcal{H}_G$ and defined analogues of the Hecke operators acting on a dense subspace of $\mathcal{H}_G$. We conjecture that they can be extended to mutually commuting normal compact operators on $\mathcal{H}_G$.

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on Bun_G introduced by Beilinson and Drinfeld, as well as their complex conjugates.

We conjecture that the joint spectrum of this commutative algebra (properly understood) can be identified with the set of ${}^LG$-opers on X whose monodromy is in the *split real form* of ${}^LG$, up to conjugation (these play the role of the Galois representations).

This statement may be viewed as an analytic Langlands correspondence for complex curves.

The spectral problem may be viewed as a quantum integrable system.

Basic definitions:

X – smooth projective irreducible curve over $\mathbb{C}$

$S \subset X(\mathbb{C})$ – finite subset

K_X – canonical line bundle on X

G – connected simple algebraic group over $\mathbb{C}$

${}^L G$ – the Langlands dual group

$\mathrm{Bun}_G = \mathrm{Bun}_G(X, S)$ – algebraic stack of pairs $(\mathcal{F}, r_S)$, where $\mathcal{F}$ is a G-bundle on X and r_S is a B-reduction of $\mathcal{F}|_S$

$\mathrm{Bun}_G^{\mathrm{rs}} = \mathrm{Bun}_G^{\mathrm{rs}}(X, S) \subset \mathrm{Bun}_G(X, S)$ – substack of those stable pairs $(\mathcal{F}, r_S)$ whose group of automorphisms is the center $Z(G)$ of G

Assumption:

$\mathrm{Bun}_G^{\mathrm{rs}}(X,S)$ is *open and dense* in $\mathrm{Bun}_G(X,S)$, i.e. one of the following cases:

1. the genus of X is greater than 1, and S is arbitrary;
2. X is an elliptic curve and $|S| \geq 1$;
3. $X = \mathbb{P}^1$ and $|S| \geq 3$.

The stack $\mathrm{Bun}_G^{\mathrm{rs}}(X,S)$ is a $Z(G)$-gerbe over a smooth algebraic variety $Bun_G^{\mathrm{rs}}(X,S)$ (coarse moduli space).

For our purposes, $Bun_G^{\mathrm{rs}}(X,S)$ is a good replacement for $\mathrm{Bun}_G^{\mathrm{rs}}(X,S)$ because all objects we need descend to $Bun_G^{\mathrm{rs}}(X,S)$.

Hilbert space

K_{Bun} – the canonical line bundle on Bun_G.

Beilinson and Drinfeld have constructed a square root $K_{\mathrm{Bun}}^{1/2}$ of K_{Bun}. FTheir construction sometimes requires a choice of a square root of the canonical line bundle K_X on X (spin structure). If so, we will make such a choice (however, the line bundle $\Omega_{\mathrm{Bun}}^{1/2}$ below does not depend on this choice).

We'll use the same notation for the restriction of this $K_{\mathrm{Bun}}^{1/2}$ to Bun_G^{rs}.

Given a holomorphic line bundle $\mathcal{L}$ on a variety Y, let

$|\mathcal{L}| := \mathcal{L} \otimes \overline{\mathcal{L}}$ be the corresponding C^∞ line bundle.

Set $\Omega_{\mathrm{Bun}}^{1/2} := |K_{\mathrm{Bun}}^{1/2}|$ – the line bundle of half-densities on Bun_G^{rs}.

Hilbert space

Let V_G – space of smooth compactly supported sections of $\Omega^{1/2}_{\mathrm{Bun}}$ over Bun_G^{rs}, and let

$\langle\cdot,\cdot\rangle$ – positive-definite Hermitian form on V_G given by

$$\langle v, w\rangle := \int_{Bun_G^{\mathrm{rs}}} v \cdot \overline{w}, \qquad v, w \in V_G$$

$\mathcal{H}_G$ – the Hilbert space completion of V_G

What kind of operators could act on the Hilbert space $\mathcal{H}_G$?

1. holomorphic differential operators;
2. anti-holomorphic differential operators;
3. Hecke (integral) operators.

Challenges: Differential operators are unbounded. It is a highly non-trivial task to define their self-adjoint (or normal) extensions, which is necessary to be able to make sense of the notion of their joint spectra on $\mathcal{H}_G$ (and there could be different choices).

Hecke operators are also initially defined on a dense subspace of $\mathcal{H}_G$. But we conjecture that they extend by continuity to normal compact operators on the entire $\mathcal{H}_G$. If one proves this, one gets a good spectral problem for both Hecke & differential operators since one can show that they commute.

Holomorphic differential operators

Consider the case of simply-connected G and $|S| = \emptyset$ (so $g > 1$). Let $\mathcal{D}_G$ be the sheaf of algebraic (hence holomorphic) differential operators acting on the line bundle $K_{\mathrm{Bun}}^{1/2}$ on Bun_G.

$D_G := \Gamma(\mathrm{Bun}_G, \mathcal{D}_G)$ – global holomorphic diff. operators on $K_{\mathrm{Bun}}^{1/2}$

Theorem 1 (Beilinson & Drinfeld)

$D_G \simeq \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(X)$, *where* $\mathrm{Op}_{{}^L G}(X)$ *– space of* ${}^L G$*-opers on* X*.*

Definition. An ${}^L G$-oper on a curve X is a holomorphic ${}^L G$-*bundle* with a holomorphic *connection* ∇ and a *reduction to a Borel subgroup* ${}^L B$ which is in a special relative position with ∇.

Example. A PGL_2-oper on X is a *projective connection*, i.e. a second-order holomorphic differential operator of the form
$\partial_z^2 - v(z)\colon\ K_X^{-1/2} \to K_X^{3/2}$.

Beilinson and Drinfeld derived their theorem from a **local result**:

Fix $x \in X$, and let $F_x \simeq \mathbb{C}((t))$ – completion of $F = \mathbb{C}(X)$

$\mathfrak{g}$ – simple Lie algebra, and $\widehat{\mathfrak{g}}_x$ – affine Kac–Moody algebra:

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}}_x \to \mathfrak{g} \otimes \mathbb{C}((t)) \to 0$$

Let $V_k(\mathfrak{g})$ be the corresponding *chiral (or vertex) algebra* at level $k \in \mathbb{C}$. It is generated by the holomorphic Kac–Moody currents

$$J^a(z) := \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}, \qquad J^a_n := J^a \otimes t^n.$$

Elements of $V_k(\mathfrak{g})$ are *normally ordered* differential polynomials in $J^a(z)$.

Let $Z(V_k(\mathfrak{g}))$ be the center of the chiral algebra $V_k(\mathfrak{g})$.

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$.

Theorem 2 (Boris Feigin & E.F.)

(1) If $k \neq -h^\vee$, then $Z(V_k(\mathfrak{g})) = \mathbb{C}$.

(2) $Z(V_{-h^\vee}(\mathfrak{g})) \simeq \operatorname{Fun} \operatorname{Op}_{{}^L G}(D_x)$,

where D_x is the disc around $x \in X$ and $\operatorname{Op}_{{}^L G}(D_x)$ is the space of ${}^L G$-opers on D_x.

$k = -h^\vee$ is called the critical level.

Example. At the critical level, the Sugawara current

$$S(z) = \frac{1}{2} \sum_a : J^a(z) J_a(z) := \sum_{n \in \mathbb{Z}} S_n z^{-n-2}$$

commutes with the Kac–Moody currents $J^a(z)$, and hence belongs to the center $Z(V_{-h^\vee}(\mathfrak{g}))$.

Let $G = SL_2, {}^L G = PGL_2$.

Then the center $Z(V_{-2}(\mathfrak{sl}_2))$ is equal to $\mathbb{C}[\partial_z^m S(z)]_{m \geq 0} \simeq \mathbb{C}[S_n]_{n \leq -2}$

On the other hand, PGL_2-oper on D_x is the same as a *projective connection*, i.e. a second-order holomorphic differential operator of the form $\partial_z^2 - v(z)$: $K_X^{-1/2} \to K_X^{3/2}$, where $v(z) = \sum_{n \leq -2} v_n z^{-n-2}$

So, $\mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x) = \mathbb{C}[v_n]_{n \leq -2}$

The isomorphism $Z(V_{-2}(\mathfrak{sl}_2)) \simeq \mathrm{Fun}\,\mathrm{Op}_{PGL_2}(D_x)$

sends $\qquad S_n \mapsto v_n$

From local to global

$\mathrm{Bun}_G \simeq G(\mathbb{C}[X \backslash x]) \backslash G(F_x)/G(\mathcal{O}_x)$

$\widehat{\mathfrak{g}}_x$ acts (from the right) on sections of a $G(\mathcal{O}_x)$-equivariant line bundle on $G(X \backslash x) \backslash G(F_x)$, which descends to a square root $K^{1/2}_{\mathrm{Bun}}$ of the canonical line bundle on Bun_G. Central element $\mathbf{1} \mapsto -h^\vee$.

Hence $Z(V_{-h^\vee}(\mathfrak{g})) \to D_G$, global hol. diff. operators on $K^{1/2}_{\mathrm{Bun}}$.

Moreover, we have the following commutative diagram:

$$
\begin{array}{ccc}
Z(V_{-h^\vee}(\mathfrak{g})) & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(D_x) \\
\downarrow & & \downarrow \\
D_G & \xrightarrow{\sim} & \mathrm{Fun}\,\mathrm{Op}_{{}^L G}(X)
\end{array}
$$

Hitchin system

Since $D_G \simeq \mathrm{Fun}\,\mathrm{Op}_{^LG}(X)$, and $\mathrm{Op}_{^LG}(X)$ is an affine space, choosing a system of coordiates on $\mathrm{Op}_{^LG}(X)$ we can identify D_G with $\mathbb{C}[P_1, \ldots, P_N]$, where $N = \dim(\mathrm{Op}_{^LG}(X)) = \dim G(g-1)$.

Since the P_i's commute with each other, the following system of equations makes sense for any collection $\{\chi_i \in \mathbb{C}, i = 1, \ldots, N\}$ of eigenvalues:

$$P_i \cdot f = \chi_i f, \qquad f \in \Gamma(Bun_G^{\mathrm{rs}}, K^{1/2}), \quad i = 1, \ldots, N.$$

This system is known as the *quantum Hitchin system*.

The collection $\{\chi_i\}$ naturally corresponds to a ^LG-oper $\chi \in \mathrm{Op}_{^LG}(X)$.

This system has holomorphic solutions on an open dense subset $Bun_G^{\rm vs} \subset Bun_G^{\rm rs}$. However, because $Bun_G^{\rm vs}$ is non-compact, these solutions (which locally span a vector bundle of a rank that grows exponentially with the genus of X) are *multi-valued* with complicated monodromy around the compactification divisor.

Because of that, there is no natural way in general to attach to a given χ a specific holomorphic half-form. (Even if there were single-valued solutions, it wouldn't be clear which one to choose.)

Instead, Belinson and Drinfeld attach to χ the following $\mathcal{D}_G$-module on Bun_G:

$$\Delta_\chi := \mathcal{D}_G/(\mathcal{D}_G \cdot (P_i - \chi_i)_{i=1,\ldots,N})$$

Beilinson and Drinfeld have proved that Δ_χ is a *Hecke eigensheaf* corresponding to the ${}^L G$-oper χ under the geometric/categorical Langlands correspondence.

In order to get *single-valued* solutions, we need to couple holomorphic and anti-holomorphic Hitchin systems.

Anti-holomorphic differential operators

Complex conjugates of elements of D_G are global anti-holomorphic differential operators acting on $\overline{K}_{\mathrm{Bun}}^{1/2}$.

They generate a commutative algebra $\overline{D}_G$.

$$\overline{D}_G \simeq \mathrm{Fun}\, \overline{\mathrm{Op}}_{^LG}(X)$$

$\mathcal{A}_G := D_G \otimes \overline{D}_G$ is a commutative algebra acting on C^∞ sections of the line bundle $\Omega_{\mathrm{Bun}}^{1/2} = K_{\mathrm{Bun}}^{1/2} \otimes \overline{K}_{\mathrm{Bun}}^{1/2}$ on Bun_G^{rs}.

Let $\widetilde{V}_G$ be the space of smooth sections of $\Omega_{\mathrm{Bun}}^{1/2}$ on $Bun_G^{\mathrm{vs}} \subset Bun_G^{\mathrm{rs}}$, the moduli space of *very stable G-bundles* (i.e. those $\mathcal{F}$ which do not admit non-zero $\phi \in \Gamma(X, \mathfrak{g}_{\mathcal{F}} \otimes K_X)$ taking nilpotent values everywhere).

“Doubling” of the quantum Hitchin system

Given a homomorphism $\Lambda : \mathcal{A}_G \to \mathbb{C}$, denote by $\widetilde{V}_{G,\Lambda}$ the corresponding eigenspace of $\mathcal{A}_G$ in $\widetilde{V}_G$.

$\Lambda = (\chi, \mu)$, where $\chi \in \mathrm{Op}_{^LG}(X), \mu \in \overline{\mathrm{Op}}_{^LG}(X)$.

If f is a non-zero element of $\widetilde{V}_{G,(\chi,\mu)}$, then it satisfies two systems of differential equations:

(1) $P \cdot f = \chi(P) f, \qquad P \in D_G$

(2) $Q \cdot f = \mu(Q) f, \qquad Q \in \overline{D}_G$

This is a “double” of the Hitchin system, with both holomorphic and anti-holomorphic degrees of freedom.

Now we look for *smooth solutions* of the combined system (1) and (2). It is possible that for some χ and μ there will be a single-valued solution, which can be written locally in bilinear form

$f = \sum_{i,j} a_{ij} \; \phi_i(\mathbf{z})\overline{\psi}_j(\overline{\mathbf{z}})$

$\{\phi_i\}$ – local solutions of (1)

$\{\overline{\psi}_j\}$ – local solutions of (2)

This actually implies that $\dim \widetilde{V}_{G,(\chi,\mu)} < \infty$.

Moreover, if Δ_χ is irreducible and has regular singularities (which is known for $G = SL_n$) and $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\dim \widetilde{V}_{G,(\chi,\mu)} = 1$.

Conjecture 3

1. All $\widetilde{V}_{G,(\chi,\mu)} \subset \mathcal{H}_G$
2. There is an orthogonal decomposition

   $$\mathcal{H}_G = \widehat{\bigoplus}_{(\chi,\mu)} \widetilde{V}_{G,(\chi,\mu)}$$

3. If $\widetilde{V}_{G,(\chi,\mu)} \neq 0$, then $\mu = \tau(\overline{\chi})$, where τ is the Chevalley involution on ${}^LG$ and $\chi \in \mathrm{Op}_{{}^LG}(X)_{\mathbb{R}}$.

Definition. $\mathrm{Op}_{{}^LG}(X)_{\mathbb{R}}$ is the set of ${}^LG$-opers on X such that the *monodromy representation* $\rho_\chi : \pi_1(X, p_0) \to {}^LG(\mathbb{C})$ is isomorphic to its complex conjugate, i.e. $\rho_\chi \simeq \overline{\rho}_\chi$.

We expect that $\mathrm{Op}_{{}^LG}(X)_{\mathbb{R}}$ is a *discrete subset* of $\mathrm{Op}_{{}^LG}(X)$. This is known for $G = SL_2$ (G. Faltings).

For $G = SL_2$, Conjecture 3 implements ideas of J. Teschner.

We expect that $\mathrm{Op}_{^L G}(X)_{\mathbb{R}}$ coincides with the set of all $^L G$-opers on X with *real monodromy*, i.e. such that the image in $^L G(\mathbb{C})$ of the monodromy representation

$$\rho_\chi : \pi_1(X, p_0) \to {}^L G$$

associated to χ is contained, up to conjugation, in the split real form $^L G(\mathbb{R})$ of $^L G(\mathbb{C})$.

This is known for $G = SL_2$ and we can prove it for general G in the case when there is at least one point with Borel reduction (i.e. $|S| \neq \emptyset$).

Quantum integrable system

In some cases, the global differential operators (and the Hecke operators) can be written down explicitly, and then one obtains interesting quantum integrable systems. Our results and conjectures give a description of the spectra of the quantum Hamiltonians in these models.

Specifically, consider the case of $X = \mathbb{P}^1$ and

$$S = \{z_1, \ldots, z_N, \infty\}$$

Then the corresponding quantum integrable system is a **double** of the *Gaudin model* combining both holomorphic and anti-holomorphic degrees of freedom.

Let $G = SL_2$. Then the moduli space $Bun^{\mathrm{rs}}_{SL_2}$ is an open dense subspace of

$$(\mathbb{P}^1)^{N+1}/SL_2^{\mathrm{diag}} = (\mathbb{P}^1)^N/B^{\mathrm{diag}}$$

We have the Gaudin operators

$$H_i = \sum_{j \neq i} \frac{e^{(i)} \otimes f^{(j)} + f^{(i)} \otimes e^{(j)} + \frac{1}{2} h^{(i)} \otimes h^{(j)}}{z_i - z_j}, \quad i = 1, \ldots, N$$

which commute with the diagonal action of SL_2. They give rise holomorphic differential operators on $Bun^{\mathrm{rs}}_{SL_2}$.

In the past, looked at their action on the space of global sections of the line bundle $\boxtimes_{i=1}^N \mathcal{L}_{\lambda_i} \boxtimes \mathcal{L}_{\lambda_\infty}$, which is $\otimes_{i=1}^N V_{\lambda_i} \otimes V_{\lambda_\infty}$.

The joint eigenvalues of the H_i correspond to PGL_2-opers with regular singularities at $z_1, \ldots, z_N, \infty$ and *trivial monodromy*.

Now we look instead at the Hilbert space $\mathcal{H}$, which is the space of L^2 sections of the line bundle $\boxtimes_{i=1}^N |\mathcal{L}_{-1}| \boxtimes |\mathcal{L}_{-1}|$ of half-densities on $(\mathbb{P}^1)^{N+1}/SL_2^{\mathrm{diag}}$.

It carries an action of the Gaudin Hamiltonians $H_i, i = 1, \ldots, N$ and their anti-holomorphic analogues $\overline{H}_i, i = 1, \ldots, N$.

The double of the Gaudin system (which is what the Hitchin system is called in this case) is

$$H_i \cdot \Psi = \chi_i \Psi, \qquad \overline{H}_i \cdot \Psi = \mu_i \Psi.$$

Theorem 4 (EFK3)

This system has a single-valued solution if and only if (1) $\mu_i = \overline{\chi_i}$ *and (2) the second order Fuchsian differential operator on* $\mathbb{P}^1$

$$\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{N} \frac{1}{4(z-z_i)^2} - \sum_{i=1}^{N} \frac{\chi_i}{z-z_i}$$

has real monodromy representation $\pi_1(\mathbb{P}^1 \backslash S) \to PGL_2(\mathbb{R})$.

The converse statement is also true.

This result was previously conjectured by J. Teschner.

We have conjectured (and proved for small N) that these single-valued solutions are always *square-integrable* (i.e. belong to the Hilbert space), so they are the eigenfunctions of a suitable self-adjoint extension of the algebra $\mathcal{A}_{\mathbb{R}} = \mathbb{C}[H_i + \overline{H}_i, (H_i - \overline{H}_i)/\imath]_{i=1,\dots,N}$.

Hecke operators

Proving Conjecture 3 directly is a daunting task. This is where the third set of operators on $\mathcal{H}_G$ – integral Hecke operators – comes in handy.

Though they are also initially defined on a dense subspace of $\mathcal{H}_G$ (like diff. operators), Etingof, Kazhdan, and I conjecture that, unlike the differential operators, they extend to (mutually commuting) continuous operators on the entire $\mathcal{H}_G$, which are moreover *normal* and *compact* with trivial common kernel.

If so, then by a general result of functional analysis, $\mathcal{H}_G$ decomposes into a (completed) direct sum of mutually orthogonal *finite-dimensional eigenspaces* of the Hecke operators. Moreover, we can show that they commute with the differential operators, and so the Compactness Conjecture can be used to prove Conjecture 3.

In fact, Etingof, Kazhdan, and I have defined the Hecke operators for curves over any local field.

For non-archimedian local fields, these operators were essentially defined earlier by A. Braverman and D. Kazhdan in *Some examples of Hecke algebras for two-dimensional local fields*, Nagoya Math. J. Volume 184 (2006), 57-84.

For $G = PGL_2, X = \mathbb{P}^1$, Hecke operators were studied by M. Kontsevich in his paper *Notes on motives in finite characteristic* (2007). In his letters to us (2019) he conjectured compactness of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over $\mathbb{C}$ exist and could be used to construct an analogue of the Langlands correspondence is due to R.P. Langlands himself. This was one of the motivations for our joint with P. Etingof and D. Kazhdan.

For a dominant coweight λ of G, denote by

$q : Z(\lambda) \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X$

the *Hecke correspondence* attached to λ. Let

$p_{1,2} : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to \mathrm{Bun}_G, \quad p_3 : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X \to X$

be the projections, and set $q_i := p_i \circ q$.

The following is due to Beilinson–Drinfeld and Braverman–Kazhdan.

Theorem 5

There exists an isomorphism

$$a : q_1^*(K_{\mathrm{Bun}}^{1/2}) \simeq q_2^*(K_{\mathrm{Bun}}^{1/2}) \otimes \omega_2 \otimes q_3^*(K_X^{-\langle \lambda, \rho \rangle})$$

where ω_2 is the relative canonical bundle along the fibers of $q_2 \times q_3$ and ρ is the half sum of positive roots.

The isomorphism a gives rise to an isomorphism

$$|a| : q_1^*(\Omega_{\mathrm{Bun}}^{1/2}) \simeq q_2^*(\Omega_{\mathrm{Bun}}^{1/2}) \otimes \Omega_2 \otimes q_3^*(|K_X|^{-\langle\lambda,\rho\rangle})$$

where $\Omega_2 := |\omega_2|$ is the relative line bundle of densities along the fibers of $q_2 \times q_3$. Let

$$U_G(\lambda) := \{\mathcal{F} \in Bun_G^{\mathrm{rs}} | (q_2(q_1^{-1}(\mathcal{F})) \subset Bun_G^{\mathrm{rs}}\}$$

This is an open subset of Bun_G^{rs}, which is dense if

$$\dim \mathrm{Bun}_G = \dim G \cdot (g-1) + \dim G/B \cdot |S| \quad (g > 1)$$

is sufficiently large. (For example, for $G = PGL_2, \lambda = \omega_1$, this is so if $\dim \mathrm{Bun}_G > 1$.)

Assume that $U_G(\lambda) \subset Bun_G^{\mathrm{rs}}$ is dense and let $V_G(\lambda) \subset V_G$ be the subspace of half-densities f such that $\mathrm{supp}(f) \subset U_G(\lambda)$.

$Z_{\mathcal{G},x} := (q_2 \times q_3)^{-1}(\mathcal{G} \times x), \qquad \mathcal{G} \in \mathrm{Bun}_G(\mathbb{C}), \quad x \in X(\mathbb{C})$

It is compact and isomorphic to the closure $\overline{\mathrm{Gr}}_\lambda$ of the $G[[z]]$-orbit Gr_λ in the affine Grassmannian of G.

The results of Braverman–Kazhdan imply that for any $f \in V_G(\lambda)$ and $x \in X(\mathbb{C})$, the restriction of the pull-back $q_1^*(f)$ to $Z_{\mathcal{G},x}$ is a well-defined measure with values in the line $|\Omega_{\mathrm{Bun}}|_{\mathcal{G}}^{1/2} \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$.

Hence for any $f \in V_G(\lambda)$, the integral

$$(\widehat{H}_\lambda(x) \cdot f)(\mathcal{G}) := \int_{Z_{\mathcal{G}}^x(F)} q_1^*(f)$$

is absolutely convergent for all $\mathcal{G} \in Bun_G^{\mathrm{rs}}(\mathbb{C})$ and belongs to the space V_G of compactly supported smooth sections on $Bun_G^{\mathrm{rs}}(\mathbb{C})$.

Therefore this integral defines a *Hecke operator*

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to V_G \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$$

Thus, we obtain an operator

$$\widehat{H}_\lambda(x) : V_G(\lambda) \to \mathcal{H}_G \otimes |K_X|_x^{-\langle\lambda,\rho\rangle}$$

Conjecture 6 (Compactness Conjecture)

1. For any identification $(K_X^{1/2})_x \cong \mathbb{C}$, the corresponding operators $V_G(\lambda) \to \mathcal{H}_G$ extend to a family of commuting compact normal operators on $\mathcal{H}_G$, which we denote by $H_\lambda(x)$.
2. $H_\lambda(x)^\dagger = H_{-w_0(\lambda)}(x)$.
3. $\bigcap_{\lambda,x} \mathrm{Ker} H_\lambda(x) = \{0\}$.

Remark. We expect that integrals defining Hecke operators $H_\lambda(x)$ are absolutely convergent for all $f \in V_G$.

From now on we **assume** that Compactness Conjecture holds.

Let $\mathbb{H}_G$ be the commutative algebra generated by operators $H_\lambda(x), \lambda \in \check{P}^+, x \in X$. Denote by $\mathrm{Spec}(\mathbb{H}_G)$ its spectrum.

Corollary 7

There is an orthogonal decomposition

$$\mathcal{H}_G = \widehat{\bigoplus}_{s \in \mathrm{Spec}(\mathbb{H}_G)} \mathcal{H}_G(s)$$

where $\mathcal{H}_G(s), s \in \mathrm{Spec}(\mathbb{H}_G)$, *are the finite-dimensional joint eigenspaces of* $\mathbb{H}_G$ *in* $\mathcal{H}_G$.

At the moment, we only have a conjectural description of $\mathrm{Spec}(\mathbb{H}_G)$ for $F = \mathbb{C}$ (and, in some cases, for $F = \mathbb{R}$).

So, let's go back to the case $F = \mathbb{C}$. Then we also have the algebra $\mathcal{A}_G = D_G \otimes \overline{D}_G$ of differential operators.

Observe that $\mathcal{A}_G$ acts on the space $V_G^\vee$ of distributions on Bun_G^{rs}, and $\mathcal{H}_G$ is naturally realized as a subspace of $V_G^\vee$. Hence we can apply elements of $\mathcal{A}_G$ to vectors in the eigenspaces $\mathbb{H}_G(s)$ of the Hecke operators, viewed as *distributions*.

Conjecture 8

Every $\mathbb{H}_G(s)$ is an eigenspace of $\mathcal{A}_G$.

Corollary 9

If $(\chi, \mu) \in \mathrm{Spec}\, \mathcal{A}_G$, *then* $\mu = \tau(\overline{\chi})$ *and* $\chi \in \mathrm{Op}^\gamma_{{}^L G}(X)_{\mathbb{R}}$.

Recall that $\mathrm{Op}_{{}^L G}(X)_{\mathbb{R}}$ is the subset of real ${}^L G$-opers in $\mathrm{Op}_{{}^L G}(X)$.

Remark. Recall that first we defined a Hecke operator $\widehat{H}_\lambda(x) : V_G(\lambda) \to V_G$.

The algebra $\mathcal{A}_G$ naturally acts on both $V_G(\lambda)$ and V_G. Hence the commutators $[P, \widehat{H}_\lambda(x)], P \in \mathcal{A}_G$, make sense.

We have $[P, \widehat{H}_\lambda(x)] = 0, \quad \forall P \in \mathcal{A}_G$.

To see this, realize Bun_G as $G(X \backslash x) \backslash G(F_x)/G(\mathcal{O}_x)$.

Then $\widehat{H}_\lambda(x)$ acts from the *right*, whereas $\mathcal{A}_G$ can be obtained from the action of the center of $\widetilde{U}(\widehat{\mathfrak{g}})_{\mathrm{crit}}$ from the *left*.

However, to prove Conjecture 8 we need a stronger form of commutativity, and a crucial element in proving it is the system of differential equations satisfied by $\widehat{H}_\lambda(x)$.

The case of $G = PGL_2$, so ${}^L G = SL_2$

Consider SL_2-opers on X (following Beilinson and Drinfeld):

$$\mathrm{Op}_{SL_2}(X) = \bigsqcup_{\gamma \in \theta(X)} \mathrm{Op}^{\gamma}_{SL_2}(X)$$

where $\theta(X)$ is the set of isomorphism classes of square roots of K_X.

Pick a square root $K_X^{1/2}$ of K_X. An SL_2-oper in the corresponding component $\mathrm{Op}^{\gamma}_{SL_2}(X)$ is a holomorphic connection on the rank 2 vector bundle $\mathcal{V}_{\omega_1}$

$$0 \to K_X^{1/2} \to \mathcal{V}_{\omega_1} \longrightarrow K_X^{-1/2} \to 0$$

satisfying a *transversality condition*.

Here's an alternative description of this component.

A *projective connection* associated to $K_X^{1/2}$ is a second-order differential operator $P : K_X^{-1/2} \to K_X^{3/2}$ such that

1. $\mathrm{symb}(P) = 1 \in \mathcal{O}_X$, and
2. P is algebraically self-adjoint.

They form an affine space $\mathcal{P}roj_\gamma(X)$. Locally, $P = \partial_z^2 - v(z)$.

Lemma 10

There is a bijection $\mathrm{Op}^\gamma_{SL_2}(X) \simeq \mathcal{P}roj_\gamma(X)$

$$\chi \in \mathrm{Op}^\gamma_{SL_2}(X) \quad \mapsto \quad P_\chi \in \mathcal{P}roj_\gamma(X)$$

such that the section $s_{\omega_1} \in \Gamma(X, K_X^{-1/2} \otimes \mathcal{V}_{\omega_1})$ *corresponding to the embedding* $K_X^{1/2} \hookrightarrow \mathcal{V}_{\omega_1}$ *satisfies* $P_\chi \cdot s_{\omega_1} = 0$
(here we use the $\mathcal{D}_X$*-module structure on* $\mathcal{V}_{\omega_1}$ *corresponding to* ∇_χ*).*

Let $\mathcal{V}^{\mathrm{univ}}_{\omega_1}$ be the universal vector bundle over $\mathrm{Op}^{\gamma}_{SL_2}(X) \times X$ with a partial connection ∇^{univ} along X, such that

$$(\mathcal{V}^{\mathrm{univ}}_{\omega_1}, \nabla^{\mathrm{univ}})|_{\chi \times X} = (\mathcal{V}_{\omega_1}, \nabla_{\chi})$$

Let $\mathcal{V}^{\mathrm{univ}}_{\omega_1,X} := \pi_*(\mathcal{V}^{\mathrm{univ}}_{\omega_1})$, where $\pi : \mathrm{Op}^{\gamma}_{SL_2}(X) \times X \to X$. The connection ∇^{univ} makes $\mathcal{V}^{\mathrm{univ}}_{\omega_1,X}$ into a left $\mathcal{D}_X$-module.

The algebra $D_{PGL_2} \simeq \mathrm{Fun}\, \mathrm{Op}^{\gamma}_{SL_2}(X)$ acts on $\mathcal{V}^{\mathrm{univ}}_{\omega_1,X}$ and commutes with the action of $\mathcal{D}_X$.

Lemma 11

There is a unique second-order differential operator

$$\sigma : K_X^{-1/2} \to D_{PGL_n} \otimes K_X^{3/2}$$

satisfying the following property: for any $\chi \in \mathrm{Op}^{\gamma}_{SL_2}(X)$, applying the corresponding homomorphism $D_{PGL_2} \to \mathbb{C}$ we obtain P_{χ}.

Differential equation on Hecke operators

As x varies along X, the Hecke operators $\widehat{H}_{\omega_1}(x)$ combine into a section of the C^∞ line bundle $|K_X|^{-1/2}$ on X with values in operators $\mathcal{H}_{PGL_2} \to \mathcal{H}_{PGL_2}$. We denote it by $\widehat{H}_{\omega_1}$.

Theorem 12

The Hecke operator $\widehat{H}_{\omega_1}$, viewed as an operator-valued section of $|K_X|^{-1/2}$, satisfies the system of differential equations

$$\sigma \cdot \widehat{H}_{\omega_1} = 0, \qquad \overline{\sigma} \cdot \widehat{H}_{\omega_1} = 0$$

This is a system of second-order differential equations (one holomorphic and one anti-holomorphic).

Explicitly, pick a point $\chi_0 \in \mathrm{Op}^{\gamma}_{SL_2}(X)$ and use it to identify $\mathrm{Op}^{\gamma}_{SL_2}(X)$ with $H^0(X, K_X^2)$.

Pick a basis $\{\varphi_i, i = 1, \ldots, 3g-3\}$ of $H^0(X, K_X^2)$

Let $\{F_i, i = 1, \ldots, 3g-3\}$ be the dual set of generators of the polynomial algebra $\mathrm{Fun}\,\mathrm{Op}^{\gamma}_{SL_2}(X) = D_{PGL_2}$ dual to this basis.

Each F_i is a global holomorphic diff. operator on Bun_{PGL_2}.

Locally on X, $P_{\chi_0} = \partial_z^2 - v_0(z)dz^2$. Then

$$\sigma = \partial_z^2 - v_0(z)dz^2 - \sum_{i=1}^{3g-3} F_i \otimes \varphi_i \ : \ K_X^{-1/2} \to D_{PGL_2} \otimes K_X^{3/2}$$