Kazhdan's property (T) for Aut($\mathbf{F}_n$) and $\mathrm{EL}_n(\mathcal{R})$

Narutaka OZAWA (小澤 登高)

RIMS, Kyoto University

Harvard Math Picture Language Seminar, 2022.10.11

C*-algebras are an esoteric subject —"the most abstract nonsense that exists in mathematics," in Casazza's words. "Nobody outside the area knows much about it."

Quanta Magazine: *'Outsiders' Crack 50-Year-Old Math Problem.*

`http://www.quantamagazine.org/computer-scientists-solve-kadison-singer-problem-20151124`

Kazhdan's property (T)

Theorem (Kazhdan 1967)

Any simple Lie group G of real rank ≥ 2 (e.g., $G = \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$) and its lattice Γ (e.g., $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) have **property (T)**.

$\rightsquigarrow$ Γ is finitely generated and has finite abelianization.

Definition (for a discrete group Γ)

Γ has (T) $\overset{\text{def}}{\iff} \exists S \subset \Gamma$ finite $\exists \kappa > 0$ s.t. $\forall (\pi, \mathcal{H})$ unitary rep'n and $\forall v \in \mathcal{H}$

$$d(v, \mathcal{H}^\Gamma) \leq \kappa^{-1} \max_{s \in S} \|v - \pi(s)v\|.$$

$\iff$ Γ is f.g. & $\forall S \subset \Gamma$ generating $\exists \kappa = \kappa(\Gamma, S) > 0$ s.t. $\cdots$

The optimal $\kappa(\Gamma, S)$ is called the **Kazhdan constant** for (Γ, S).

- Property (T) inherits to finite-index subgroups and quotient groups.
- $\mathbb{Z}$ (or any infinite amenable group) does not have property (T).
  $\because \frac{1}{\sqrt{2k+1}} 1_{[-k,k]} \in \ell^2(\mathbb{Z})$ is asymp. $\mathbb{Z}$-invariant, but $\ell^2(\mathbb{Z})^{\mathbb{Z}} = \{0\}$.

$\rightsquigarrow$ Any f.i. subgroup with property (T) has finite abelianization.

An application of property (T): Expander graphs

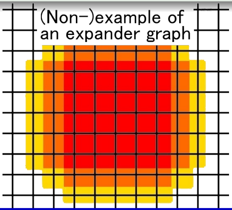
Explicit construction of expanders (Margulis 1973)

$\Gamma = \langle S \rangle$, X a finite set, and $\Gamma \curvearrowright X$ transitively

$\rightsquigarrow$ Schreier graph: Vertices $= X$ and Edges $= \{\{x, sx\} : x \in X,\ s \in S\}$ is a ε-**expander** for $\varepsilon = \frac{\kappa(\Gamma,S)^2}{2}$. Namely, for $\forall A \subset X$ one has

$$|\partial A| \geq \varepsilon |A|(1 - \frac{|A|}{|X|}).$$

- For $\mathcal{N}_k(A) := \{x \in X : \mathrm{d}(x, A) \leq k\}$, $|\mathcal{N}_k(A)| \geq (1+\varepsilon)^k |A|$ until it reaches $\frac{1}{2}|X|$. After that $|\mathcal{N}_k(A)^c|$ decreases by a factor $1+\varepsilon$.
- Random walk on X has mixing time $O(\log |X|)$.
- Want a large ε-expander with fixed degree and ε.



Expander Sampling Theorem (Gillman 1998):

For $\forall f \in \ell_\infty(X)$ with $\|f\|_\infty \leq 1$ and $m = \frac{1}{|X|} \sum f(x)$,
Simple Random Walk $x_0, x_1, \ldots$ on X satisfies

$$\mathbb{P}(|\frac{1}{T} \sum_{k=0}^{T-1} f(x_k) - m| > \delta) < \exp(-C_{d,\varepsilon} \delta^2 T).$$

Some examples of property (T) groups

- $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, (Kazhdan 1967), but not $\mathrm{SL}_2(\mathbb{Z})$.
  - $\mathrm{EL}_n(\mathcal{R}) = \langle e_{ij}(r) : i \neq j, r \in \mathcal{R} \rangle \subset \mathrm{GL}_n(\mathcal{R})$, $n \geq 3$, where $\mathcal{R}$ finitely generated ring and $e_{ij}(r) := I_n + rE_{ij}$ (Shalom & Vaserstein, Ershov–Jaikin-Zapirain 2006–08).
  - $\mathrm{Aut}(\mathbf{F}_n)$, $n \geq 4$. (Kaluba–Nowak–O., K–Kielak–N., Nitsche 17–20).
    $\mathrm{Aut}(\mathbf{F}_n)$ is the noncommutative analogue of $\mathrm{GL}_n(\mathbb{Z})$.
    $\mathbf{F}_n \twoheadrightarrow \mathbb{Z}^n$ abelianization $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.
    $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_2)$ does not have (T). Neither $\mathrm{Aut}(\mathbf{F}_3)$ (McCool 1989).

Product Replacement Algorithm (Celler et al. 95, Lubotzky–Pak 01)

$\mathrm{Aut}^+(\mathbf{F}_n) = \langle R^{\pm}_{i,j}, L^{\pm}_{i,j} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbf{F}_n)$, where $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$ and

$$R^{\pm}_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i g_j^{\pm}, g_{i+1}, \ldots, g_n),$$
$$L^{\pm}_{i,j} \colon (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_j^{\pm} g_i, g_{i+1} \ldots, g_n).$$

PRA is a practical algorithm to obtain "random" elements in a given finite group Λ of rank $< n$ via the PRA random walk

$$\mathrm{Aut}^+(\mathbf{F}_n) \curvearrowright \{(h_1, \ldots, h_n) \in \Lambda^n : \Lambda = \langle h_1, \ldots, h_n \rangle\}.$$

Hilbert's 17th Pb: $f \in \mathbb{R}(x_1, \ldots, x_d)$, $f \geq 0$ on $\mathbb{R}^d$
(E. Artin 1927) $\Longrightarrow f = \sum_i g_i^2$ for some $g_1, \ldots, g_k \in \mathbb{R}(x_1, \ldots, x_d)$.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$ **positive cone**

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

- $\mathbb{B}(\mathcal{H})^+ := \{A = A^* : \langle Av, v\rangle \geq 0 \ \forall v \in \mathcal{H}\} = \Sigma^2\mathbb{B}(\mathcal{H})$ psd operators.
- $\forall(\pi, \mathcal{H})$ unitary rep'n, $\pi(\sum_i f_i^* f_i) = \sum_i \pi(f_i)^* \pi(f_i) \geq 0$ in $\mathbb{B}(\mathcal{H})$.
- $C^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Laplacian: For $\Gamma = \langle S\rangle$ with $S = S^{-1}$ finite,

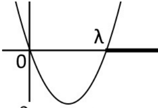
$$\Delta := \tfrac{1}{2}\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s\in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

Then, $\langle \pi(\Delta)v, v\rangle = \frac{1}{2}\sum_{s\in S}\|v - \pi(s)v\|^2$ and

Γ has (T) $\Longleftrightarrow \exists\lambda > 0 \quad \forall(\pi, \mathcal{H}) \quad \mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\lambda, \infty)$

$\Longleftrightarrow \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \geq 0$ in $C^*[\Gamma]$

$\rightsquigarrow \kappa(\Gamma, S) \geq \sqrt{2\lambda/|S|}$



$t^2 - \lambda t \geq 0$

Algebraic characterization of property (T)

Let $\Gamma = \langle S \rangle$ with $S = S^{-1}$ finite.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$$

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

$$\Delta := \frac{1}{2}\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s\in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

$\mathrm{C}^*[\Gamma]$ the universal enveloping C^*-algebra of $\mathbb{R}[\Gamma]$.

Then,

$$\Gamma \text{ has (T)} \iff \exists \lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \geq 0 \text{ in } \mathrm{C}^*[\Gamma]$$

$$\rightsquigarrow \ \kappa(\Gamma, S) \geq \sqrt{2\lambda/|S|}$$

Theorem (O 2013)

$$\Gamma \text{ has (T)} \iff \exists \lambda > 0 \text{ such that } \Delta^2 - \lambda\Delta \succeq 0 \text{ in } \mathbb{R}[\Gamma]$$

Stability (Netzer–Thom): It suffices if $\exists \lambda > 0\ \exists \Theta \in \Sigma^2\mathbb{R}[\Gamma]$ such that

$$\|\Delta^2 - \lambda\Delta - \Theta\|_1 \ll \lambda.$$

Semidefinite Programming (SDP)

Γ has (T) $\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$

$\iff \exists E \Subset \Gamma\ \exists\lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_E^+\}$

By fixing a finite subset $E \Subset \Gamma$, we arrive at the SDP:

$$\begin{array}{ll} \texttt{minimize} & -\lambda \\ \texttt{subject to} & \Delta^2 - \lambda\Delta = \sum_{x,y\in E} P_{x,y} x^{-1} y, \quad P \in \mathbb{M}_E^+ \end{array}$$

- Due to computer capacity limitation, we almost always take

$$E := \mathrm{Ball}(2) = \{e\} \cup S \cup S^2 = \operatorname{supp}\Delta \cup \operatorname{supp}\Delta^2.$$

$\rightsquigarrow$ Size of SDP: dimension $|E|^2$ and constraints $|E^{-1}E| = |\,\mathrm{Ball}(4)|$.

Certification Procedure:

Suppose (λ_0, P_0) is a hypothetical solution obtained by a computer.
Find $P_0 \approx Q^{\mathrm{T}} Q$ (with $Q\mathbf{1} = 0$) and calculate **with guaranteed accuracy**

$$r := \|\Delta^2 - \lambda_0\Delta - \textstyle\sum_{x,y} (Q^{\mathrm{T}} Q)_{x,y} (1-x)^*(1-y)\|_1 \ll \lambda_0.$$

$\rightsquigarrow$ Γ has (T) with $\lambda = \lambda_0 - 2r$ (in the case of $E = \mathrm{Ball}(2)$).

- Solving SDP is computationally hard, but certifying (T) is relatively easy.

Results

Γ has (T) $\iff \exists E \Subset \Gamma\ \exists \lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_E^+\}$

Results of SDP for $E = \mathrm{Ball}(2)$.

- $\mathrm{SL}_n(\mathbb{Z})$ with $S = \{e_{ij}^{\pm} : i \neq j\}$: $\lambda_3 > 0.27$, $\lambda_4 > 1.3$, $\lambda_5 > 2.6$. (Netzer–Thom 2014, Fujiwara–Kabaya 2017, Kaluba–Nowak 2017)
- No response for $\mathrm{SL}_6(\mathbb{Z})$.

For $\mathrm{Aut}^+(\mathbf{F}_4)$, the size of SDP $\approx$ 10 000 000, beyond our computer's capacity. We exploited invariance under $\mathfrak{S}(n) \ltimes (\mathbb{Z}/2)^{\oplus n} \curvearrowright \mathrm{Aut}^+(\mathbf{F}_n)$.

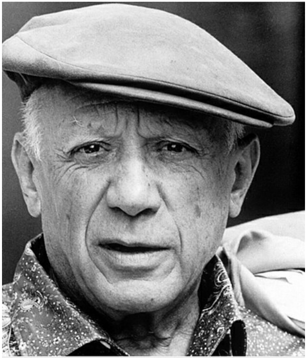
- $\mathrm{Aut}^+(\mathbf{F}_4)$: ☹ ☹ ☹ No response.
- $\mathrm{Aut}^+(\mathbf{F}_5)$: !☺⅄☺⅄☺! **YES!!!** with $\lambda > 1.2$.

Theorem

$\mathrm{Aut}^+(\mathbf{F}_n)$ has property (T) for

- $n = 5$ (Kaluba–Nowak–O. 2017)
- $n \geq 6$ (Kaluba–Kielak–Nowak 2018)
- $n = 4$ (Nitsche 2020, by a new SDP)

Reception



Revista Vea y Lea, January 1962

"But they (= computers) are useless.
They can only give you answers."
Pablo Picasso, 1968.

Property (T) for an infinite series (KKN 2018)

$\Gamma_n := \mathrm{Aut}^+(\mathbf{F}_n)$, $\quad S_n := \{R^{\pm}_{i,j}, L^{\pm}_{i,j} : i \neq j\}$, $\quad \mathrm{E}_n := \{\{i,j\} : i \neq j\}$

Want to show $\Delta_n = \sum_{s \in S_n} 1 - s$ satisfies $\Delta_n^2 - \lambda_n \Delta_n \succeq 0$.

$$\Delta_n = \sum_{\mathrm{e} \in \mathrm{E}_n} \Delta_{\mathrm{e}},$$
$$\Delta_n^2 = \sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e} \sim \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}} + \sum_{\mathrm{e} \perp \mathrm{f}} \Delta_{\mathrm{e}} \Delta_{\mathrm{f}}$$
$$=: \ \mathbf{Sq}_n \ + \ \mathbf{Adj}_n \ + \ \mathbf{Op}_n \, .$$

- $\mathbf{Sq}_n$ and $\mathbf{Op}_n$ are positive, but $\mathbf{Adj}_n$ may not.

For $n > m$,

$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\Delta_m) = m(m-1) \cdot (n-2)! \cdot \Delta_n$$
$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Adj}_m) = m(m-1)(m-2) \cdot (n-3)! \cdot \mathbf{Adj}_n$$
$$\sum_{\sigma \in \mathfrak{S}(n)} \sigma(\mathbf{Op}_m) = m(m-1)(m-2)(m-3) \cdot (n-4)! \cdot \mathbf{Op}_n$$

Trial and error on the computer has confirmed

$$\mathbf{Adj}_5 + \alpha \, \mathbf{Op}_5 - \varepsilon \Delta_5 \succeq 0$$

with $\alpha = 2$ and $\varepsilon = 0.13$. It follows that

$$0 \preceq 60(n-3)!\big(\mathbf{Adj}_n + \tfrac{2\alpha}{n-3} \mathbf{Op}_n - \tfrac{n-2}{3} \varepsilon \Delta_n\big) \preceq 60(n-3)!\big(\Delta_n^2 - \tfrac{n-2}{3} \varepsilon \Delta_n\big),$$

provided $2\alpha/(n-3) \leq 1$. $\quad \rightsquigarrow \ \kappa(\mathrm{Aut}^+(\mathbf{F}_n), S_n) \geq \sqrt{2\lambda_n/|S_n|} \geq \sqrt{\varepsilon/6n}$

Generalizing property (T) for $\mathrm{EL}_n(\mathcal{R})$ for a rng $\mathcal{R}$

The computer taught us the inequality $\mathbf{Adj}_5 + \alpha\,\mathbf{Op}_5 - \varepsilon\Delta_5 \succeq 0$ is true, useful, and even **easy to prove (!)** **when** $\alpha > 0$ **is large**.

Theorem (O. 2022)

For any f.g. **comm.** rng $\mathcal{R}$ generated by $R_0 \Subset \mathcal{R}$ and for n **large enough**,

$$\Delta := \sum_{r \in R_0} \sum_{i \neq j} (1 - e_{ij}(r))^*(1 - e_{ij}(r)) \quad \text{and}$$
$$\Delta^{(2)} := \sum_{r,s \in R_0} \sum_{i \neq j} (1 - e_{ij}(rs))^*(1 - e_{ij}(rs))$$

in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$ satisfy

$$\Delta^2 \geq \varepsilon \Delta^{(2)}$$

in $C^*[\mathrm{EL}_n(\mathcal{R})]$ for some $\varepsilon > 0$ (but not $\Delta^2 \succeq \varepsilon\Delta^{(2)}$ in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$).

! "rng" = "ring" − "i". $\mathrm{EL}_n(\mathcal{R}) \twoheadrightarrow \mathrm{EL}_n(\mathcal{R}/\mathcal{R}^2) \cong (\mathcal{R}/\mathcal{R}^2)^{\oplus n(n-1)}$ abelian.

The proof of $\mathbf{Adj}_5 + \alpha\,\mathbf{Op}_5 - \varepsilon\Delta_5^{(2)} \geq 0$ is silicon-free and relies on Boca and Zaharescu's work (2005) on the almost Mathieu operators in $\mathcal{A}_\theta$.

Corollary

$\exists n\ \exists \varepsilon > 0$ s.t. $\mathrm{Cayley}(\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}), \{e_{ij}(p) : i \neq j\})$, $p \perp q$, are ε-expanders.

! The Kazhdan constants for $(\mathrm{EL}_n(p\mathbb{Z}))_{p \in \mathbb{N}}$ are not uniform.