Time–energy uncertainty relation and quantum error correction for noisy quantum metrology

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arXiv:2207.13707

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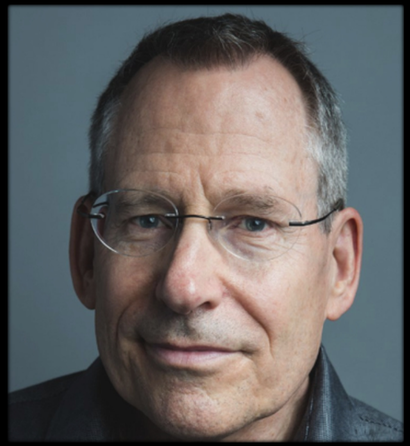

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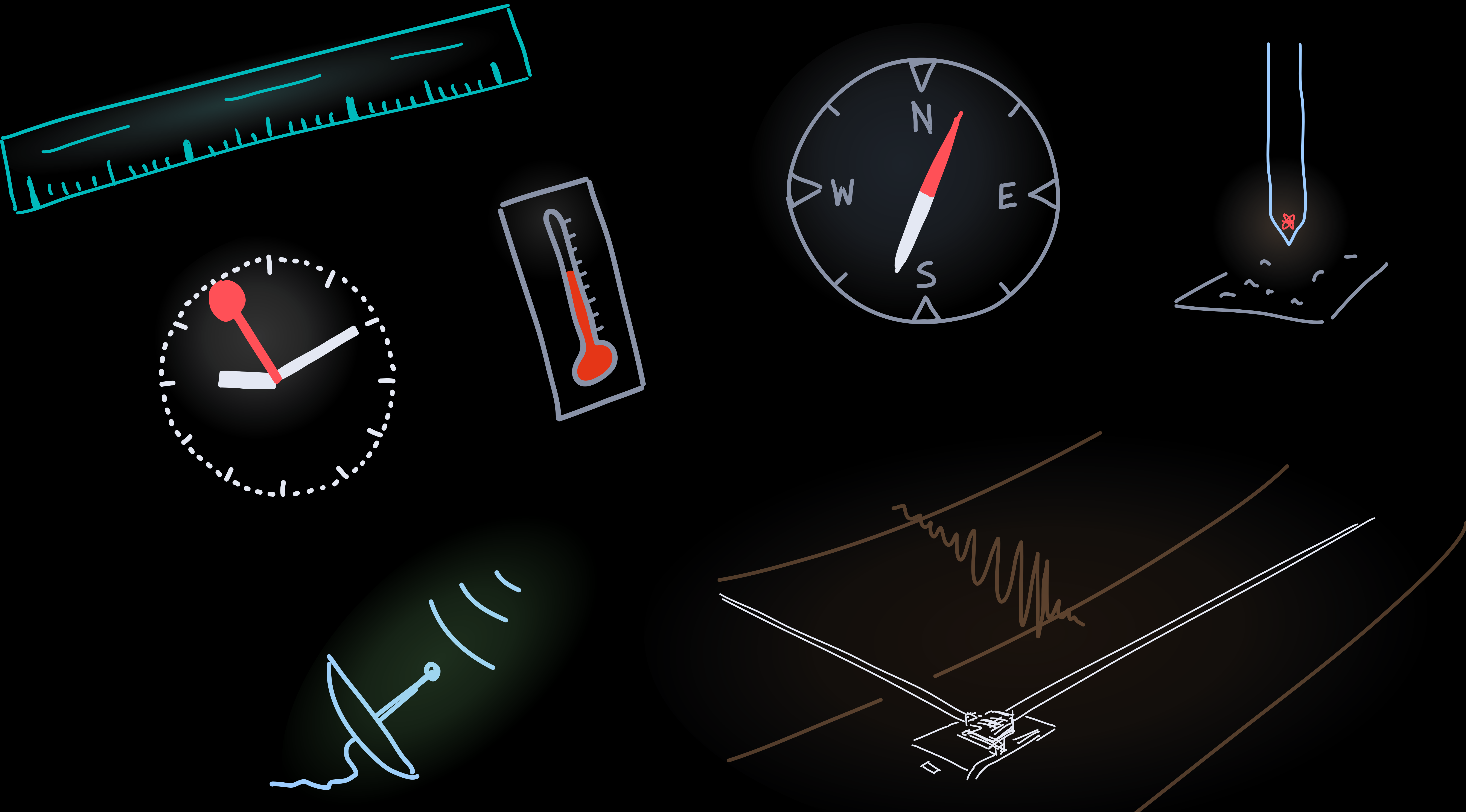
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Observables can be incompatible.

Heisenberg uncertainty principle

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

A measurement disturbs the quantum state ρ

$$\rho \rightarrow \frac{P\rho P}{\mathrm{tr}(P\rho)}$$

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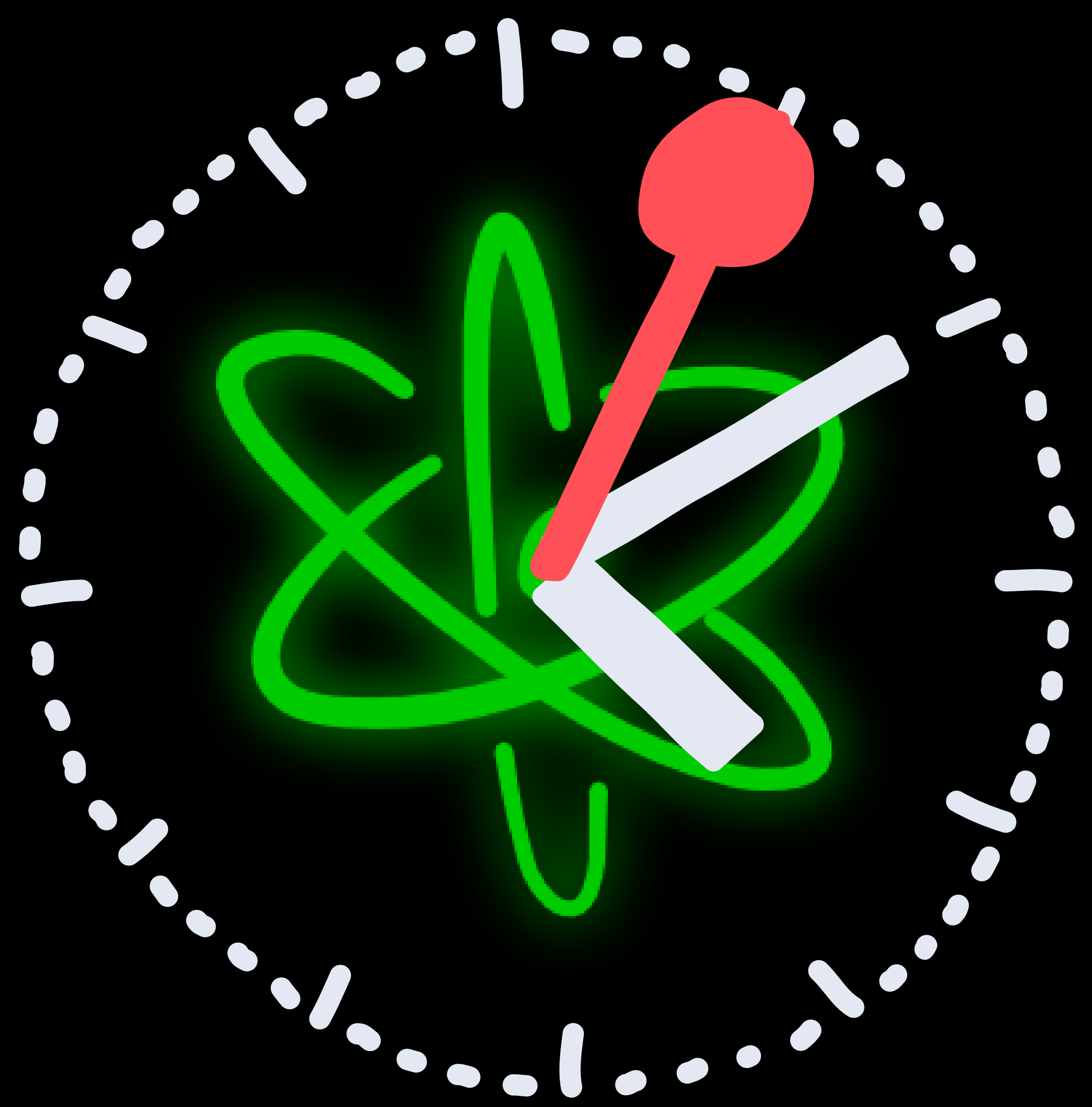
***Time* is not an observable.**

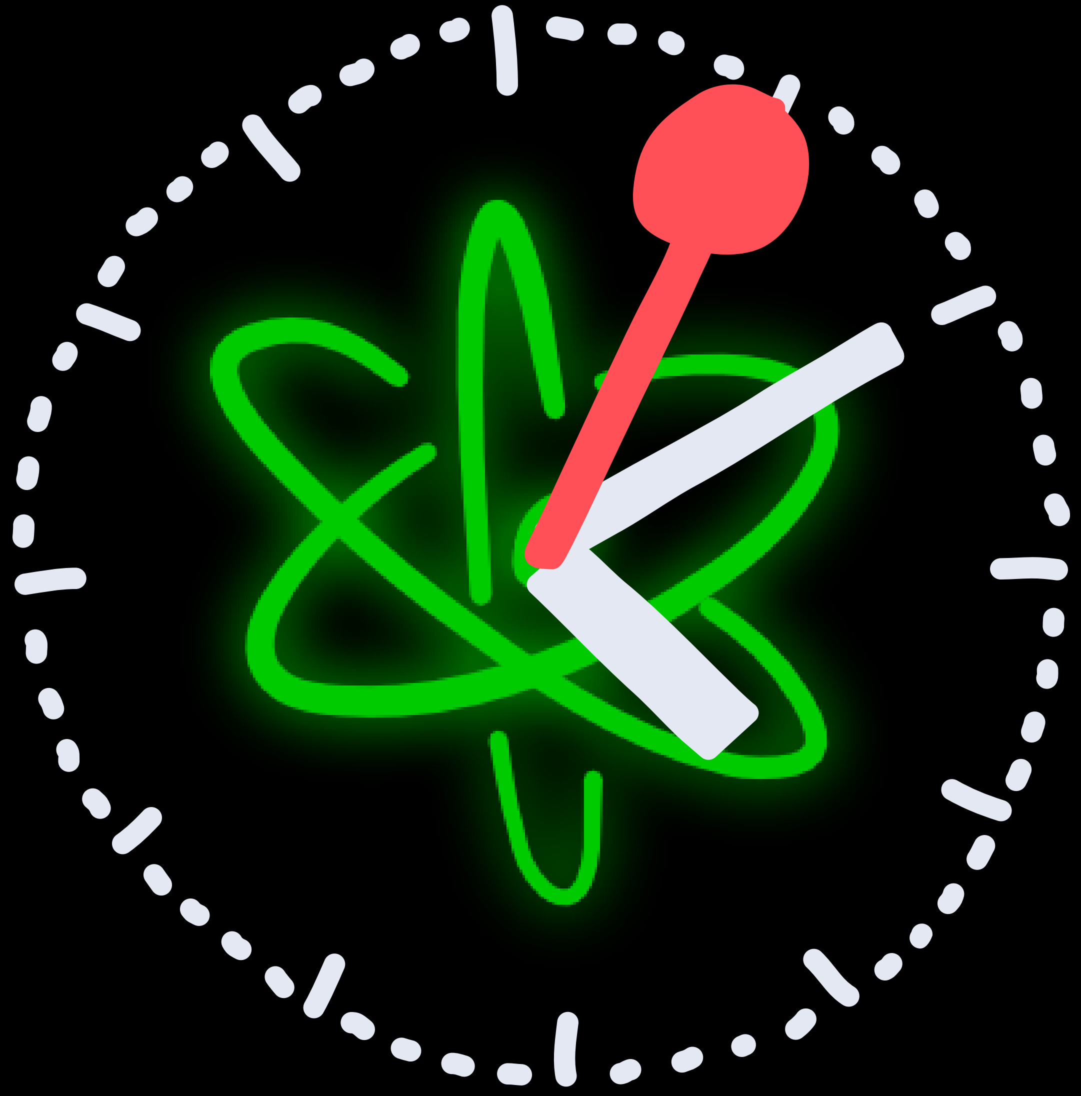
Time–Energy uncertainty principle

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$

e.g. time it takes for any observable to change by at least one standard deviation



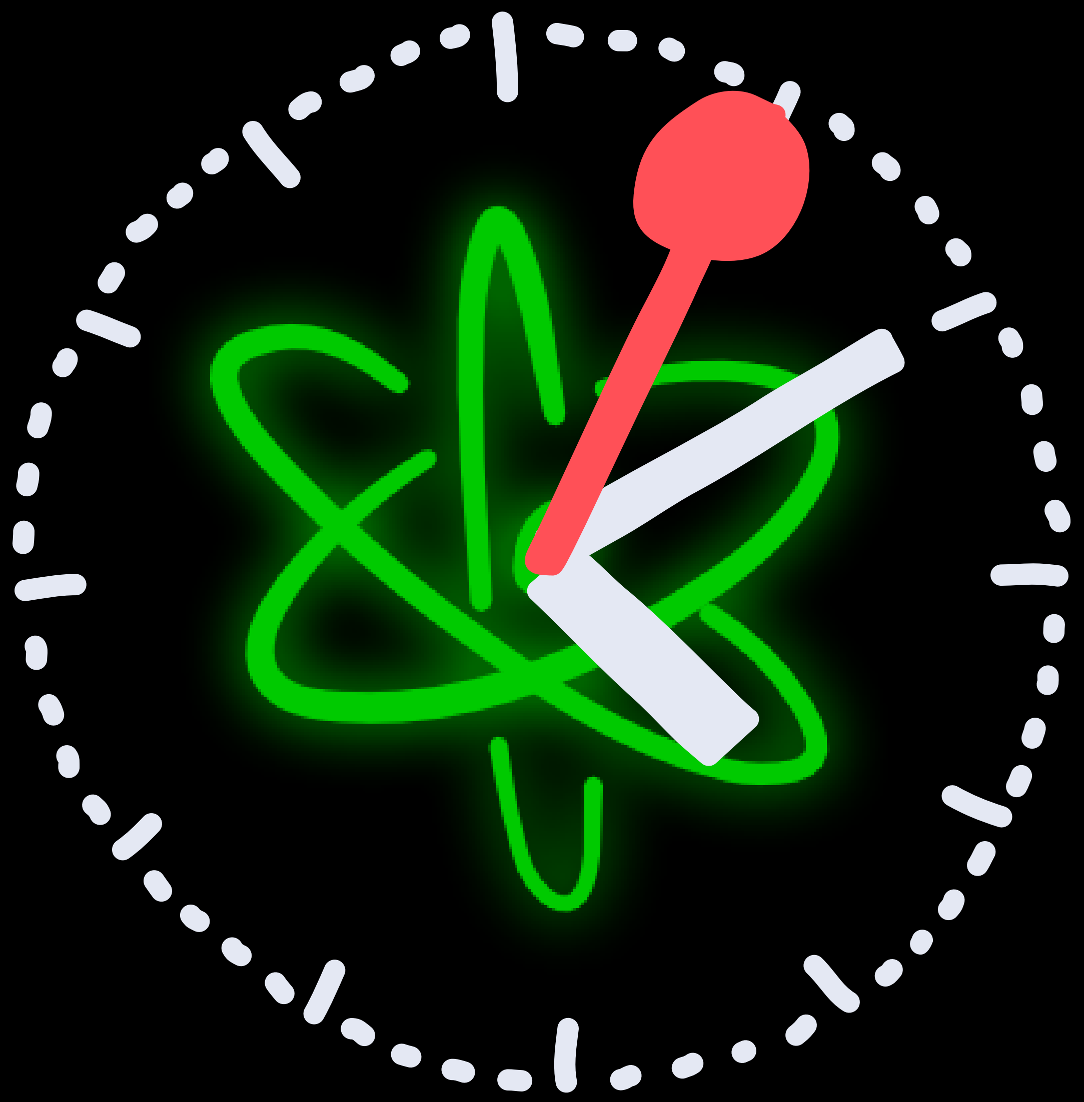
quantum state $\rho(t)$



time evolution

$$\partial_t \rho = -i[H, \rho]$$

quantum state $\rho(t)$



time evolution

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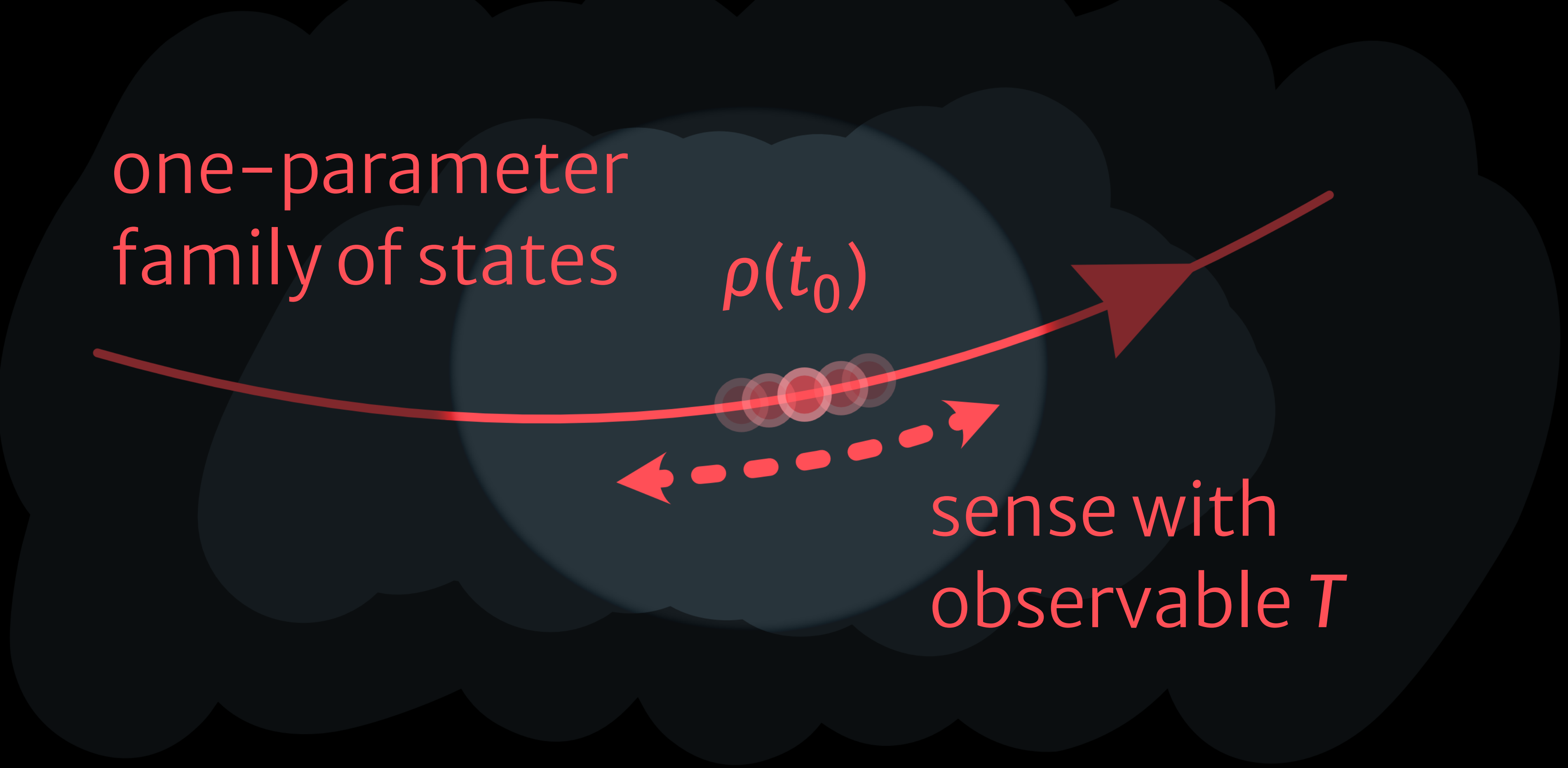
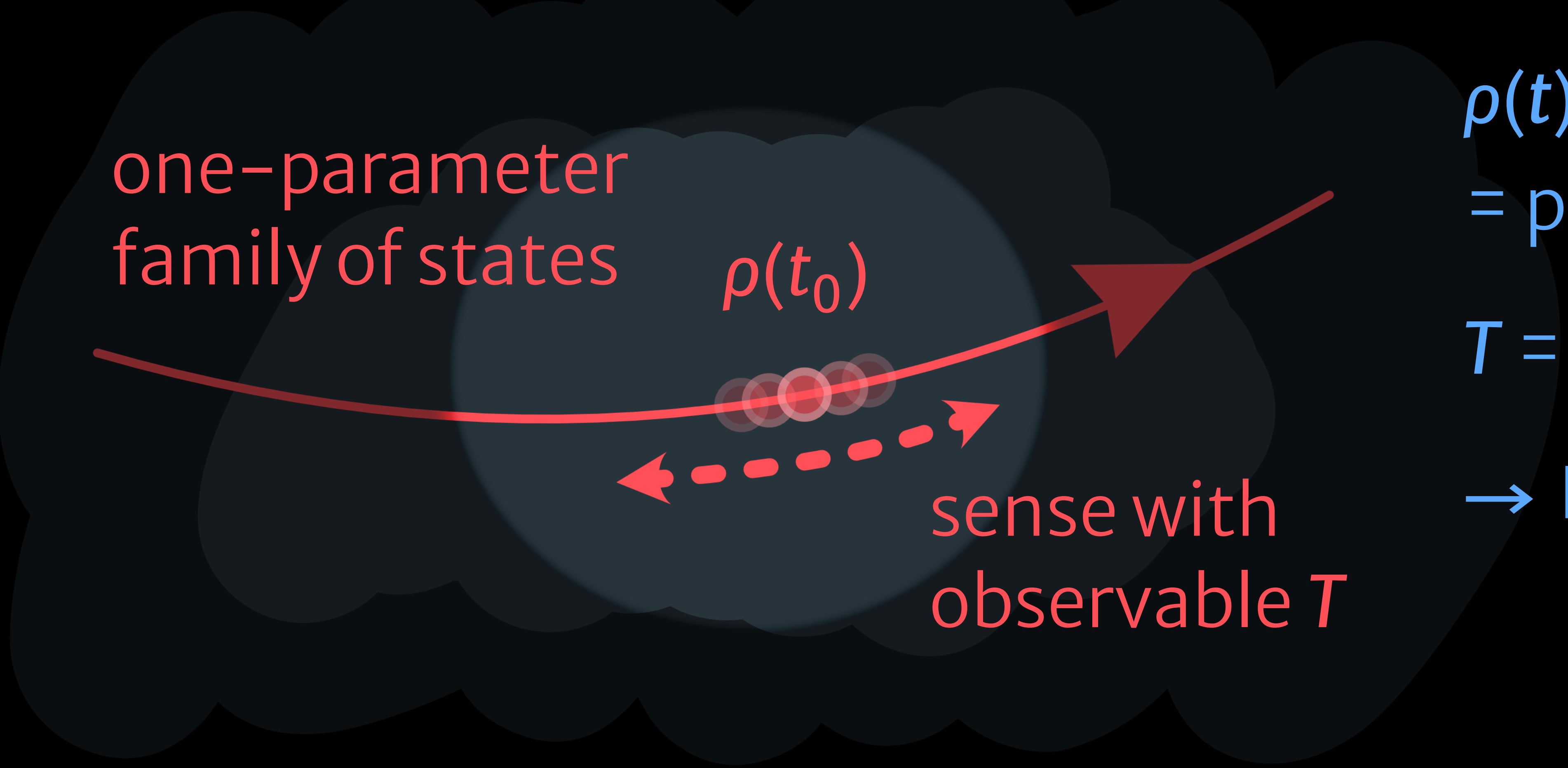
How accurately can a measurement reveal the value of t?



Braunstein/Caves PRL 1994; Holevo 2011; …



$\rho(t)$ = quantum state
= positive semidefinite operator

T = observable = Hermitian operator

→ Estimate: $t_{\mathrm{est.}} = \mathrm{tr}(\rho T) \equiv \langle T \rangle_\rho$

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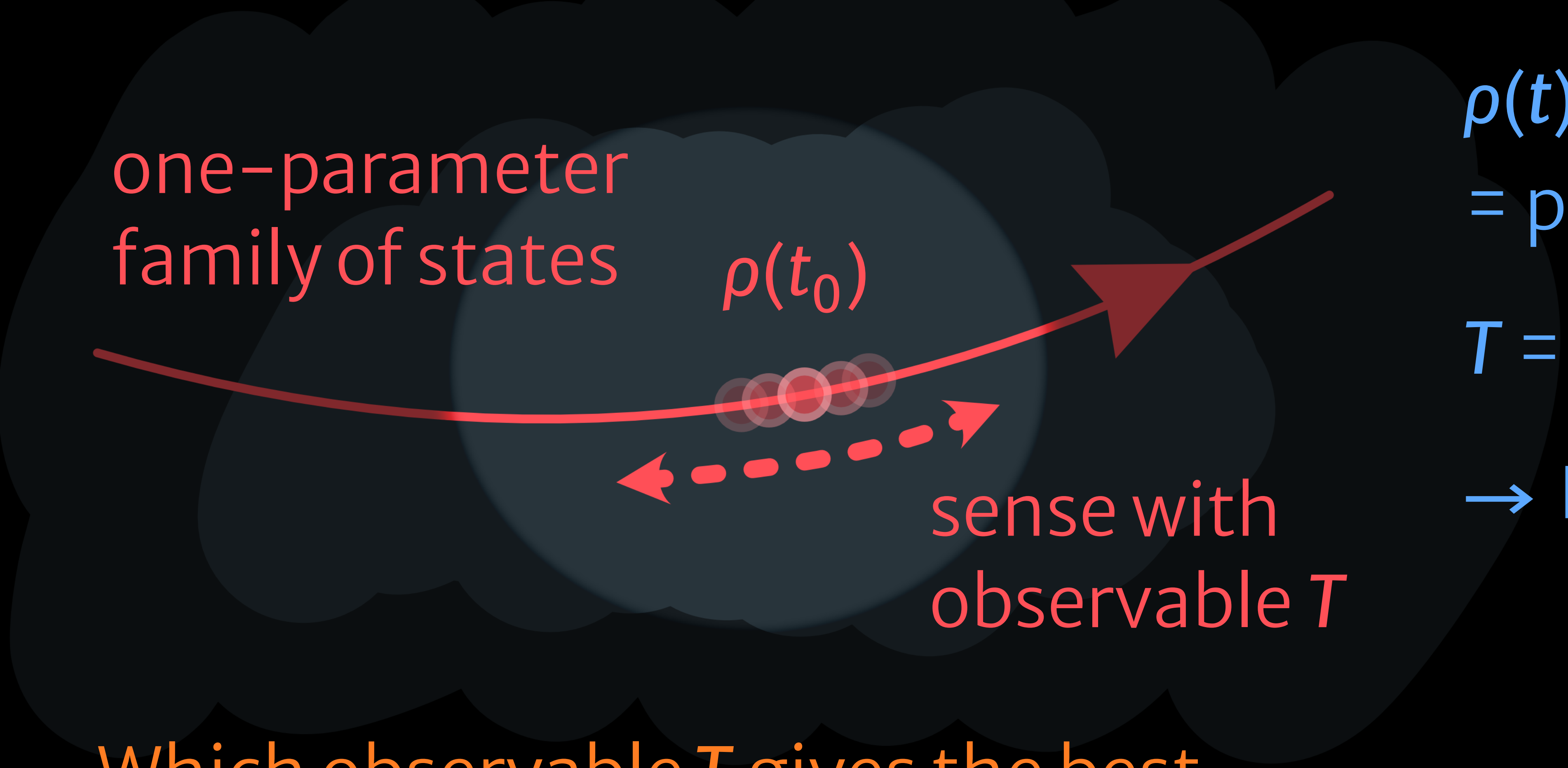

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Which observable T gives the best estimate for t around t_0?

$$\min \left\langle (T - t_0)^2 \right\rangle \quad = ?$$

$$\langle T \rangle_{\rho(t_0+dt)} \stackrel{!}{=} t_0 + dt + O(dt^2)$$

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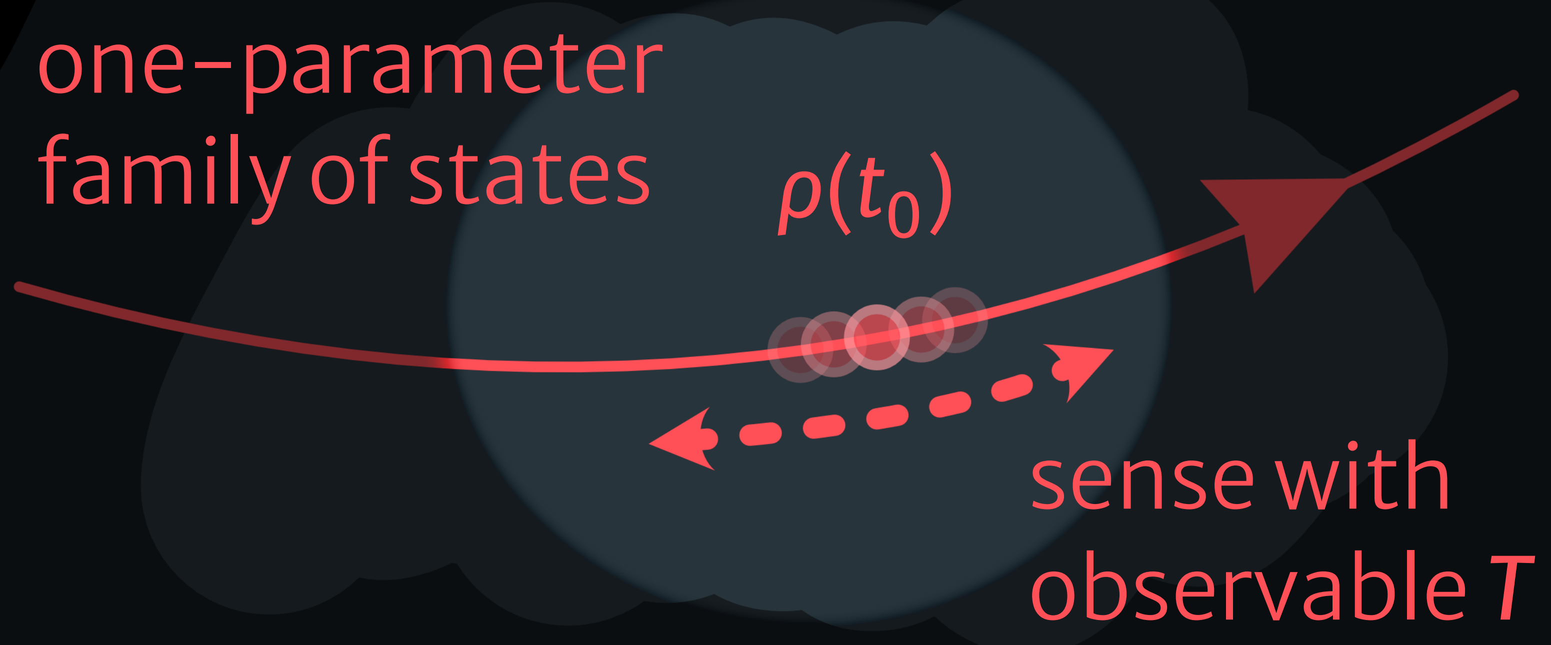

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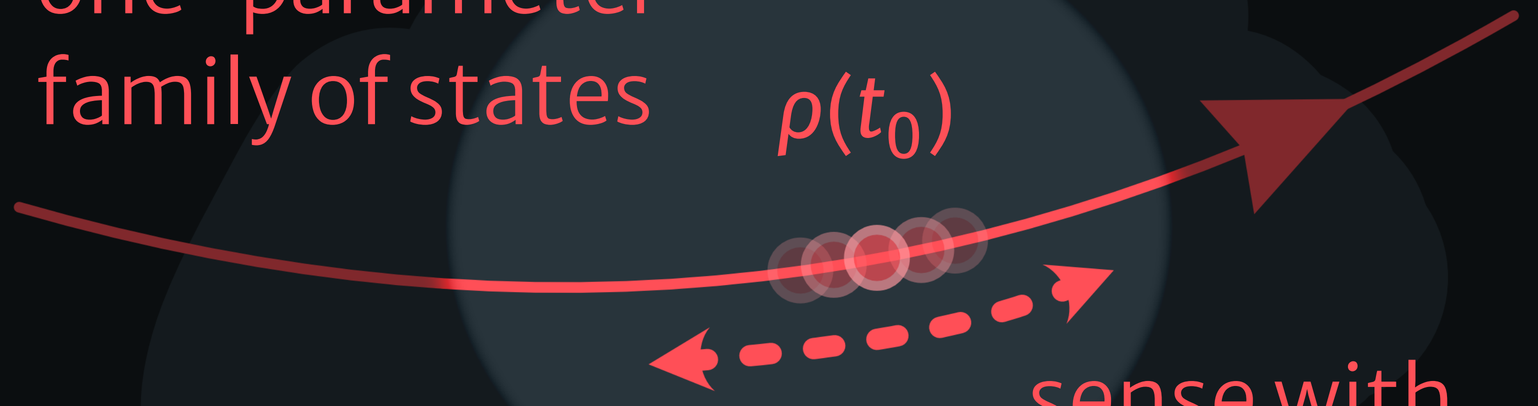

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↖ ***optimal sensing observable*** $T - t_0 \propto R$

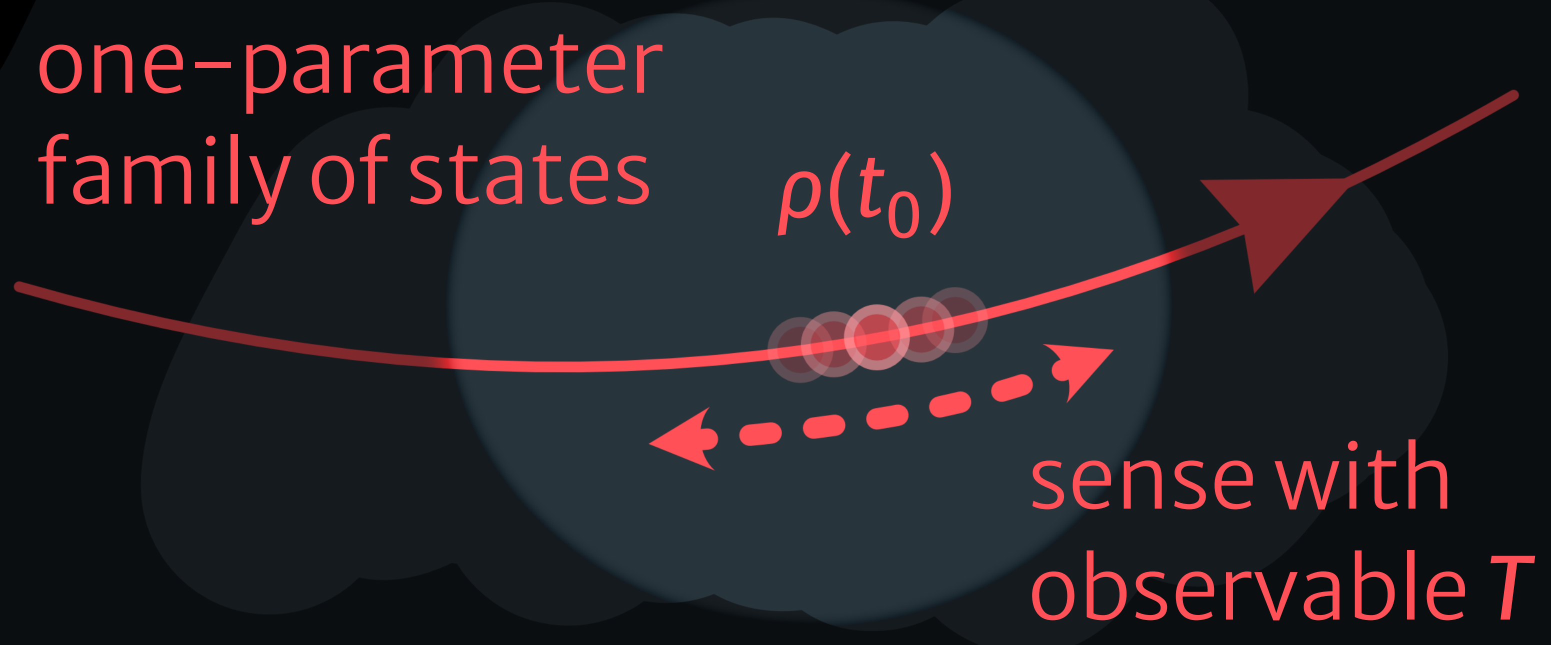
$$\min \left\langle (T - t_0)^2 \right\rangle = \frac{1}{F\big(\rho(t_0), \partial_t \rho(t_0)\big)}$$

Cramér–Rao bound

Quantum Fisher information

$F(\rho, D) = \mathrm{tr}(\rho R^2)$ where R solves $(\rho R + R\rho)/2 = D$

Braunstein/Caves PRL 1994; Holevo 2011; …



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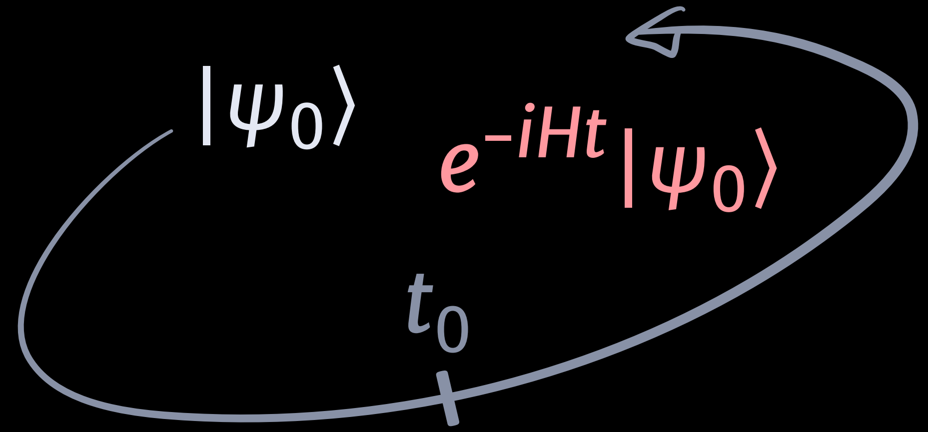
The Quantum Fisher Information $F(\rho, \partial_t \rho)$ gives the accuracy to which t can be sensed locally around t_0.

$$\min \ (\Delta T)^2 = \frac{1}{F(\rho, \partial_t \rho)}$$

Braunstein/Caves PRL 1994; Holevo 2011; …

Example: unitary pure state evolution



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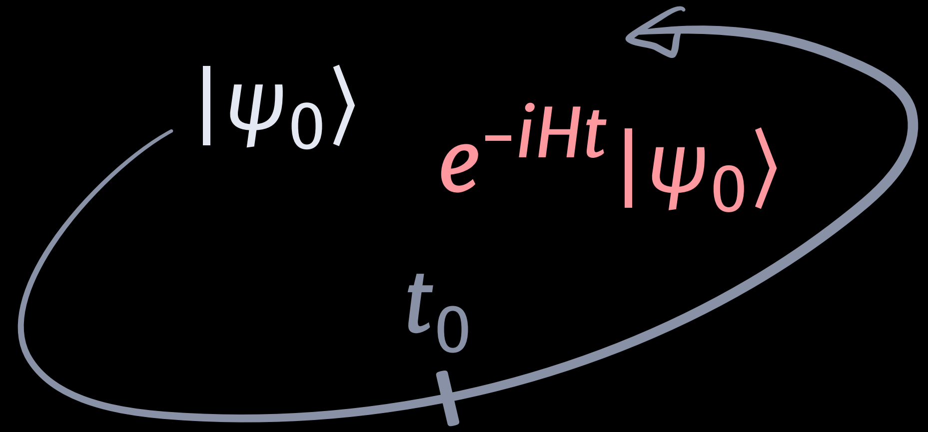

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$$\rightarrow F(\psi, \partial_t \psi) = \mathrm{tr}\left(\psi(2\partial_t \psi)^2\right) \cdots = 4\left(\langle H^2\rangle - \langle H\rangle^2\right)$$

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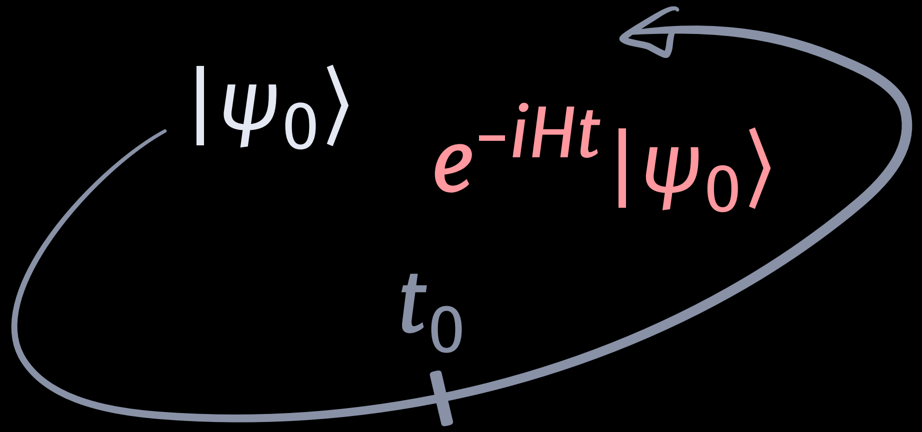

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Optimal sensing observable: $T \propto \partial_t \psi$

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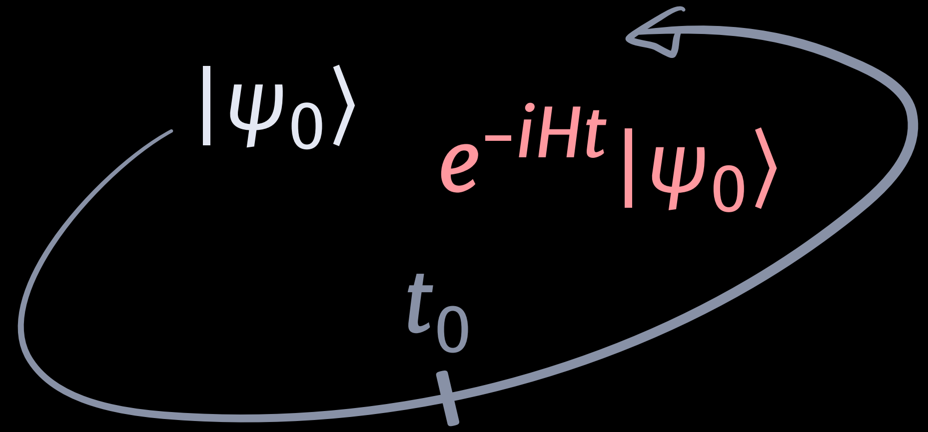

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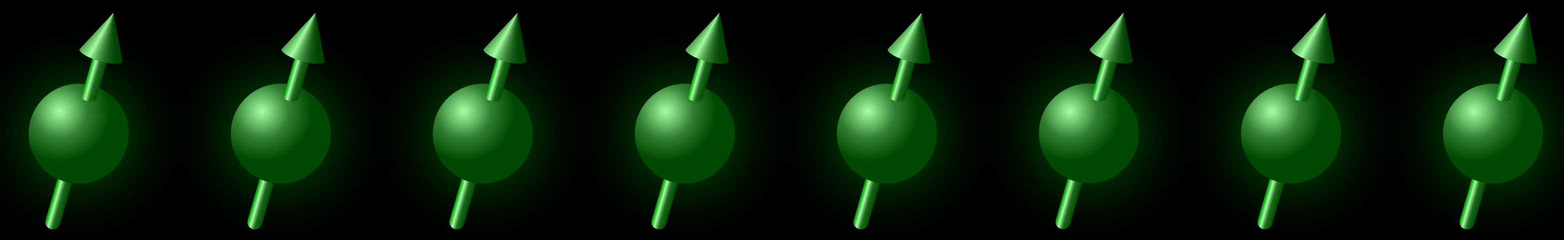
For a pure state evolving according to some Hamiltonian, the optimal sensitivity is given by the **variance of the Hamiltonian**.

Pure state accuracy $\Delta T = \frac{1}{\sqrt{4(\langle H^2 \rangle - \langle H \rangle^2)}}$

n spin–1/2 particles



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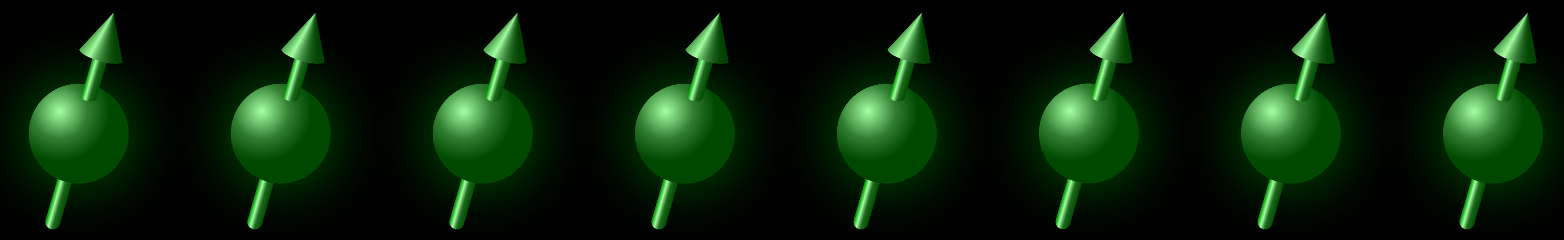

$$H = \frac{\omega\sigma_Z^{(1)}}{2} + \cdots + \frac{\omega\sigma_Z^{(n)}}{2}$$

$$|\psi\rangle = [\,|\uparrow\cdots\uparrow\rangle + |\downarrow\cdots\downarrow\rangle]/\sqrt{2}$$

$$F = 4\langle H^2\rangle = n^2\omega^2 \rightarrow \Delta T \sim 1/n$$

Heisenberg scaling

Giovannetti, Lloyd, Maccone PRL 2006

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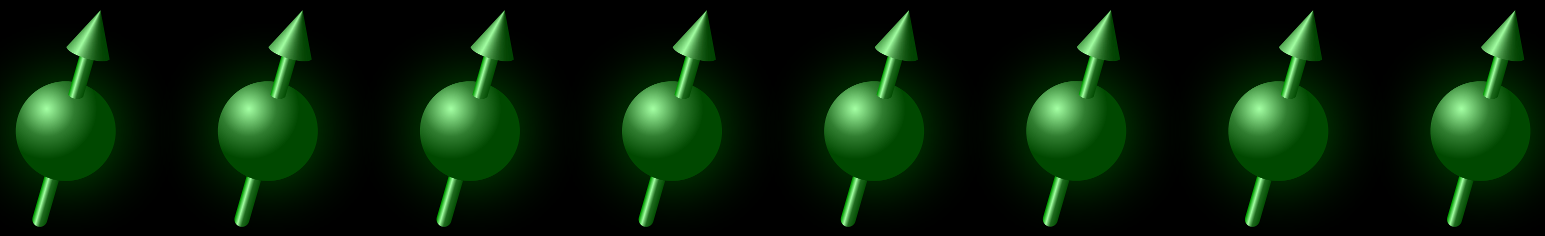

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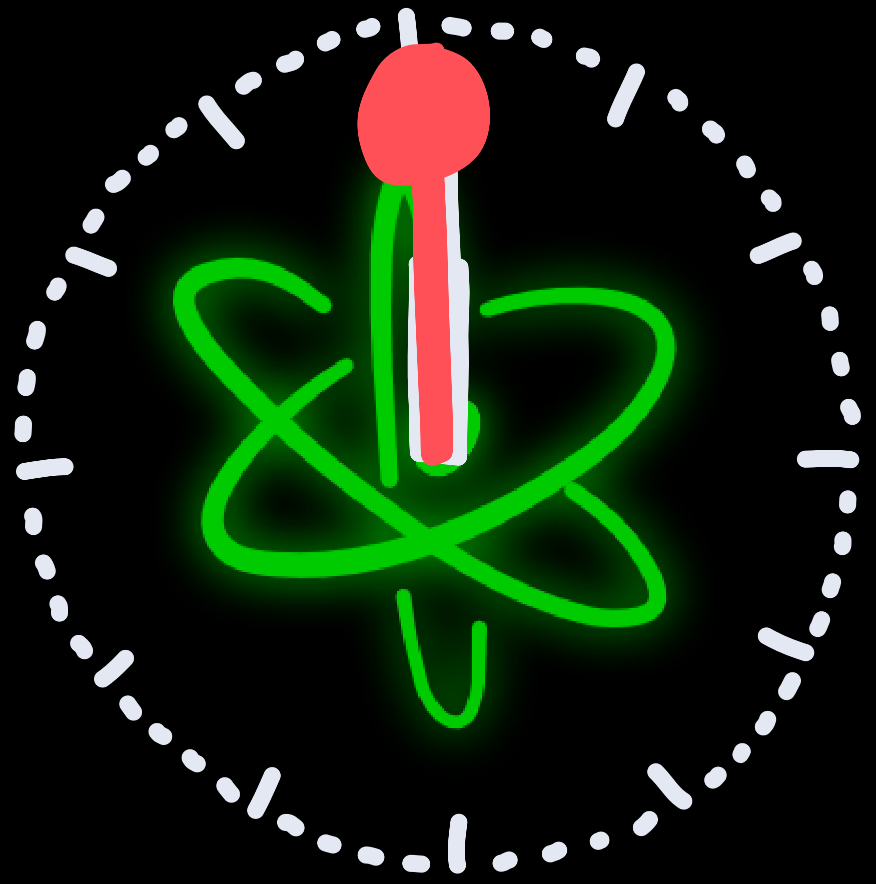
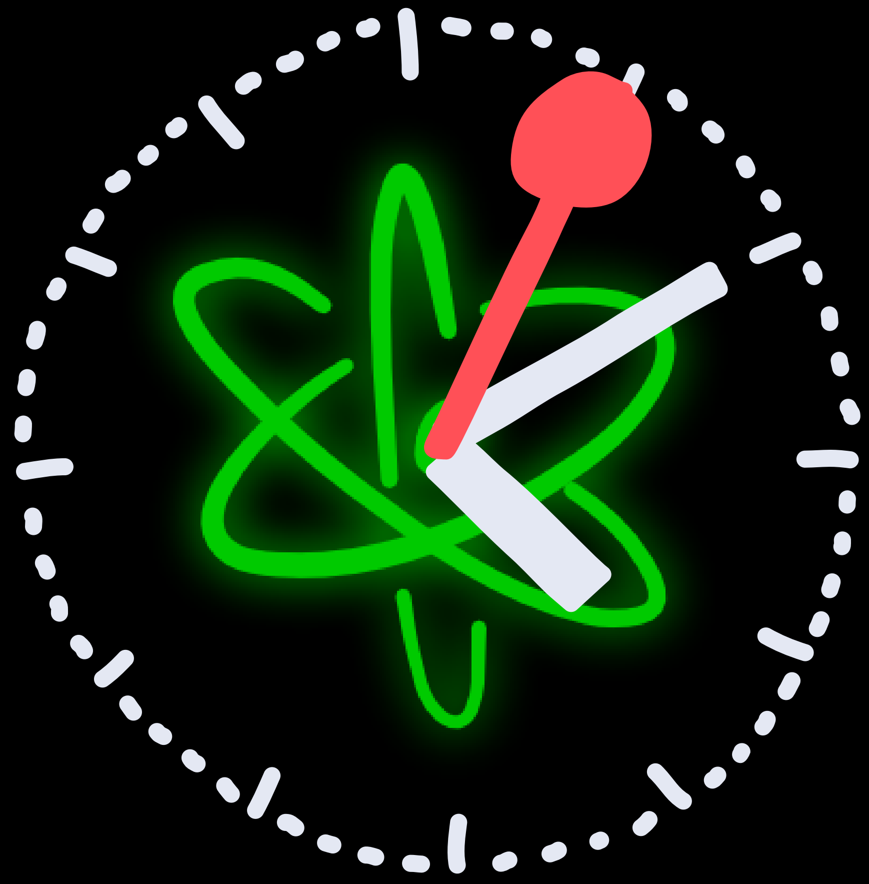
The quantum sensitivity advantage disappears in the presence of noise.

Demkowicz-Dobrzański *et al.* Nat Comm 2012

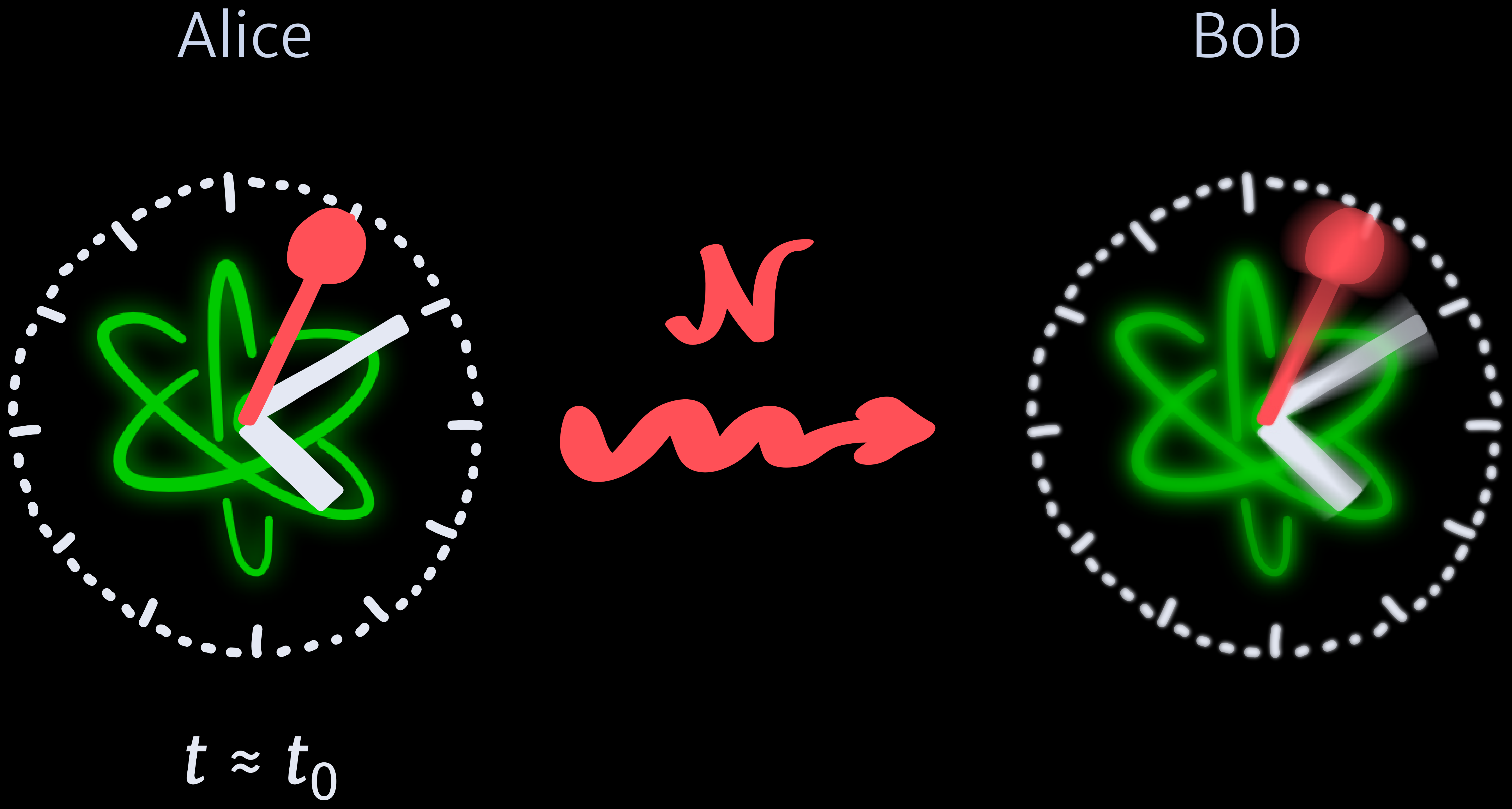
What are the ultimate sensitivity limits in the presence of noise?



$t = 0$



$t \approx t_0$


Alice
Bob
$t \approx t_0$
$\mathcal{N}$

Alice

Bob

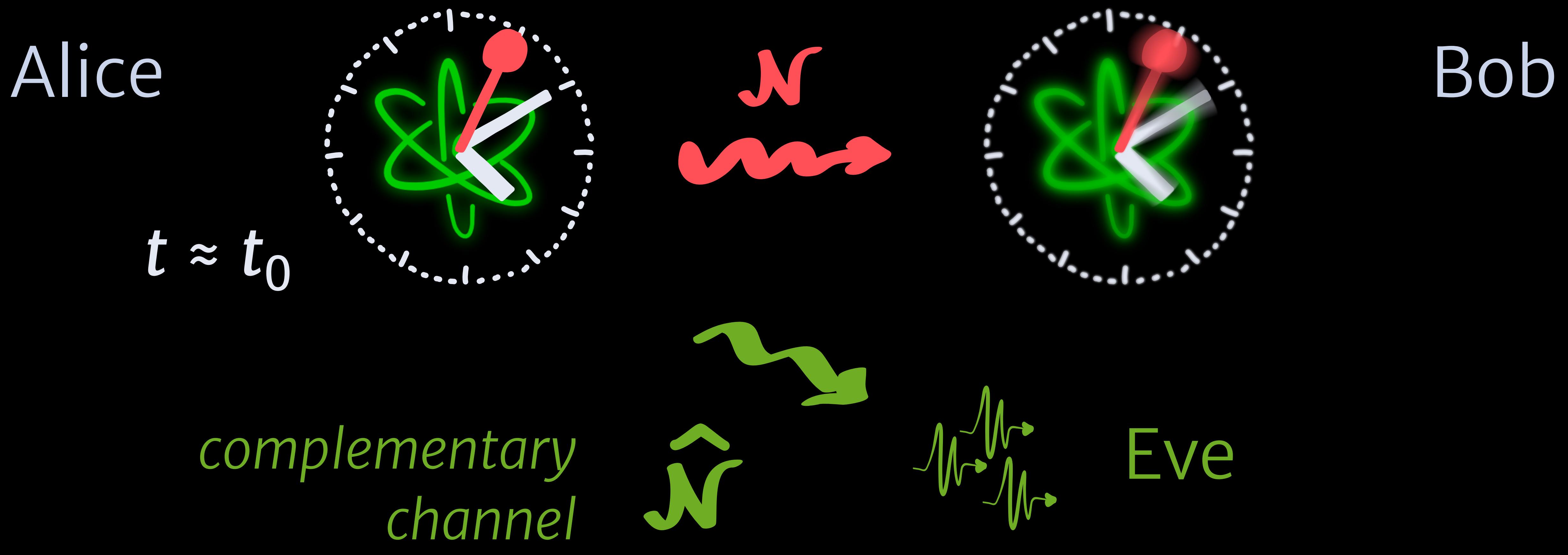
$t \approx t_0$

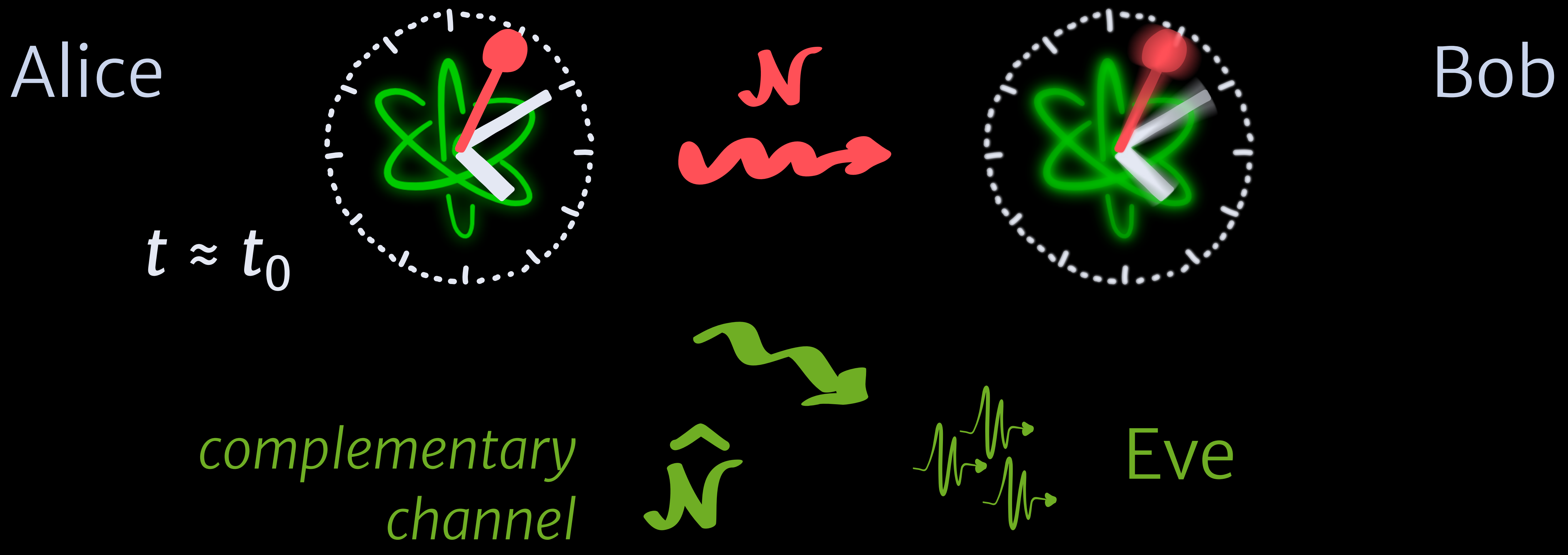
How much sensitivity is there left?

Alice

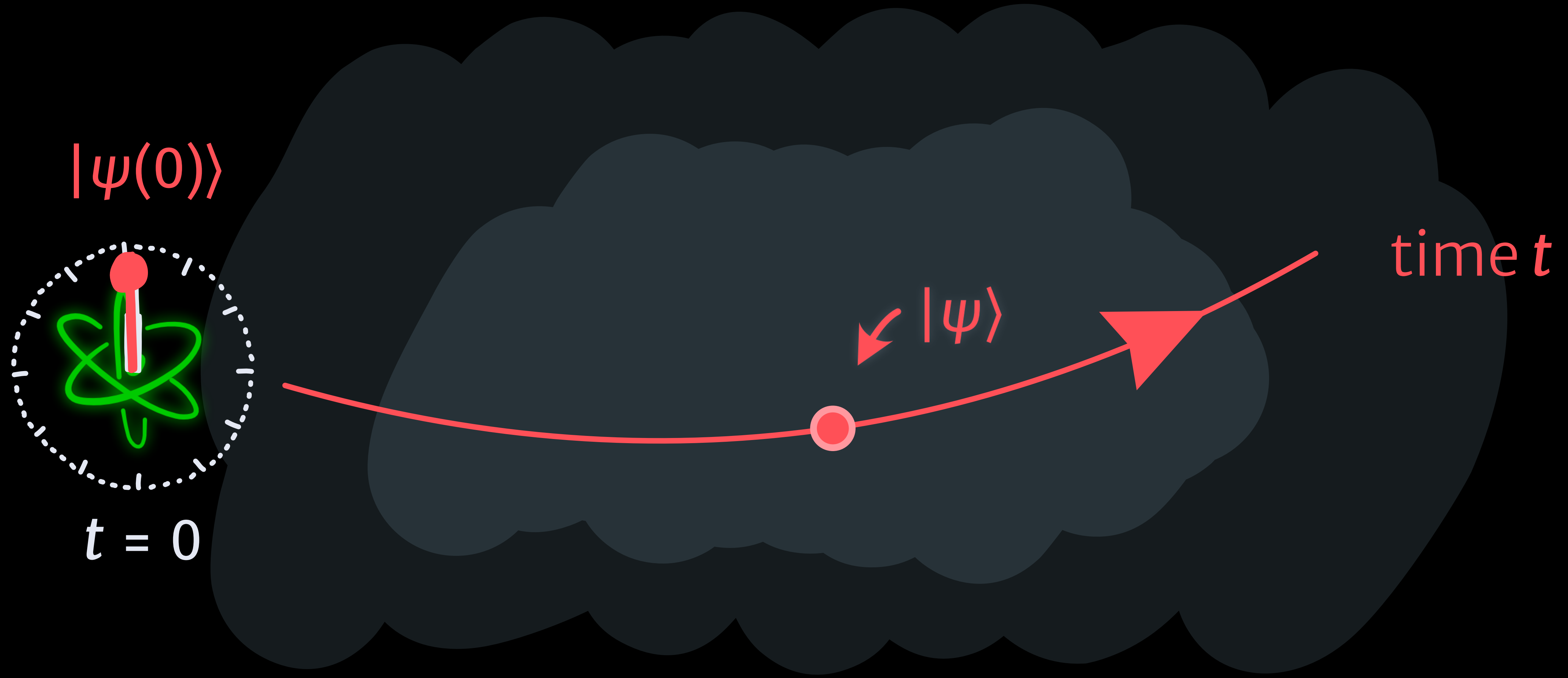
Bob

$t \approx t_0$

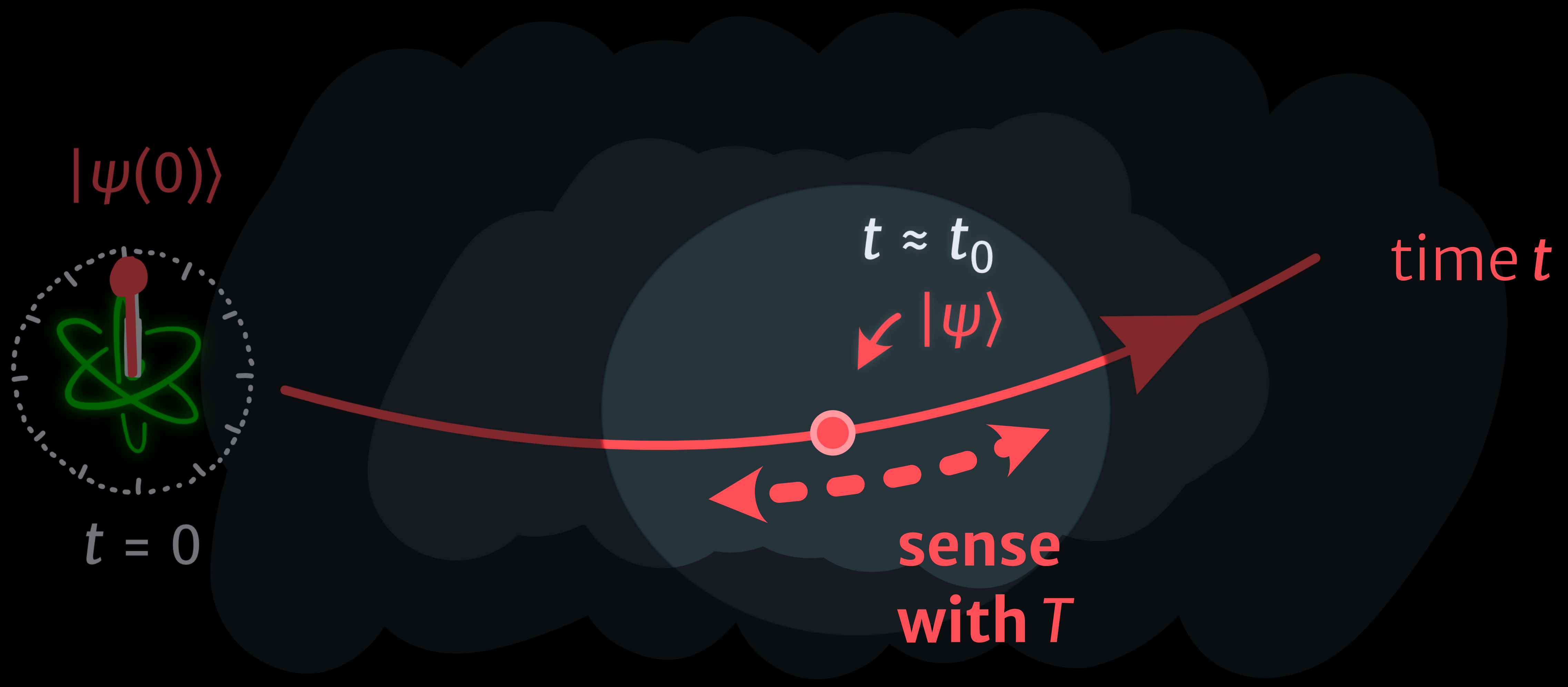

Alice
$t \approx t_0$
$\mathcal{N}$
Bob
complementary channel
$\hat{\mathcal{N}}$
Eve



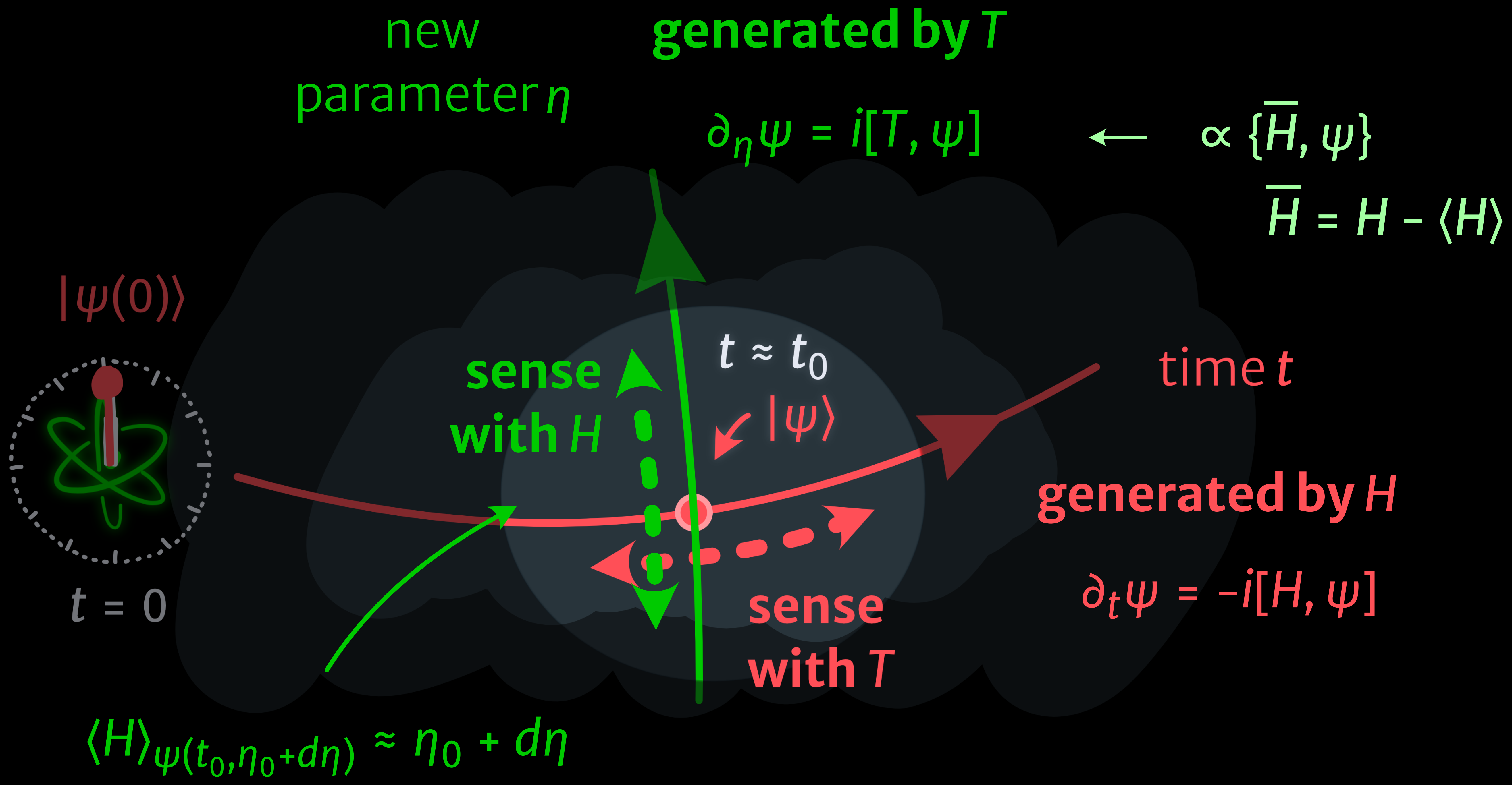
Information about energy gained by the Environment (Eve) should trade off with Bob's sensitivity to time.



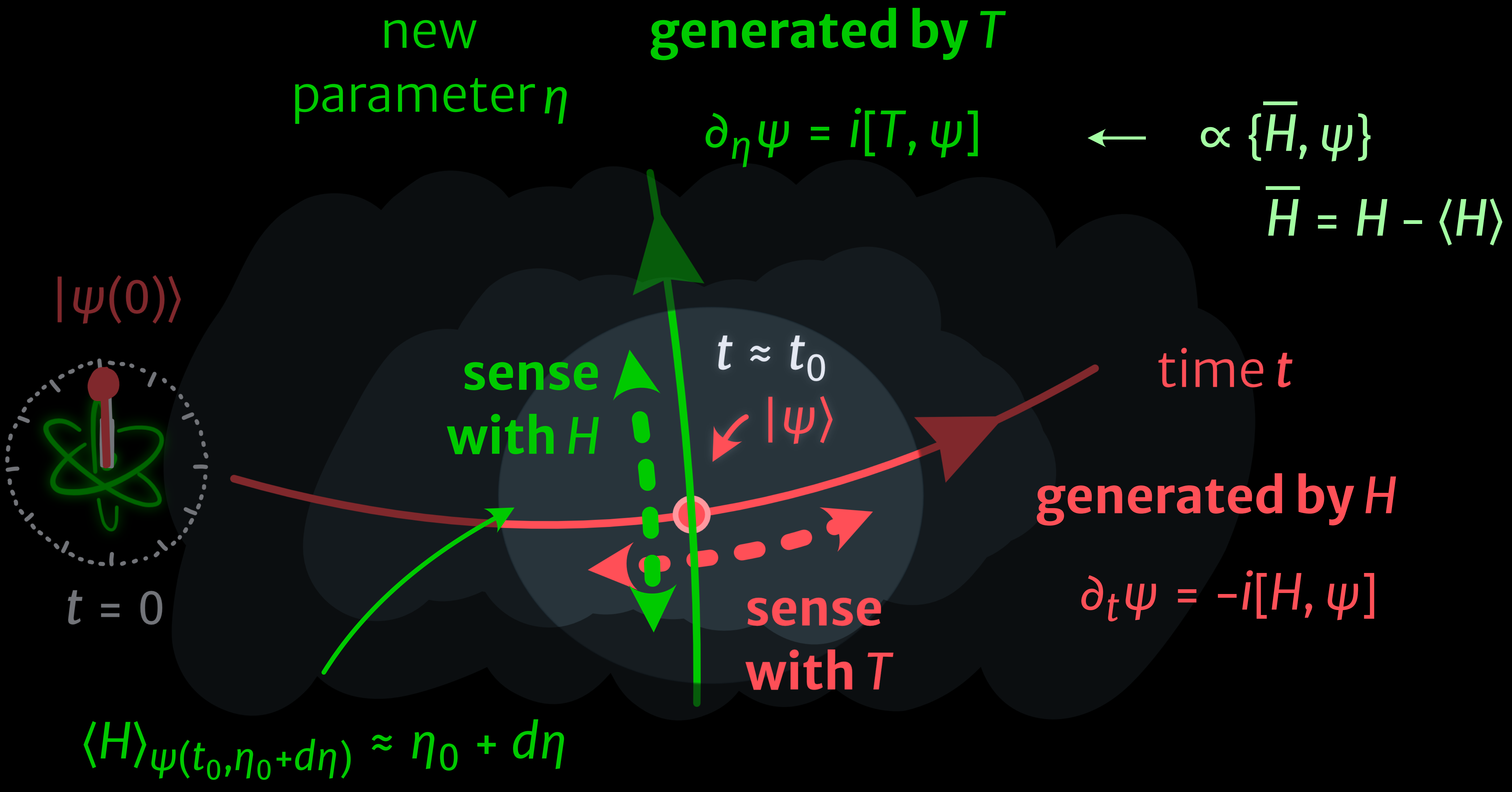
$|\psi(0)\rangle$
$|\psi\rangle$
time t
$t = 0$



$|\psi(0)\rangle$
$t = 0$
$t \approx t_0$
$|\psi\rangle$
time t
sense
with T



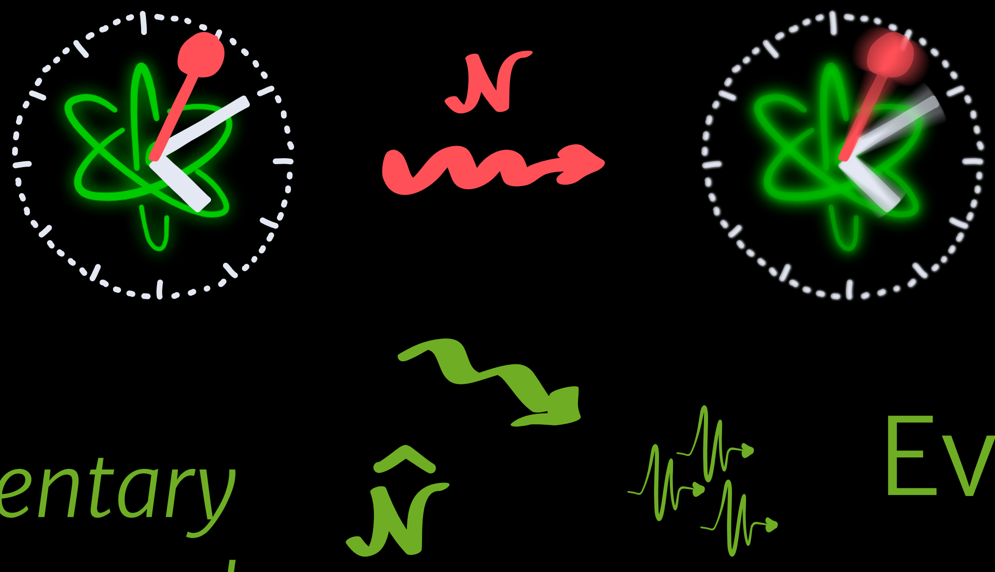
new
parameter η
generated by T
∂_η ψ = i[T, ψ]
← ∝ {H̄, ψ}
H̄ = H − ⟨H⟩
|ψ(0)⟩
sense
with H
t ≈ t_0
|ψ⟩
time t
generated by H
∂_t ψ = −i[H, ψ]
t = 0
sense
with T
⟨H⟩_ψ(t_0,η_0+dη) ≈ η_0 + dη



We identify a parameter η that represents **energy**, which is complementary to **time** t locally at $|\psi\rangle$

Alice

$t \approx t_0$



Bob

$$F_{\mathrm{Bob},t} = F\left(\mathcal{N}(\psi), \partial_t \mathcal{N}(\psi)\right)$$

complementary channel

Eve

$$F_{\mathrm{Eve},\eta} = F\left(\hat{\mathcal{N}}(\psi), \partial_\eta \hat{\mathcal{N}}(\psi)\right)$$

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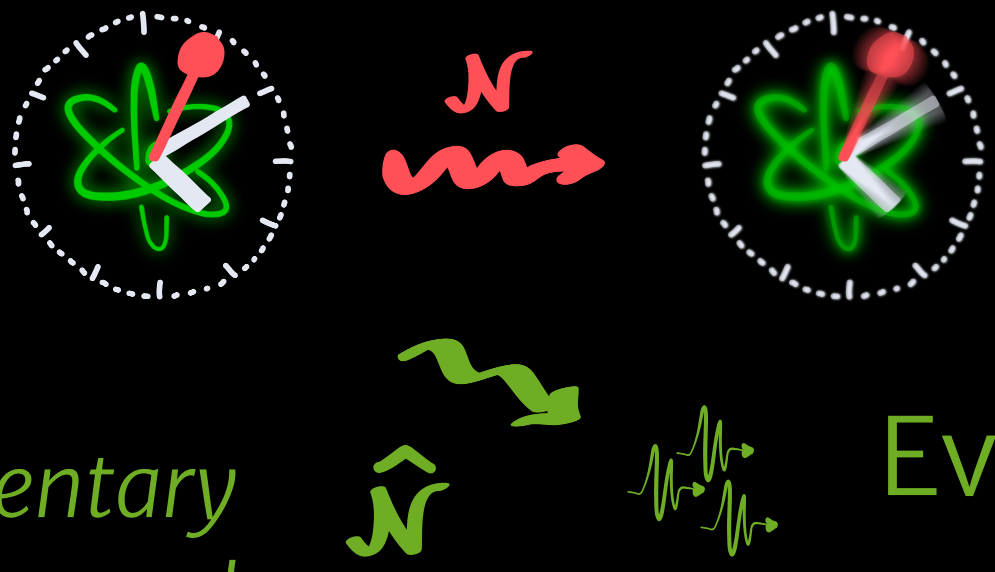
Main Result

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 1$$

Fisher information trade-off

Alice

$t \approx t_0$



Bob

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Bob's sensitivity to time

Eve's sensitivity to energy

Main Result

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Fisher information trade-off

normalization makes units consistent

Example: single qubit

Proof sketch

Useful bounds on the Fisher information of noisy states

Fisher information uncertainty relation for any two parameters

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 1$$

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: Ising spin chain with amplitude–damping noise

Example: single qubit

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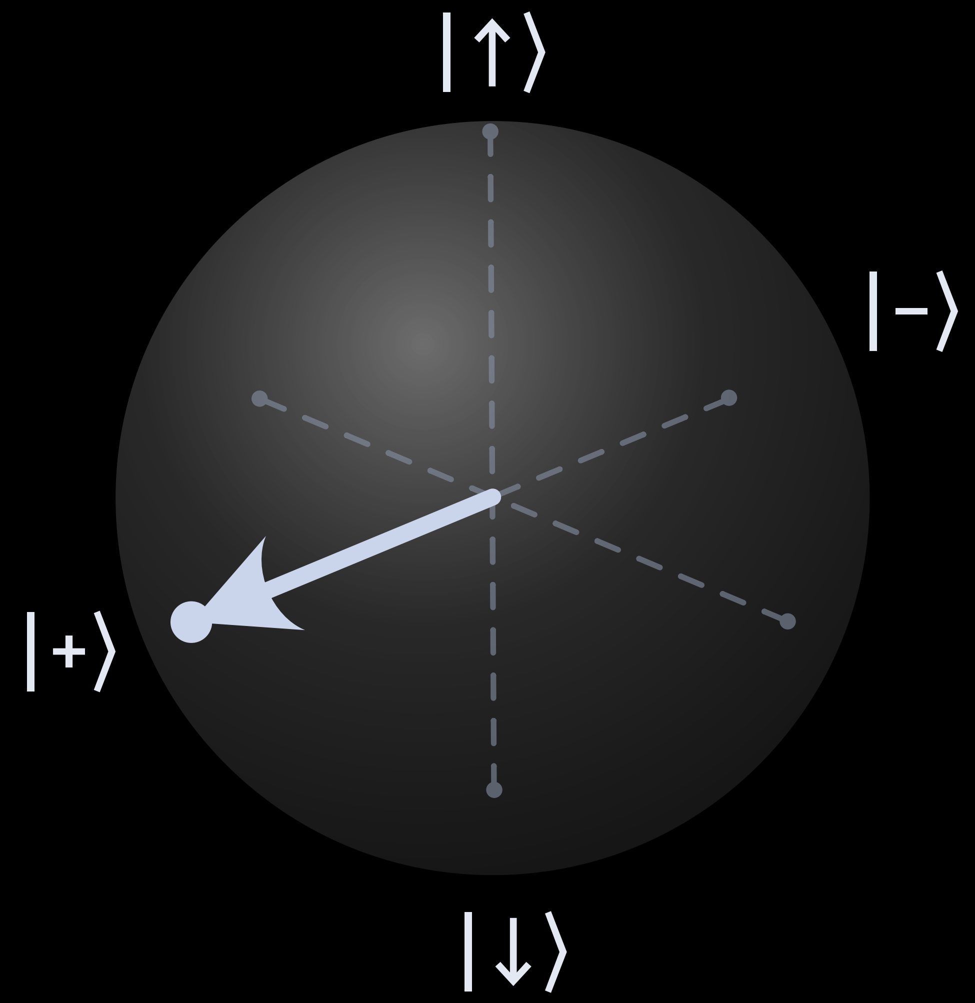
Example: 1 qubit

$$H = \frac{\omega}{2}\sigma_Z \quad |\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + |\downarrow\rangle\right] = |+\rangle$$

$$4\langle H^2\rangle = \omega^2$$



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Evolve for t_0:

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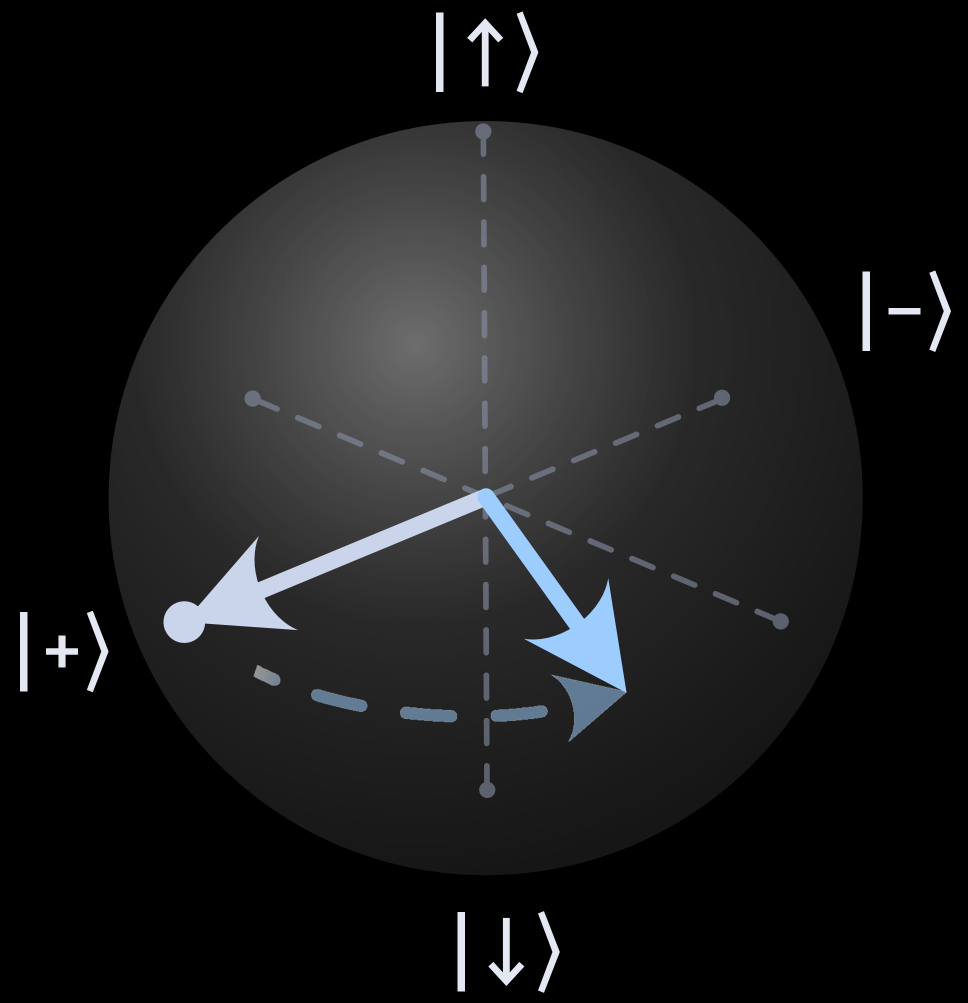

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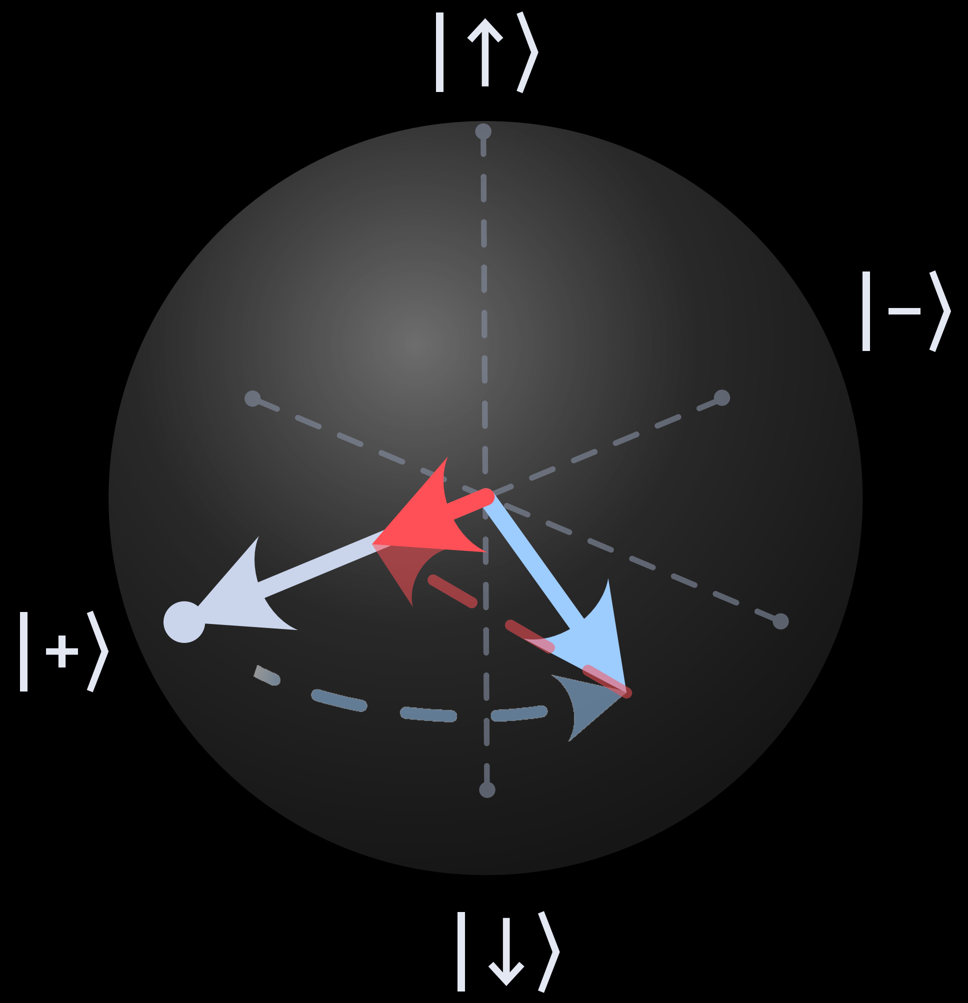
Apply dephasing along the X axis

$$\rho_{\mathrm{Bob}} = \cos^2\left(\frac{\omega t}{2}\right)|+\rangle\langle+| + \sin^2\left(\frac{\omega t}{2}\right)|-\rangle\langle-|$$



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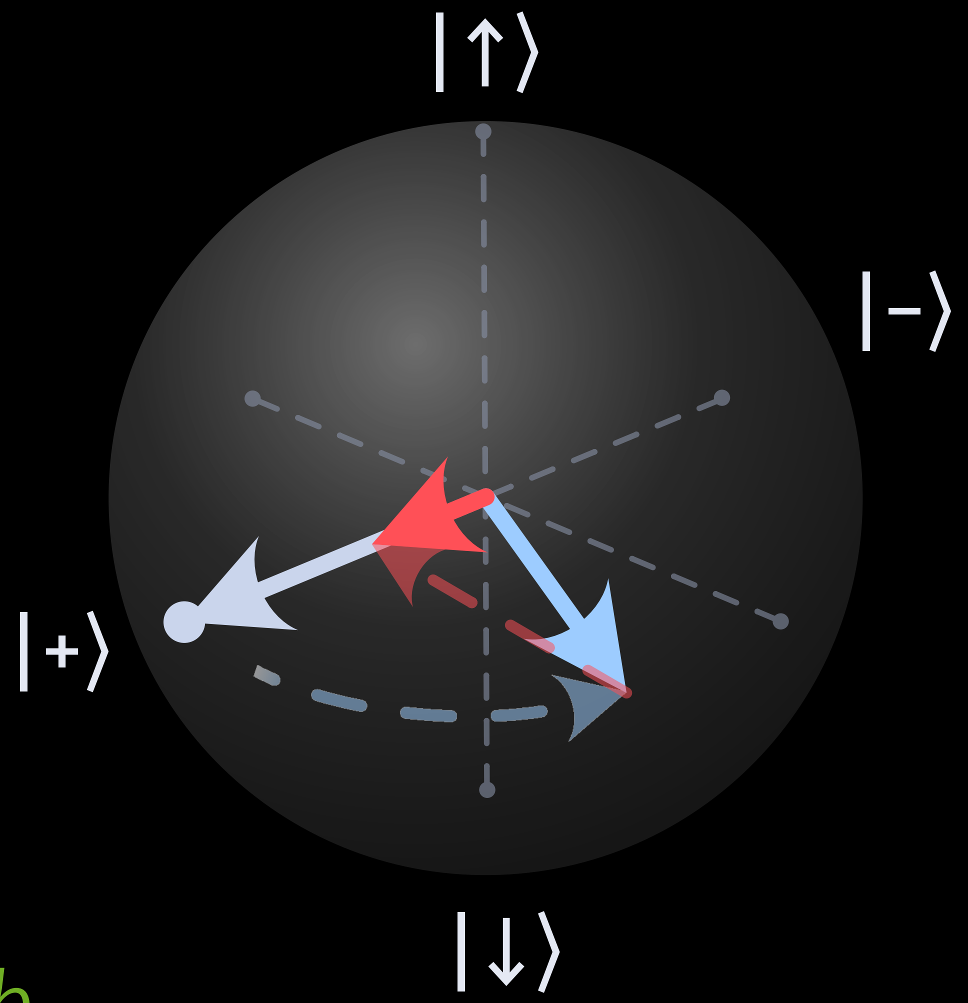
Eve's sensitivity to energy

$$F_{\mathrm{Eve},\eta} \propto F(\hat{\mathcal{N}}(\psi); \hat{\mathcal{N}}(\underbrace{\{\overline{H}, \psi\}}_{\propto\,\sigma_Z})) = 0$$

→ *zero loss of sensitivity for Bob*



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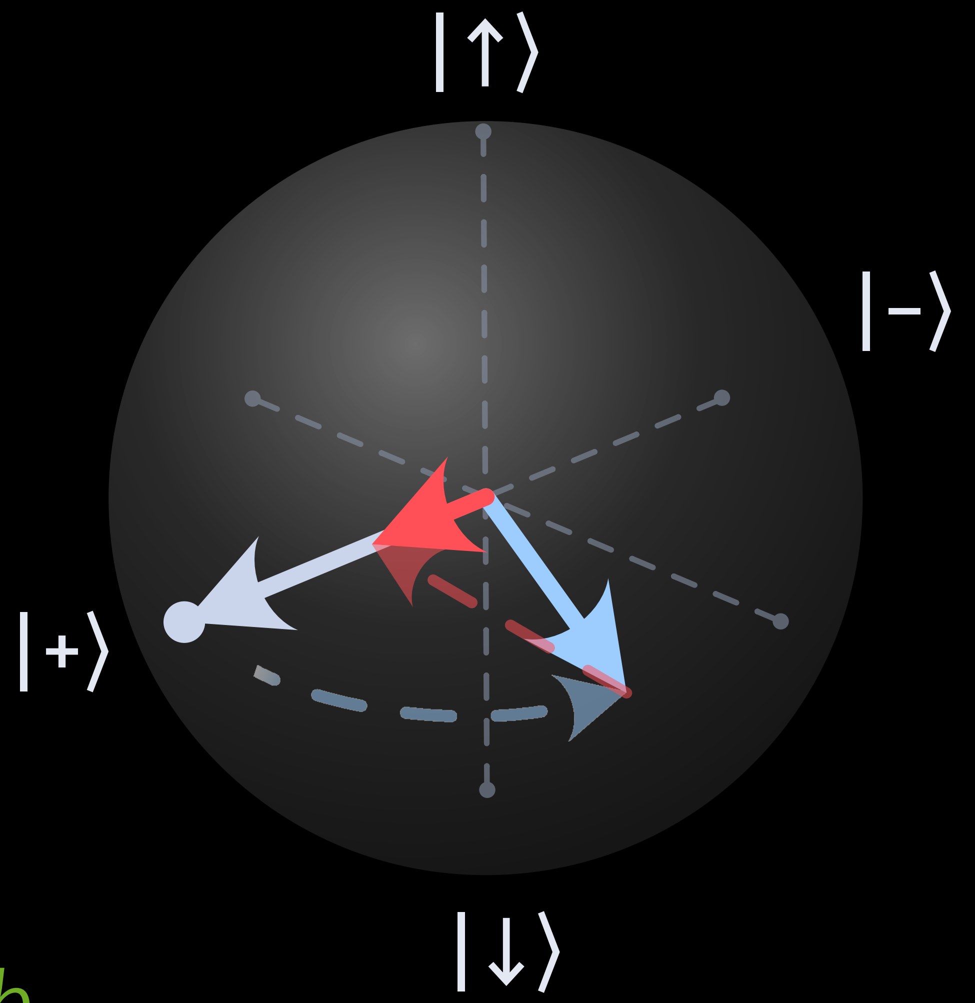

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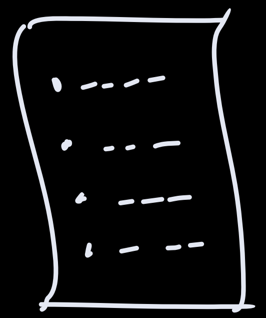
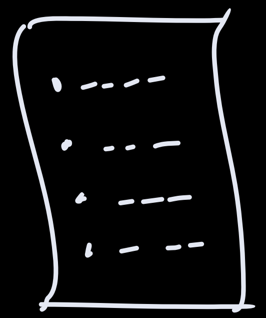
Direct calculation: $F_{\mathrm{Bob},t} = \omega^2 = F_{\mathrm{Alice},t}$ ✓



Main proof ingredients

$$\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1$$



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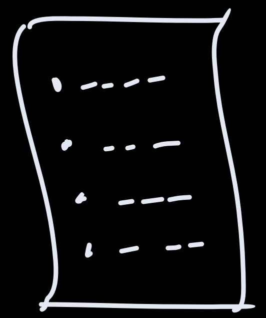
Variational characterization of the Fisher information

$$\frac{1}{4}F(\rho; D) = \max_{S=S^\dagger} \left[\mathrm{tr}(DS) - \mathrm{tr}(\rho S^2)\right]$$

$$= \min \left\{\mathrm{tr}(LL^\dagger)\colon\ \rho^{1/2}L + L^\dagger\rho^{1/2} = D\right\}$$

Macieszczak 1312.1356; Holevo 2011; …



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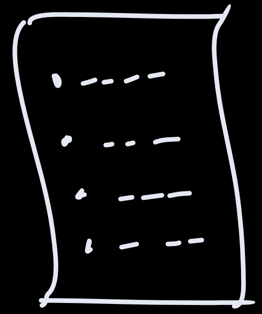
Connection to Fidelity / Bures metric

$$F(\rho(t_0); \partial_t\rho(t_0)) = -8 \left.\frac{d^2}{dt^2}\right|_{t_0} F(\rho(t_0), \rho(t))$$

Fidelity of quantum states



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Uhlmann's theorem:

$$F(\mathcal{N}(\psi), \mathcal{N}(\psi')) = \max_{W_E} \left|\langle \psi | V^\dagger (\mathbb{1} \otimes W_E) V | \psi' \rangle\right|$$

unitary on environment

dilation of $\mathcal{N}$

optim. over W_E

↔ optim. over S

Example: single qubit

Proof sketch

Useful bounds on the Fisher information of noisy states

Fisher information uncertainty relation for any two parameters

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 1$$

Necessary and sufficient conditions for zero sensitivity loss

"metrological codes"

Example: Ising spin chain with amplitude-damping noise


$\mathcal{N}$
Bob
$\hat{\mathcal{N}}$
Eve

$\mathcal{N}$
Bob
$\hat{\mathcal{N}}$
Eve
$\mathcal{N}'$
Eve'

$$F_{\mathrm{Eve}'}(\eta) \leqslant F_{\mathrm{Eve}}(\eta)$$

e.g., decohere in fixed basis
→ simpler bound for
amplitude damping noise

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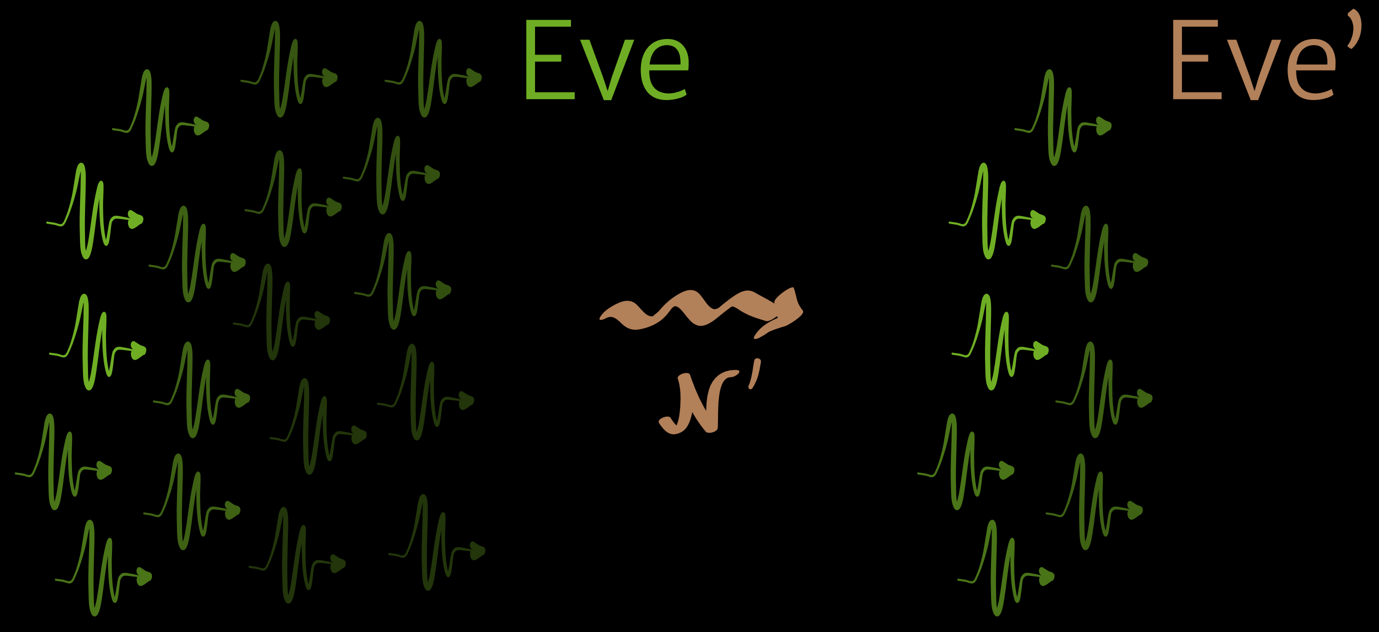
$$\rightarrow \quad \frac{F_{\text{Bob}}(t)}{F_{\text{Alice}}(t)} \leqslant 1 - \frac{F_{\text{Eve}'}(\eta)}{F_{\text{Alice}}(t)}$$

We find bounds on the quantum Fisher information of mixed states that might be simpler to compute (e.g., because the state is diagonal).

Example: keep only high probability error events



E_k: $k = 1, \dots, m$ E_k: $k = 1, \dots, m'$

$m' \ll m$

Hilbert space of Eve' is smaller than that of Eve → quantum Fisher information (QFI) is easier to compute.

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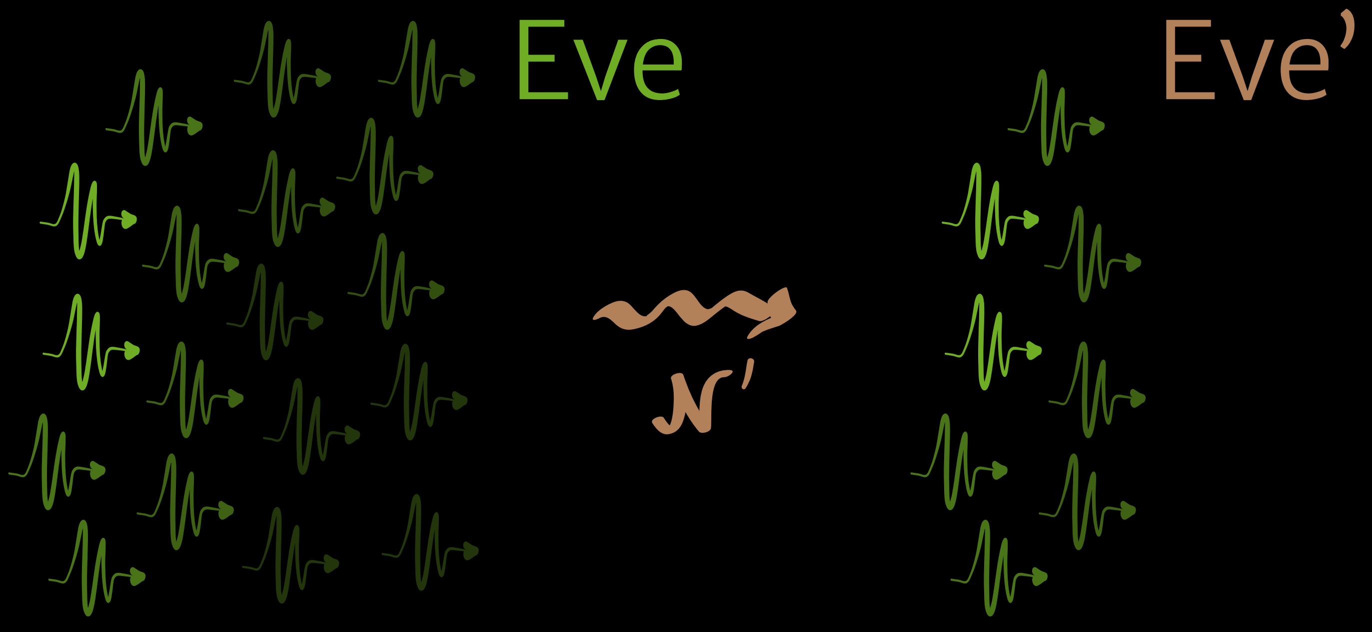

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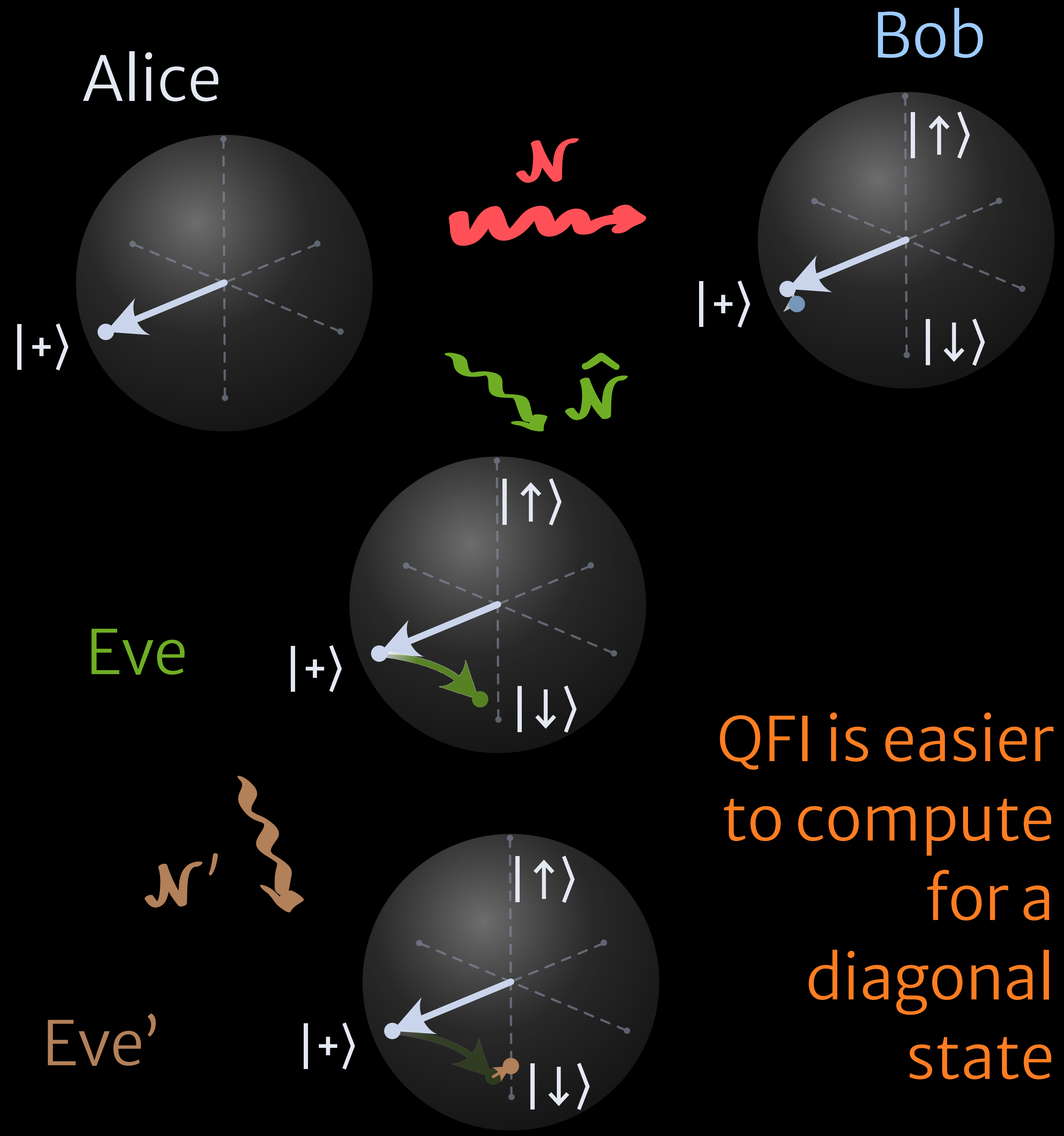
Example: dephasing for amplitude-damping noise



QFI is easier to compute for a diagonal state

Example: single qubit

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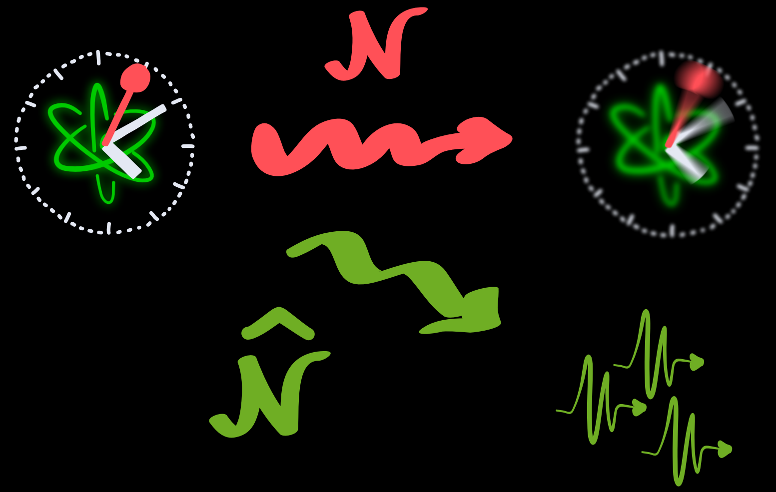
Example: Ising spin chain with amplitude-damping noise

uncertainty relation for any two generators A, B:

$$\partial_a \psi = -i[A, \psi]$$
$$\partial_b \psi = -i[B, \psi]$$



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$$\sigma_X^2 = \langle X^2\rangle - \langle X\rangle^2$$

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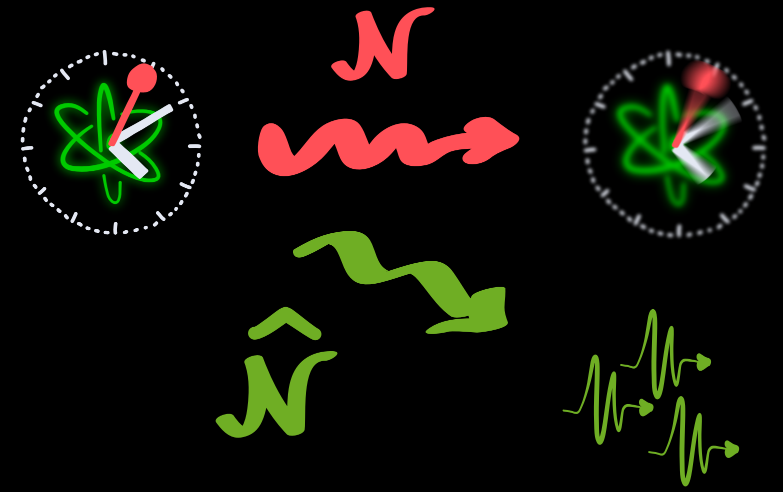

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(e.g., energy H & time T)

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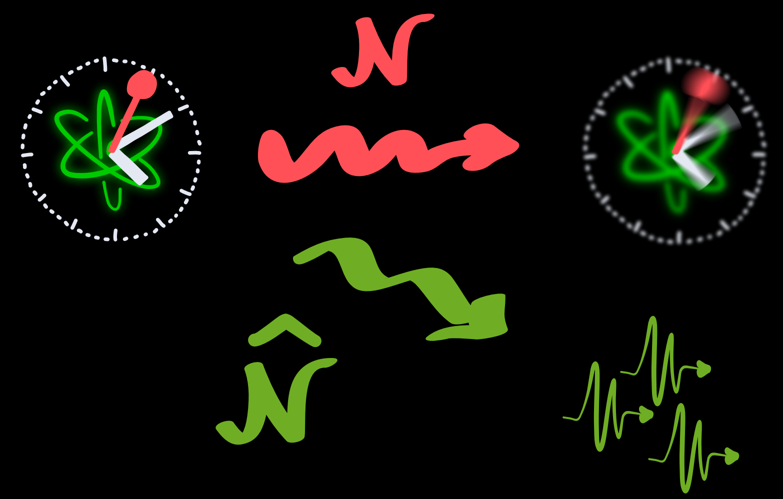

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Fisher information trade-off relation that depends on incompatibility of A,B

Example: single qubit

Proof sketch

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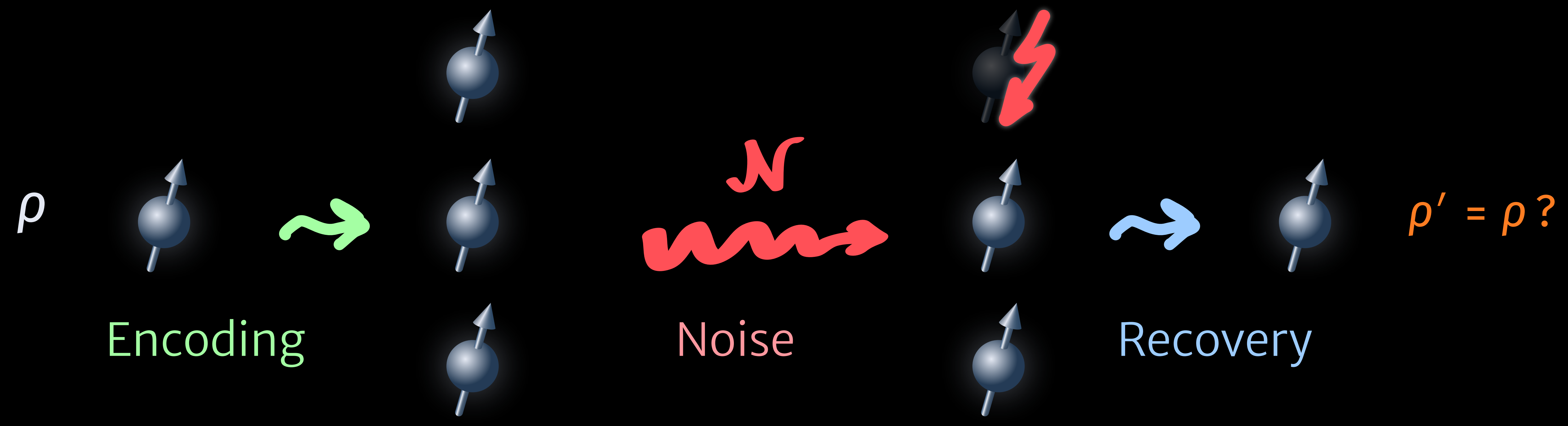
Example: Ising spin chain with amplitude–damping noise

Quantum Error Correction protects quantum states from noise.



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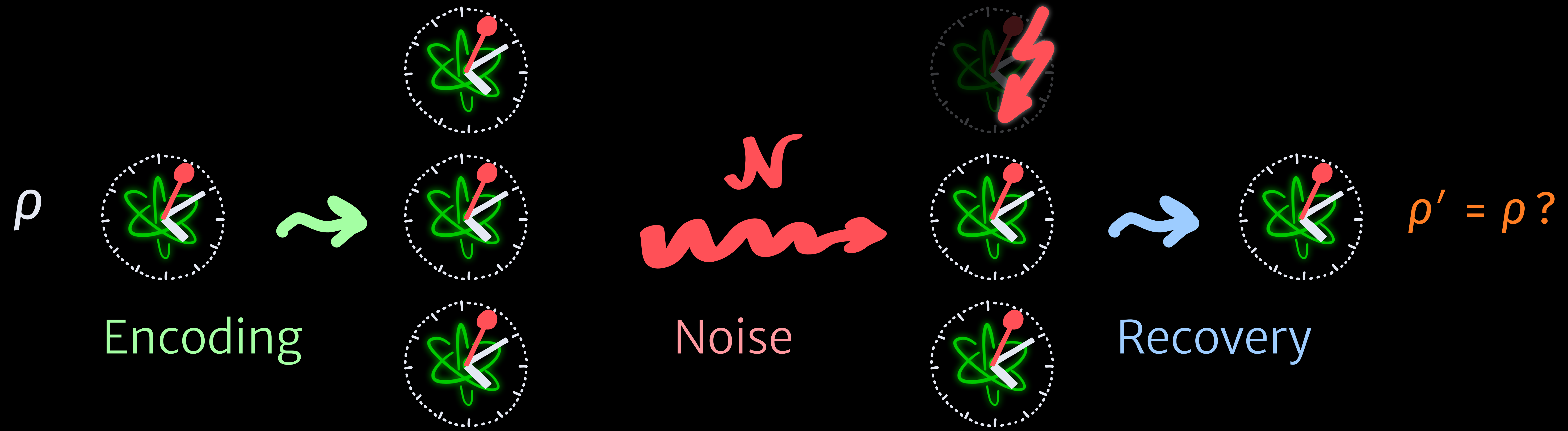

Can Quantum Error Correction protect clocks from noise?

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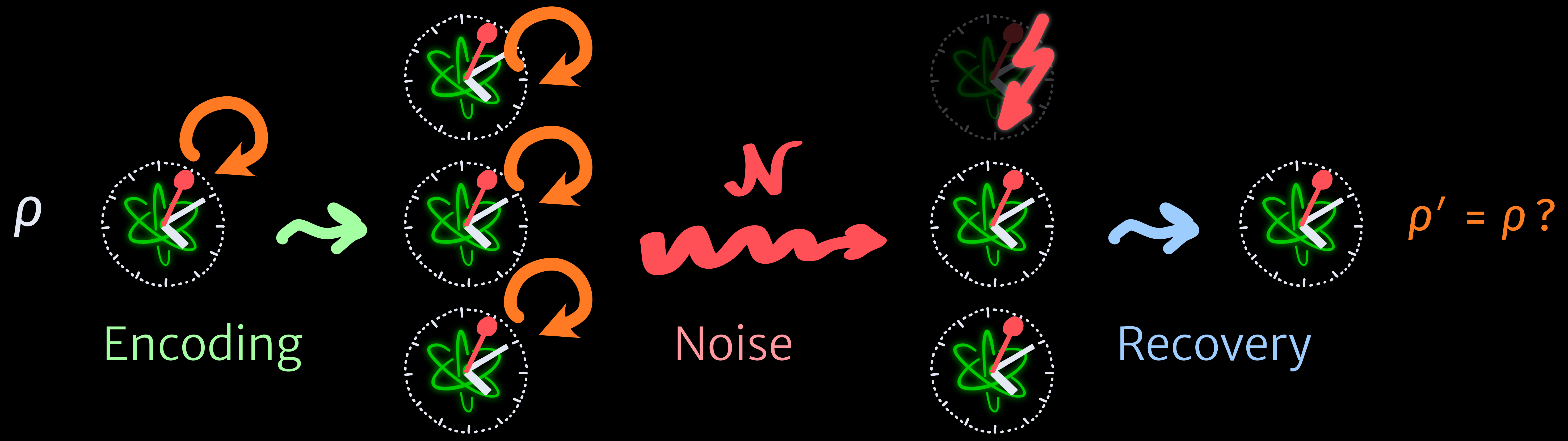

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The code must be time-covariant.

Hayden *et al.*, 1709.04471

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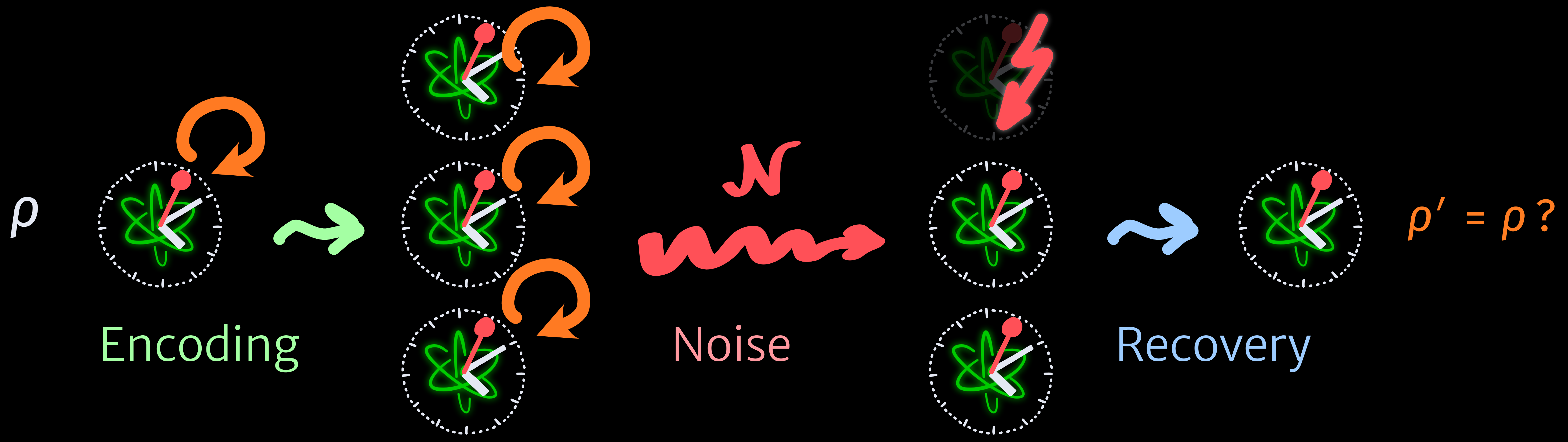

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Covariant codes that can correct local errors don't exist! (Eastin–Knill)

Eastin & Knill PRL 2009

What if we recover ρ only approximately?

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logical Hamiltonian

$$\epsilon \ge \frac{\Delta H_L}{2n \max_i \Delta H_i}$$

number of subsystems

Hamiltonian of i–th subsystem

PhF *et al.* PRX 2020
Woods & Alhambra *Quantum* 2020
Kubica *et al.* 2004.11893
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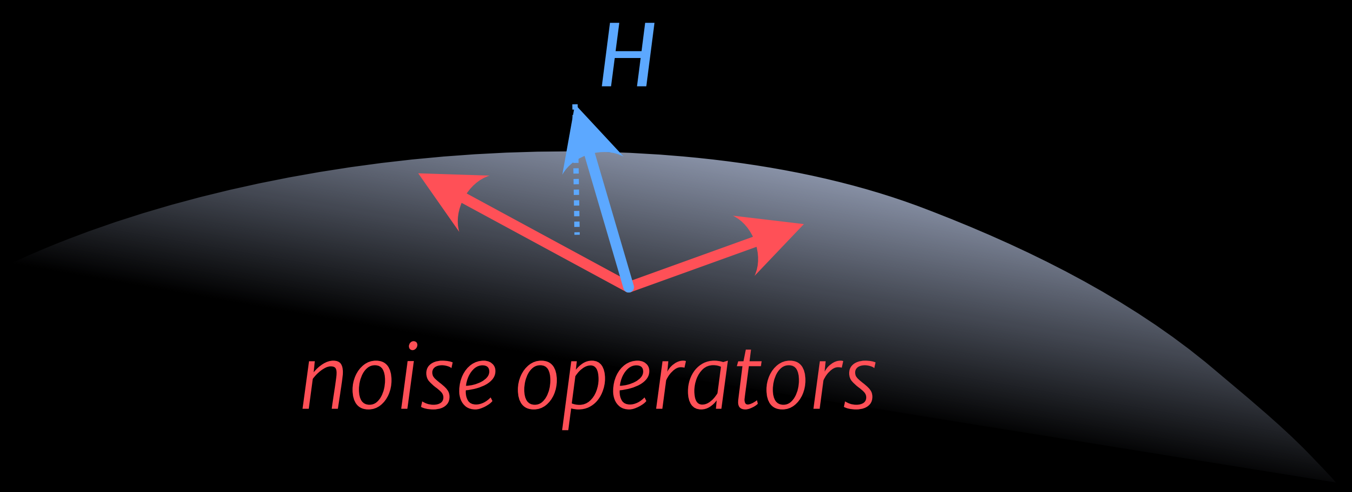
Our trade–off relation transposes this inequality to the quantum Fisher information.

$$\frac{F_{\mathrm{Bob},t}}{F_{\mathrm{Alice},t}} + \frac{F_{\mathrm{Eve},\eta}}{F_{\mathrm{Alice},\eta}} = 1$$

Quantum error correction can help if the Hamiltonian is not aligned with the noise.



Achieve same sensitivity scaling as without noise (Heisenberg scaling)

⇔

The signal Hamiltonian is not parallel to the noise (Hamiltonian not in Kraus span)

“is it the signal or is it noise?”

Demkowicz-Dobrzański+ N. Comm. 2011; Escher+ N. Phys 2011; Kessler+ / Arrad+ / Dür+ PRL 2014; Zhou+ N. Comm. 2018; Layden+ PRL 2019; …

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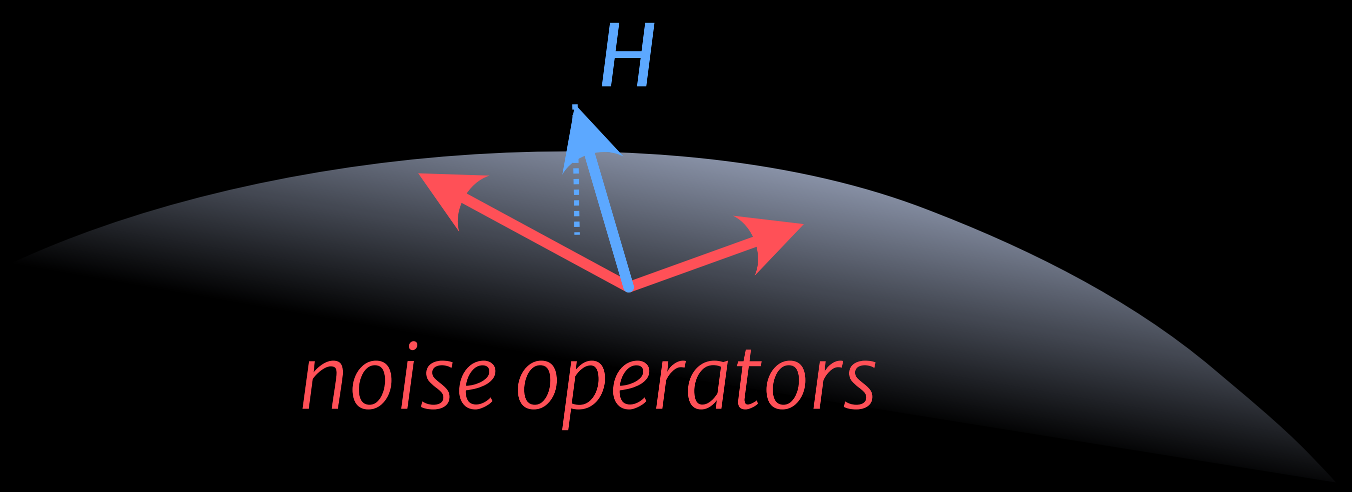

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Do we need full error correction or will a simpler scheme suffice?

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If the second term vanishes, the noisy clock has the same sensitivity as the noiseless one.

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noise operators

$$\Leftrightarrow \quad \mathrm{tr}\left[Z\,\Pi\,E_{k'}^{\dagger}E_k\,\Pi\right] = 0$$

"code qubit" spanned by

$|\psi\rangle =: |+\rangle$ and $H|\psi\rangle$

Π = projector on that space

Z = logical Pauli–Z operator

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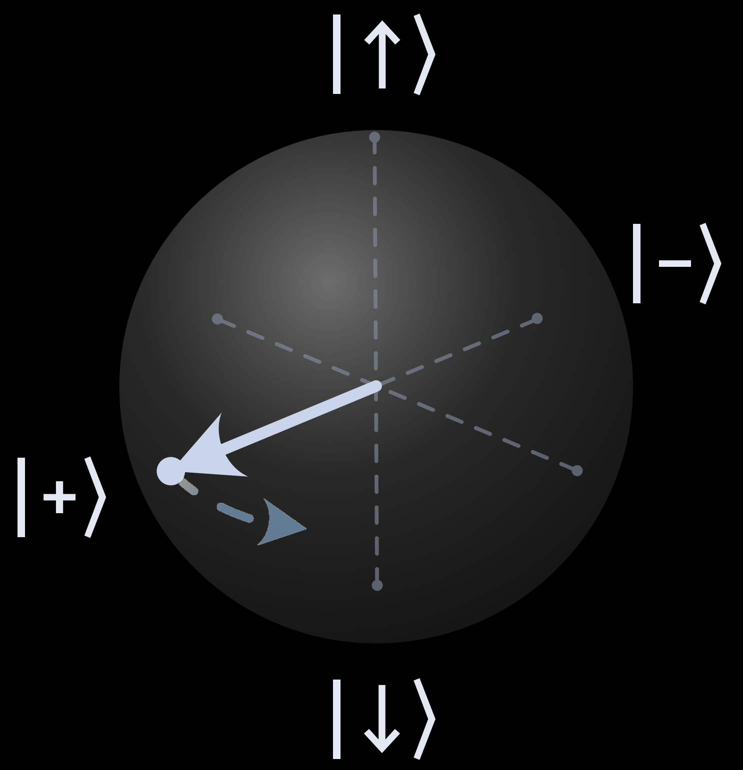

Corresponding quantum error correction conditions

$$\Pi\,E_{k'}^{\dagger}E_k\,\Pi \propto \Pi$$

Knill & Laflamme
PRA 1997

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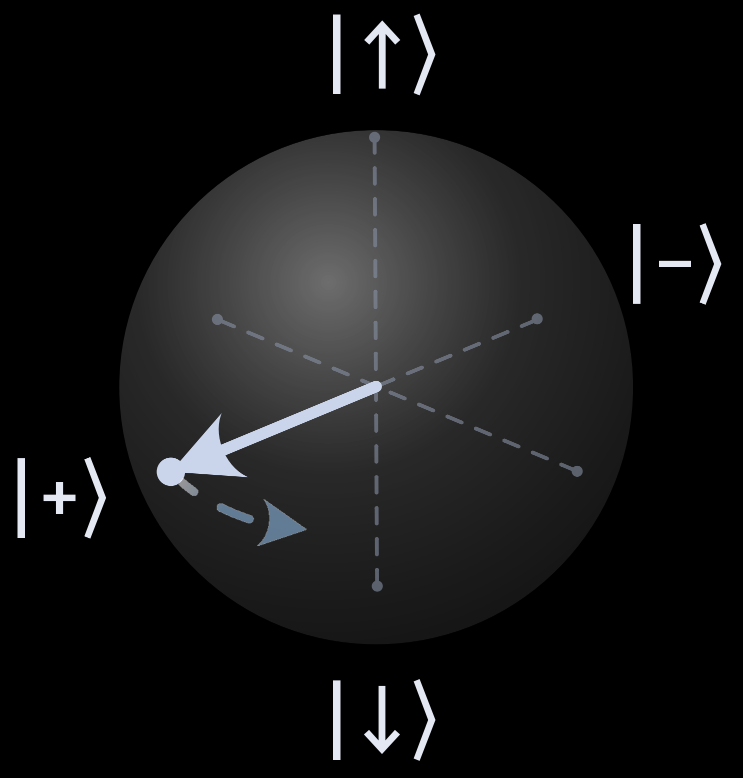

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Knill & Laflamme PRA 1997

"metrological codes" are more general than error-correcting codes.

Our earlier example was a metrological code.

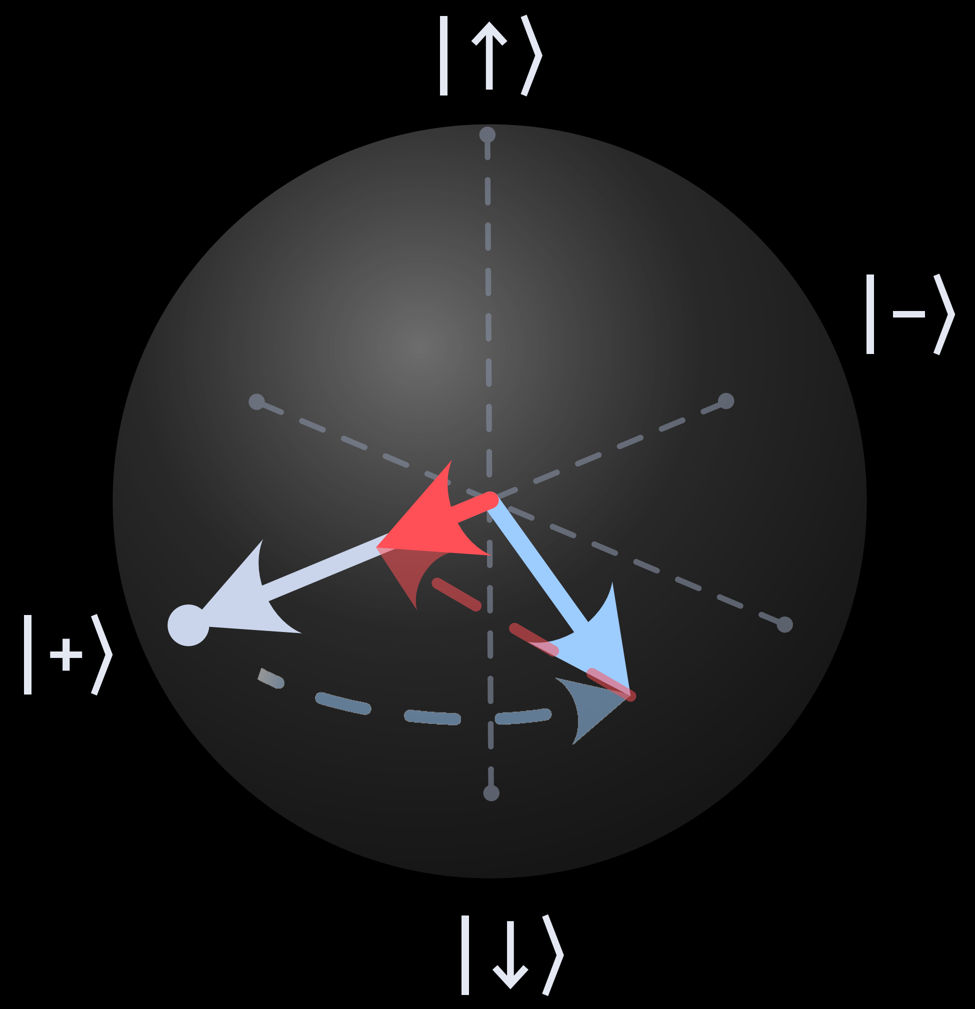
$$H = \frac{\omega}{2}\sigma_Z \quad |\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + |\downarrow\rangle\right] = |+\rangle$$

dephasing along X

$$\rho_{\mathrm{Bob}} = \cos^2\left(\frac{\omega t}{2}\right)|+\rangle\langle+| + \sin^2\left(\frac{\omega t}{2}\right)|-\rangle\langle-|$$

$$F_{\mathrm{Eve},\eta} \propto F(\hat{\mathcal{N}}(\psi); \hat{\mathcal{N}}(\underbrace{\{\overline{H}, \psi\}}_{\propto\, \sigma_Z})) = 0$$

→ *zero loss of sensitivity for Bob*



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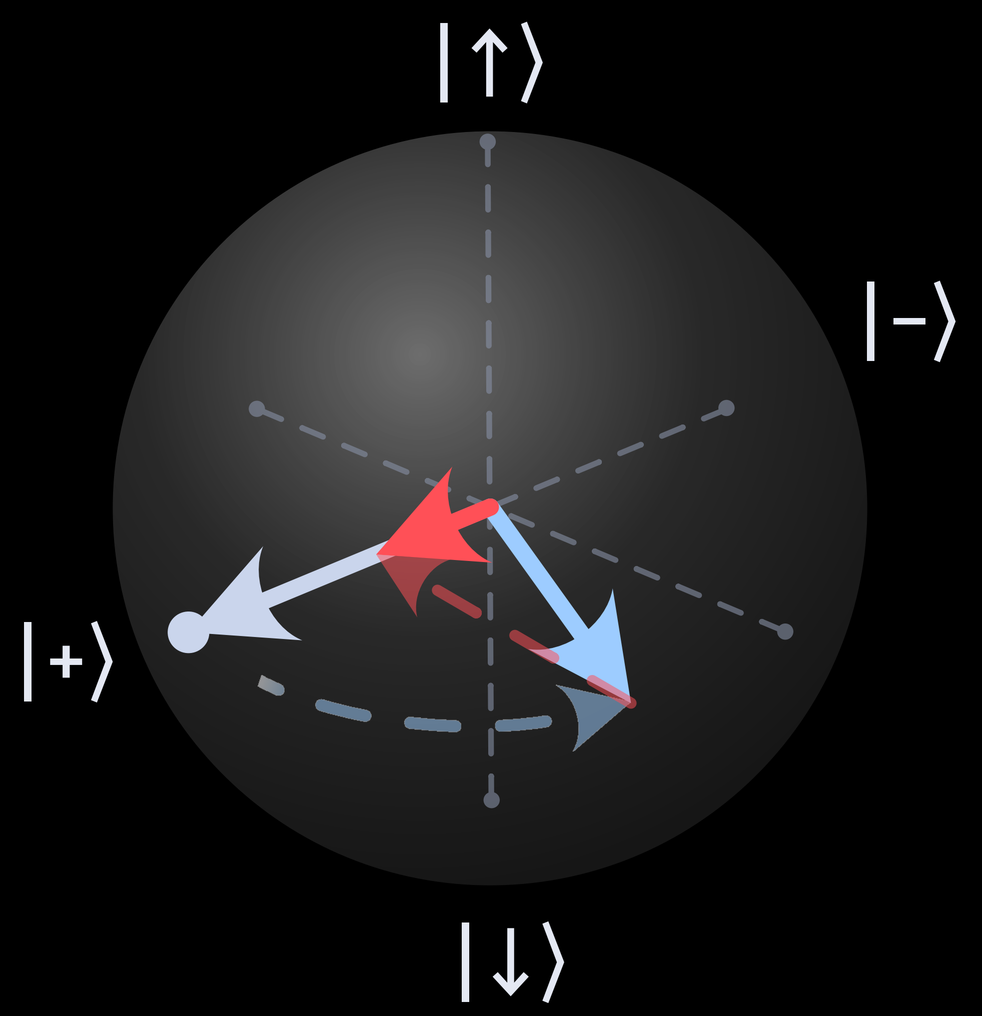
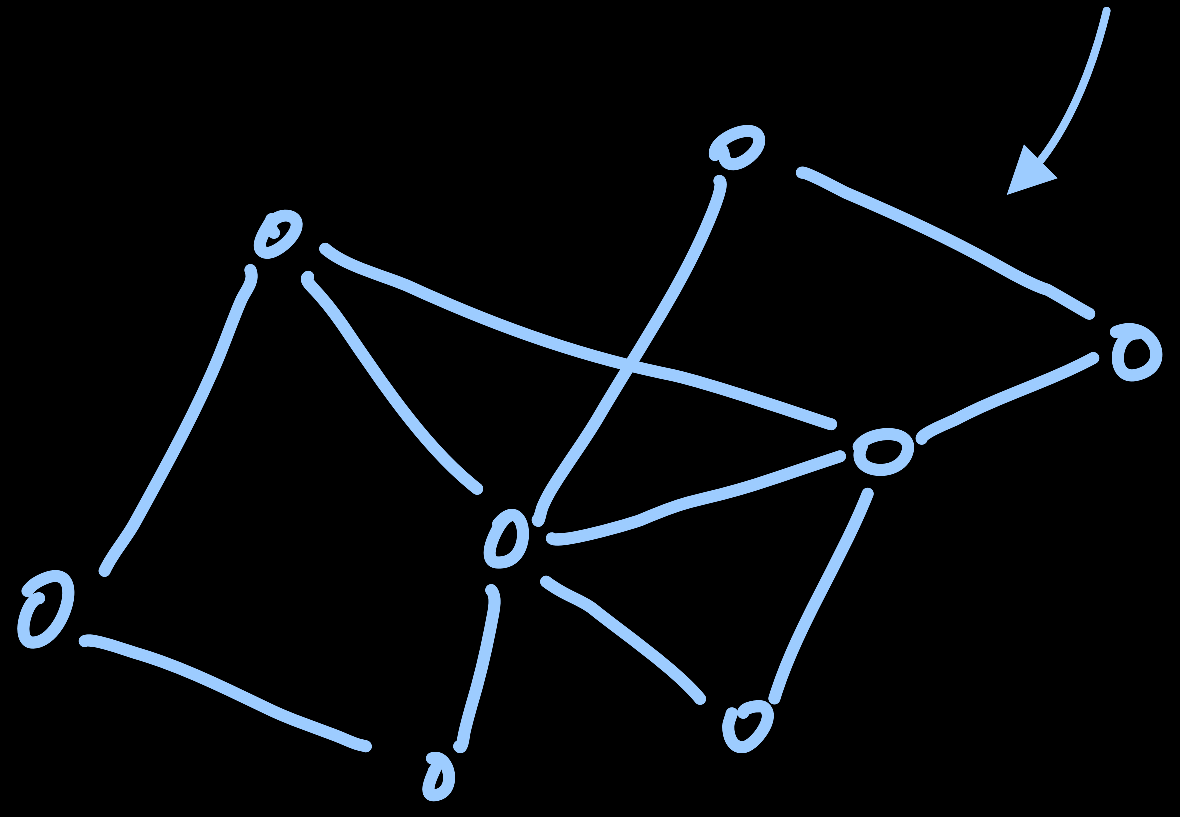


metrological code conditions

$$\mathrm{tr}\big[Z\,\Pi\,\underbrace{E_{k'}^\dagger E_k}_{\mathbb{1},\,X}\,\Pi\big] = 0$$ ✔

Example: Many-body system with Ising-type interactions

ZZ interactions
(can also include XX & YY)



cf. e.g. Ouyang IEEE TIT 2022; ...

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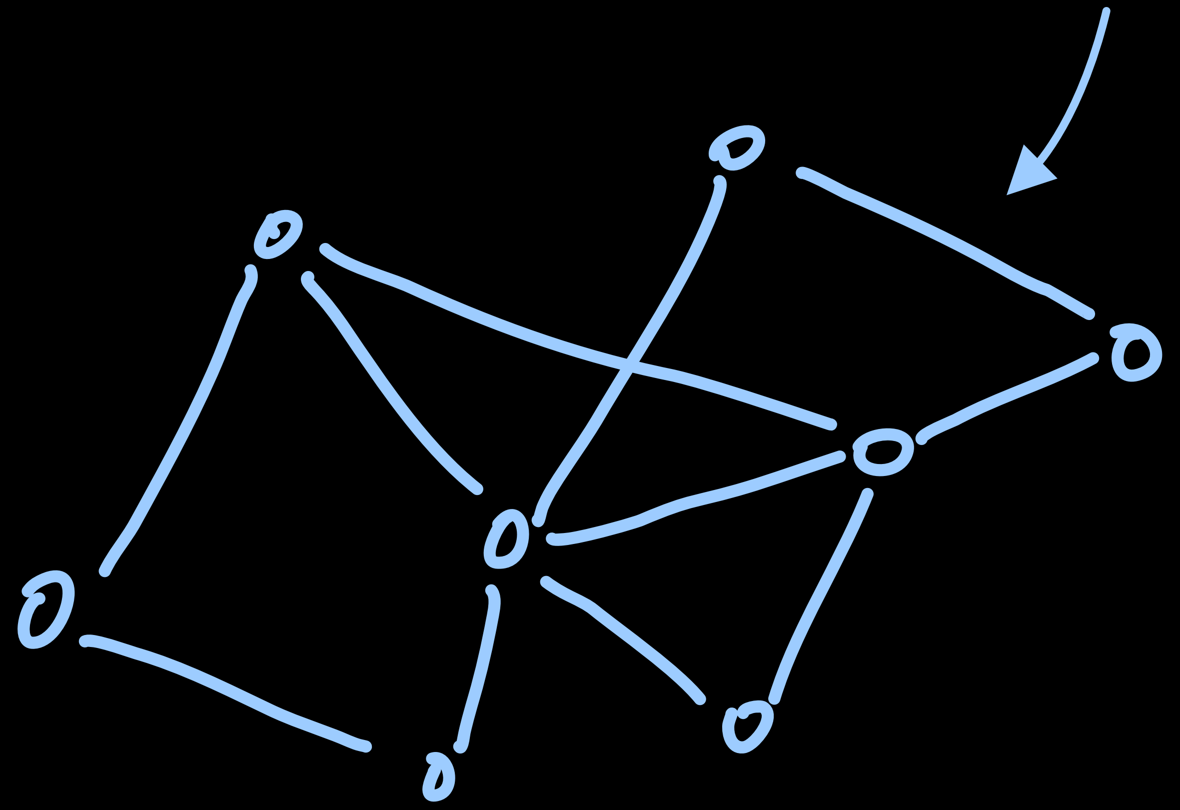

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$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\ldots0\rangle + |c^n\rangle\right]$$

c^n = configuration that violates many interaction terms

$$\mathrm{tr}\left[Z\,\Pi\,E_{k'}^\dagger E_k\,\Pi\right] \neq 0$$ ✗

prone to sensitivity loss

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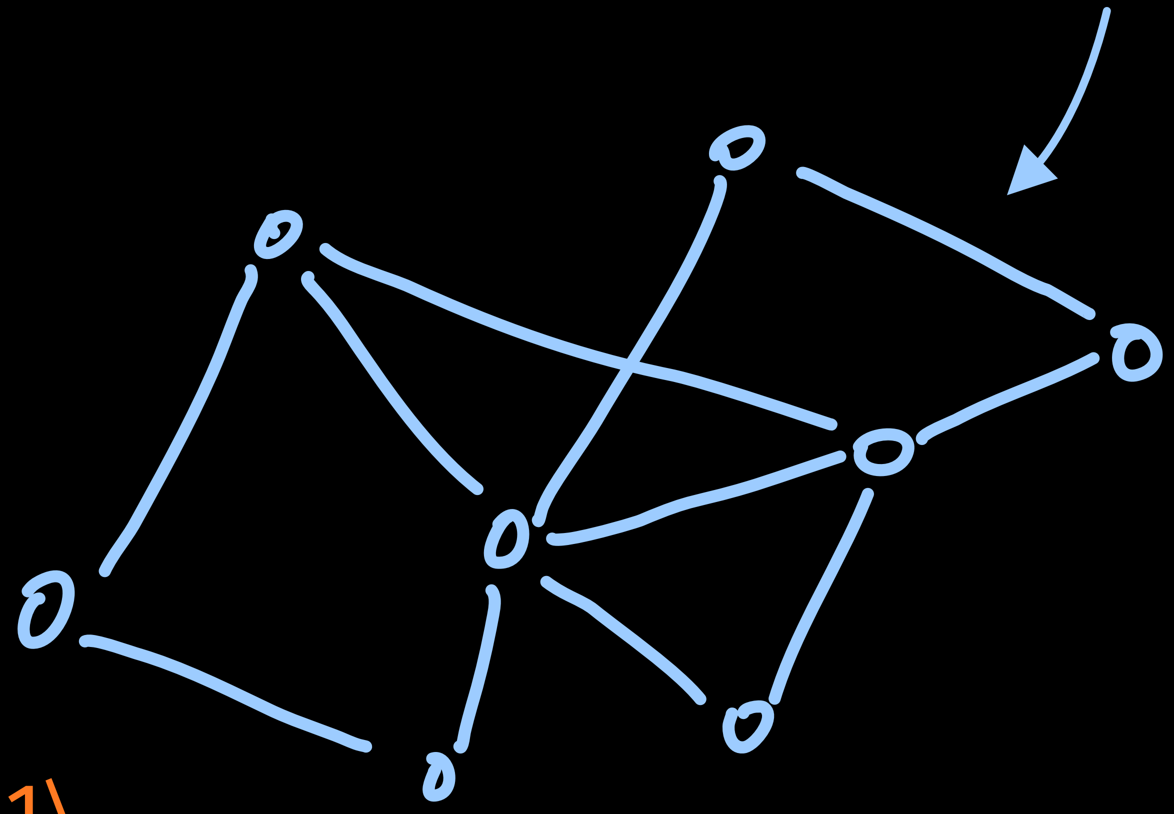

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no sensitivity loss in case of a single localized error

Example: single qubit

Proof sketch

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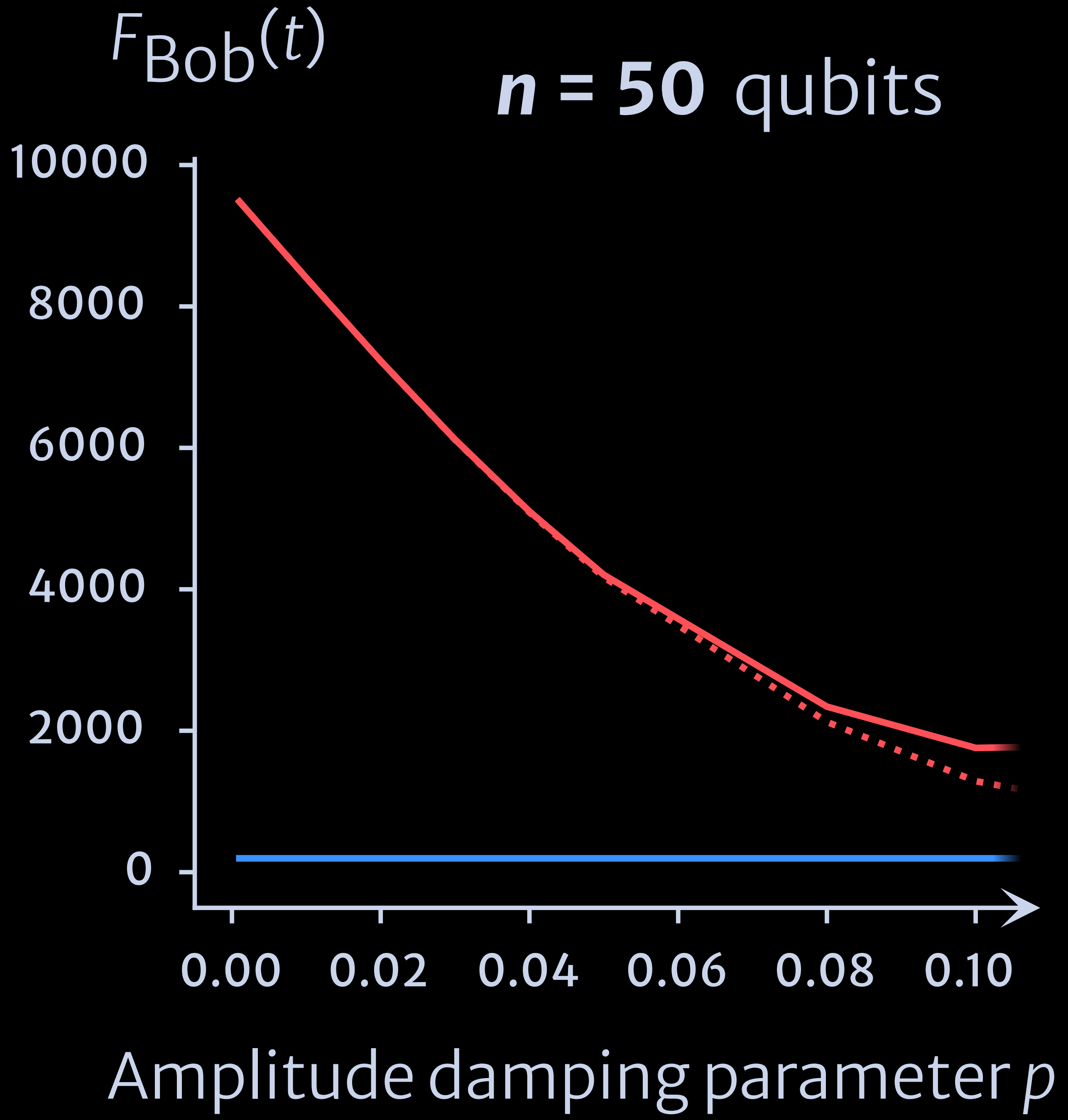
1D spin chain



ferromagnet–antiferromagnet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|00\ldots00\rangle + |01\ldots01\rangle\big)$$

spin–coherent (clock) state

$$|\psi\rangle = |+\rangle^{\otimes n}$$

$$H = \sum \sigma_Z^{(j)} \sigma_Z^{(j+1)}$$

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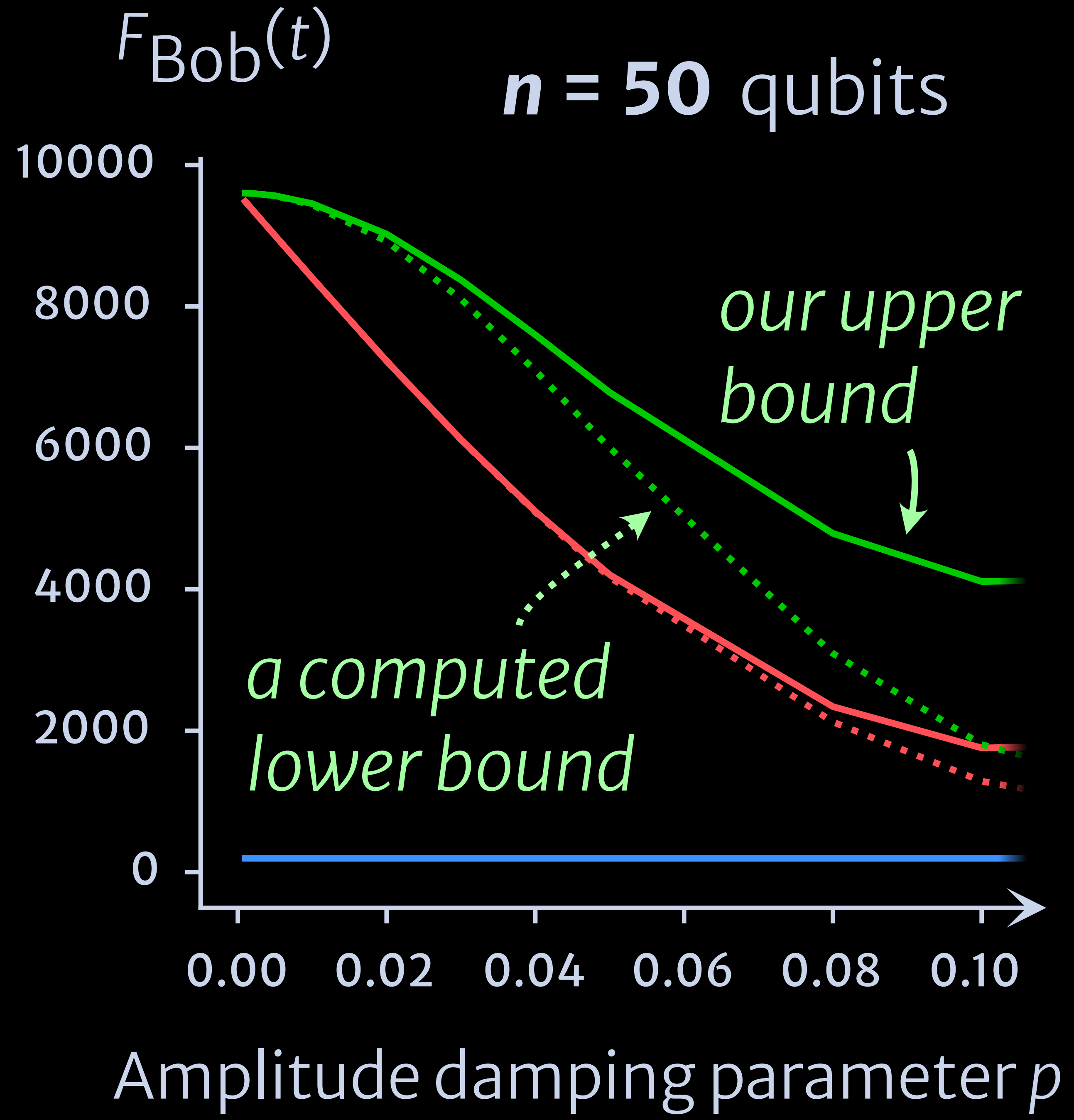

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"metrological code state"

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Example: Ising spin chain with amplitude–damping noise

Good clock state = one that hides energy from Eve

Fisher information counterpart to entropic uncertainty relations

Coles+ PRL 2019

strongly interacting many-body probes might offer better sensitivity

NV centers
Zhou+ PRX 2020

lattice atomic clocks
Goban+ Nat. 2018

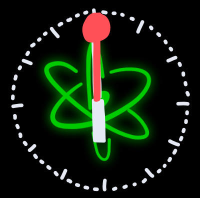
Also applicable to continuous Markovian noise, under some additional assumptions

Extension for multi-parameter metrology?

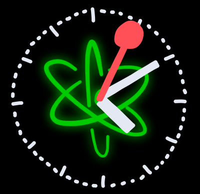
Gorecki+ Quantum 2020; ...

What are the implications of having only a finite number of samples?

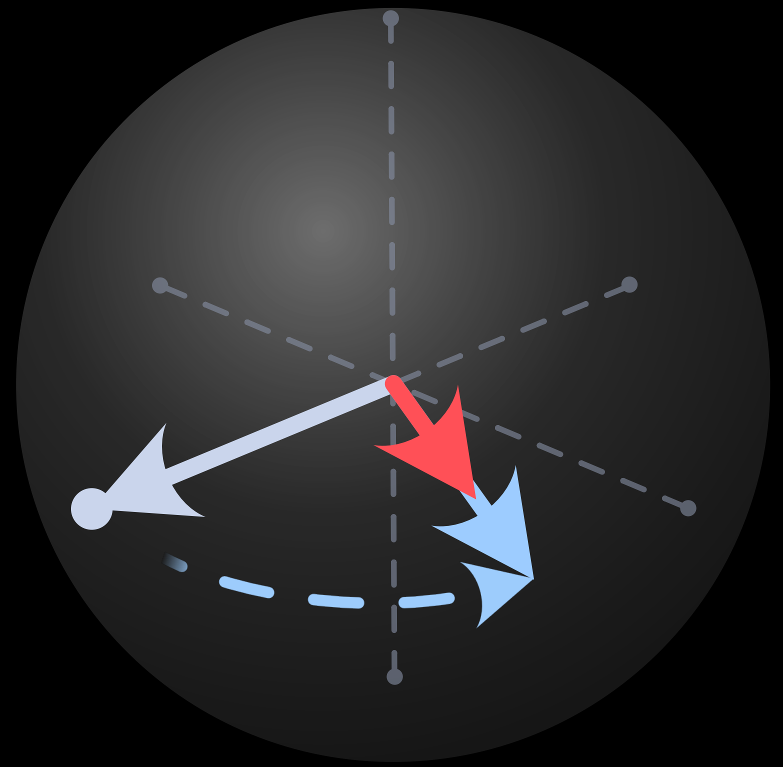
Thank you for
your attention!



Example: 1 qubit $H = \frac{\omega}{2}\sigma_Z$ $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + |\downarrow\rangle\right]$ $4\langle H^2\rangle = \omega^2$



Evolve for t_0: $\psi_t = \frac{1}{2}\begin{bmatrix} 1 & e^{-i\omega t} \\ e^{i\omega t} & 1 \end{bmatrix}$



Apply partial dephasing

$$\mathcal{D}_p[\rho] = \begin{bmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{bmatrix}$$

Eve:

$$\hat{\mathcal{D}}_p[\rho] = \begin{bmatrix} \left(1-\frac{p}{2}\right)\mathrm{tr}(\rho) & \sqrt{\frac{p}{2}\left(1-\frac{p}{2}\right)}\,\mathrm{tr}(\sigma_Z\rho) \\ \sqrt{\frac{p}{2}\left(1-\frac{p}{2}\right)}\,\mathrm{tr}(\sigma_Z\rho) & \frac{p}{2}\,\mathrm{tr}(\rho) \end{bmatrix}$$

$\{\overline{H}, \psi\} = \frac{\omega}{2}\sigma_Z$

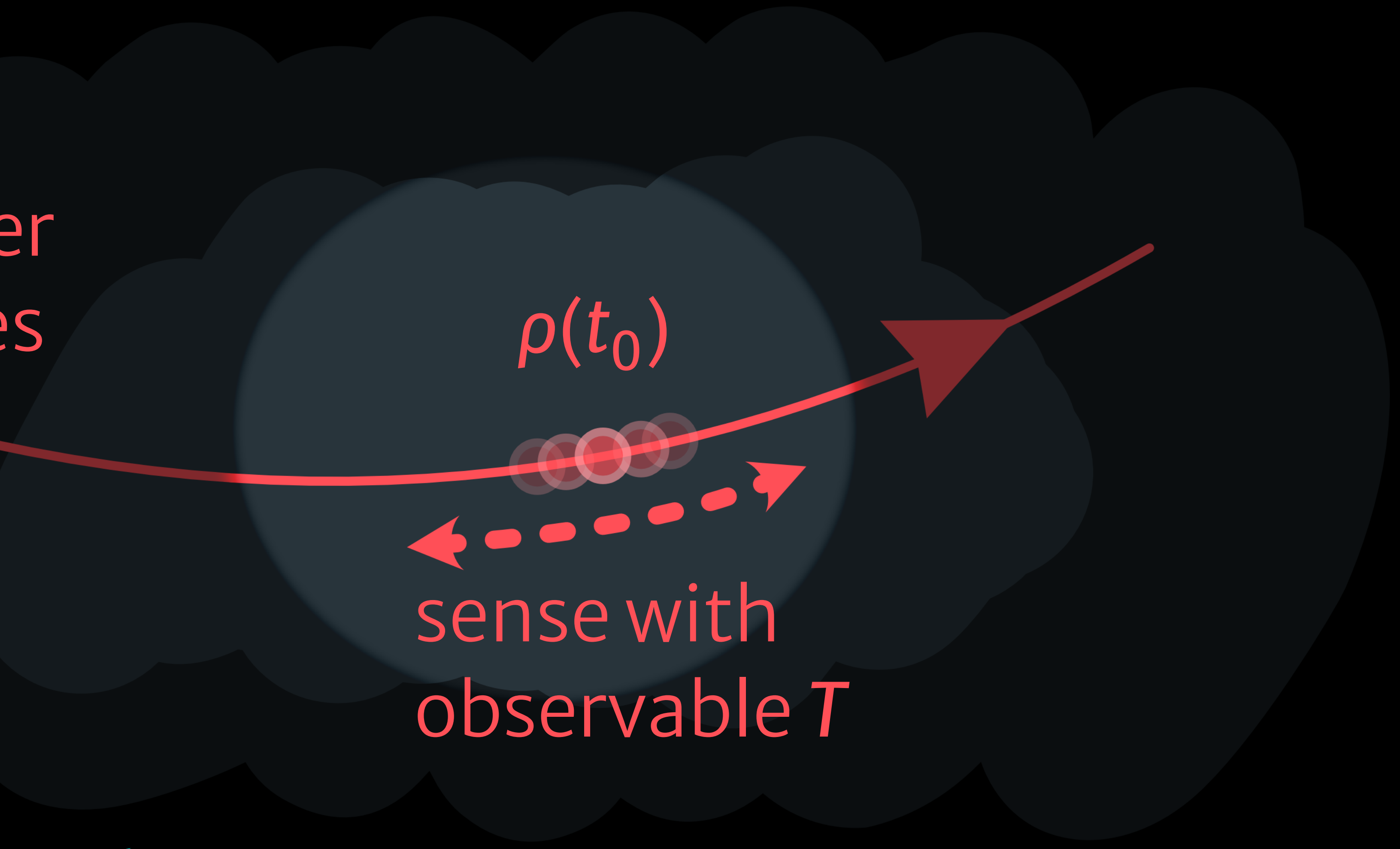
sensitivity loss: $F(\hat{\mathcal{N}}(\psi); \hat{\mathcal{N}}(\{\overline{H}, \psi\})) = \omega^2[1-(1-p)^2]$

Direct calculation: $F_{\mathrm{Bob},t} = \omega^2(1-p)^2$ $F_{\mathrm{Alice},t} = \omega^2$ ✔

Seek observable T with minimal variance at $\rho(t_0)$, such that:

$$\langle T \rangle_{\rho(t_0+dt)} = t_0 + dt + O(dt^2)$$

one-parameter family of states



Cramér–Rao bound

$$\left\langle (T - t_0)^2 \right\rangle \geq \frac{1}{F(\rho; \partial_t \rho)}$$

Fisher information

defined here via optimal T

Braunstein/Caves PRL 1994; Holevo 2011; …