# TTbar deformation of 2d quantum field theory and modular forms

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We begin by describing the problem loosely from a theoretical physicist's point of view, at some point becoming mathematically precise, hopefully while there is still something left to prove...

# Outline

- what is the TTbar deformation and why is it interesting?
- TTbar deformed massive 2d QFTs
  - geometric interpretation, *S*-matrix
- TTbar deformed CFTs
  - rectangle partition function, example of a holomorphic modular form
  - deformed version, proof of modular property
  - examples, Mellin transform
  - torus one-point function, example of a real modular form
  - deformed version, proof of modular property
  - example: Maass forms

# What is TTbar? (Zamolodchikov-Smirnov, 2004, 2017)

- a 1-parameter family of (nonlocal) 2d field theories $\mathcal{T}^\lambda$ with (on a flat euclidean manifold)

$$S^{\lambda+\delta\lambda} = S^\lambda + \delta\lambda \int \det T^\lambda \, d^2x \qquad (T^\lambda = \text{stress tensor})$$

  $\mathcal{T}^0$ a conventional local QFT. [For a CFT $\det T^0 = T_{zz} T_{\bar{z}\bar{z}}$.]
- many quantities UV finite and calculable given data of $\mathcal{T}^0$.
- correlation functions acquire only log divergences, removable by renormalization (JC, 2019)
- example of a nonlocal UV completion with a fundamental length scale $\lambda^{1/2}$
- $\lambda < 0 \Leftrightarrow$ 'going into the bulk' in AdS/CFT.

# TTbar in classical field theory

$$\delta S^\lambda = \frac{1}{2}\delta\lambda \int \epsilon^{ik}\epsilon^{jl} T^\lambda_{ij} T^\lambda_{kl} d^2x$$

Under a general infinitesimal change of metric

$$\delta S = -\int \delta g^{ij} T^\lambda_{ij} d^2x$$

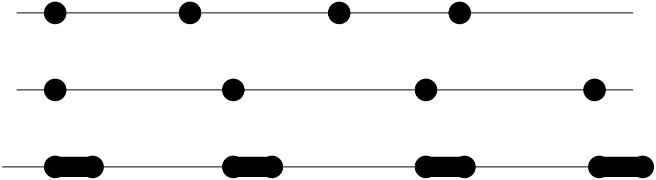
- suggests $\delta g^{ij} = -\delta\lambda\epsilon^{ik}\epsilon^{jl} T^\lambda_{kl}$
- conservation of $T^\lambda \Rightarrow$ metric remains flat
- equivalent to a diffeomorphism $x^\lambda_i \to x^\lambda_i + \delta x^\lambda_i(x)$ where

$$\partial_\lambda x^\lambda_{i,j} = \epsilon_{ik}\epsilon_{jl} {T^\lambda}^{kl}$$

$$\partial_\lambda x_i^\lambda = \epsilon_{ik} \int_{-\infty}^{x} \epsilon_{jl} T^{\lambda}{}^{kl}(x') dx'^j = -\epsilon_{ik}(\text{flux of } T^k \text{ across } (-\infty, x))$$

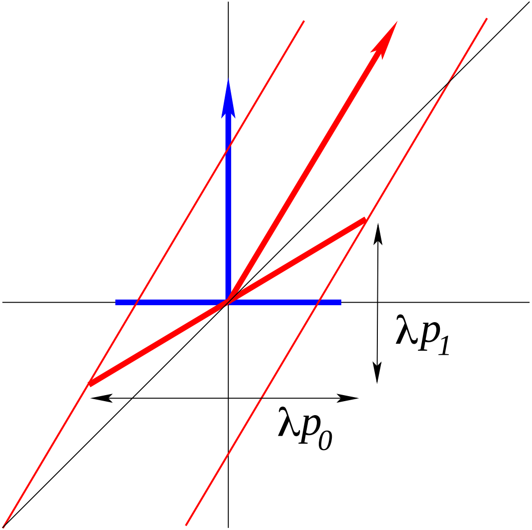
$$\partial_\lambda x_1^\lambda = (\text{energy flux across } (-\infty, x))$$

$$\partial_\lambda x_0^\lambda = -(\text{momentum flux across } (-\infty, x))$$



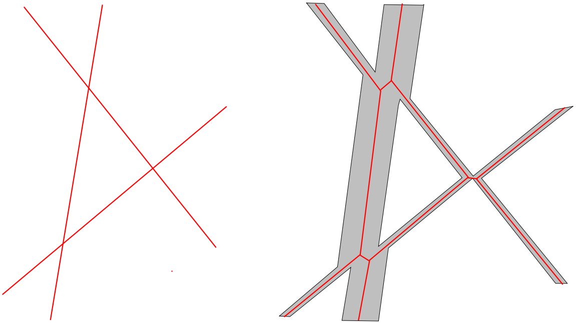
Particles at rest: shift $\equiv$ width $= \lambda \times$ rest mass

Boosted version



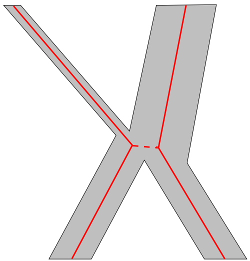
# Scattering

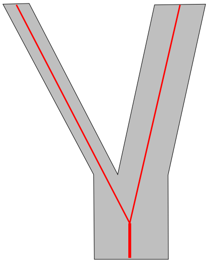
## Multiple elastic scattering



- conservation of energy ⇒ conservation of width
- conservation of momentum ⇒ extra time delay (phase shift)

# Inelastic processes





Each picture corresponds to a dissection of Minkowski space in which each tile is translated in a consistent manner

# Quantum scattering

- how does this square with $[X, P] \neq 0$?
- denote position of left and right edges by $X_L, X_R$. Then

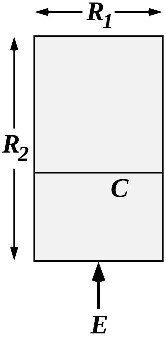
  $$[X_R - X_L, P_R + P_L] = 0$$

  so we can simultaneously specify the width and the momentum
- amounts to modifying the asymptotic states $|p_1, \ldots, p_N\rangle_{\rm in,out}$ by phase factors

  $$\exp\left(i\lambda \sum_{1\leq m<n\leq N} (p_m^0 p_n^1 - p_n^0 p_m^1)\right)$$

  or the $S$-matrix by equivalent CDD factors (Dubovsky et al 2017)
- but this finite width has a much more dramatic effect in finite volume or finite temperature...

# Rectangle partition function, a holomorphic modular form




In any CFT, with the same conformal bc on all sides,

$$Z^0_{rect}(R_1, R_2) = R_1^{c/4}\eta(q)^{-c/2}$$

where $q = e^{-2\pi R_2/R_1}$, $\eta = q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1 - q^m)$.

- $S$-symmetry $Z^0_{rect}(R_1, R_2) = Z^0_{rect}(R_2, R_1)$ is guaranteed by

$$\eta(e^{-2\pi/\delta}) = \delta^{\frac{1}{2}}\eta(e^{-2\pi\delta}) \quad (\eta \text{ has weight } \tfrac{1}{2})$$

- $q$-expansion = spectral decomposition

$$Z^0_{rect} = \sum_n |b^0_n(R_1)|^2 e^{-E^0_n(R_1)R_2} \equiv \int \rho^0(E, R_1)e^{-ER_2}dE$$

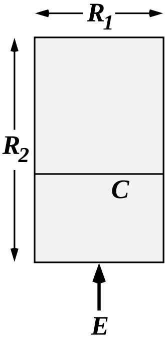
- more generally can consider

$$Z^0(R_1, R_2) = R_1^{-k} F^0(\delta = R_2/R_1)$$

where $F^0(\delta) = \sum_{n=0}^{\infty} a_n q^{\Delta+n}$ is a “modular form” of weight $k$:

$$F^0(1/\delta) = \delta^k F^0(\delta)\,, \quad F^0(\delta - i) = e^{2\pi i\Delta} F^0(\delta)$$

# TTbar deformation




$$\partial_\lambda R_1 = -\lambda E\,, \quad R_1^\lambda = R_1^0 - \lambda E \quad (\text{fixed} E)$$

$$\text{so if} \quad Z^0(R_1, R_2) = \int \rho^0(R_1, E) e^{-ER_2} dE$$

$$\text{then } Z^\lambda(R_1, R_2) \stackrel{?}{=} \int \rho^0(R_1 + \lambda E, E) e^{-ER_2} dE$$

$$\text{so formally get PDE} \quad \partial_\lambda Z = -\partial_{R_1} \partial_{R_2} Z$$

- how to make sense of this?
- if we can, does $Z^\lambda(R_1, R_2) = Z^\lambda(R_2, R_1)$?

If $Z^\lambda(R_1, R_2) = R_1^{-k} F^\lambda(\delta = R_2/R_1)\,,$ does $F^\lambda(1/\delta) = \delta^k F^\lambda(\delta)$?

Laplace transform

$$\Omega^0(R_1, s) \equiv \int_0^\infty e^{-sR_2} Z^0(R_1, R_2) dR_2$$

$$Z^0(R_1, R_2) = \int_C e^{sR_2} \Omega^0(R_1, s) \frac{ds}{2\pi i}$$

so that $\rho^0(R_1, E) = 2\,\mathrm{Im}\,\Omega^0(R_1, s = -E)$ and

$$\Omega^\lambda(R_1, s) = \Omega^0(R_1 - \lambda s, s)$$

$$= (R_1 - \lambda s)^{1-k} \phi\big((R_1 - \lambda s)s\big) \quad \text{well-defined in a CFT}$$

After some algebra...

$$\boxed{F^{\alpha}(\delta) = \int_{-i\infty}^{i\infty} e^{s\delta} \int_0^{\infty} (1 - \alpha\delta s)^{1-k} e^{-s\delta'(1-\alpha\delta s)} F^0(\delta') d\delta' \frac{ds}{2\pi i}}$$

where $\alpha = \lambda/(R_1 R_2)$.

- use this as the *definition* of $F^{\alpha}(\delta)$
- after some more algebra, completing the square in $s$,

$$F^{\alpha}(\delta) = \int_0^{\infty} K^{\alpha}(\delta, \delta')(\delta'/\delta)^{k/2} F^0(\delta') \frac{d\delta'}{\delta'}$$

  where

$$K^{\alpha}(\delta, \delta') = e^{-\frac{(\delta'-\delta)^2}{4\alpha\delta\delta'}} \int_{-\infty}^{\infty} \left( \frac{(\delta + \delta')}{2(\delta\delta')^{1/2}} - it \right)^{1-k} e^{-\alpha t^2} \frac{dt}{2\pi}$$

- gaussian smearing in moduli space
- invariance of $K^{\alpha}$ and the measure under $(\delta, \delta') \to (\delta^{-1}, \delta'^{-1}) \Rightarrow$ if $\delta'^{k/2} F^0(\delta')$ is invariant, so is $\delta^{k/2} F^{\alpha}(\delta)$. □

# Deformed spectrum

$$F^{\alpha}(\delta) = \int_{-i\infty}^{i\infty} e^{s\delta} \int_0^{\infty} (1-\alpha\delta s)^{1-k} e^{-s\delta'(1-\alpha\delta s)} F^0(\delta') d\delta' \frac{ds}{2\pi i}$$

If $F^0(\delta') = \sum_n a_n e^{-2\pi(\Delta+n)\delta'}$ we can integrate over $\delta'$ in each term to get

$$\int_{-i\infty}^{i\infty} \frac{e^{s\delta}(1-\alpha\delta s)^{1-k}}{2\pi(\Delta+n)+s-\alpha\delta s^2} \frac{ds}{2\pi i}$$

which has poles at $s = s_{\pm} = (1/2\alpha\delta)(1 \pm \sqrt{1+8\pi(\Delta+n)\alpha\delta})$. Moving contour to L we pick up only the poles at $s_-$ giving

$$F^{\alpha}(\delta) = \sum_{n=0}^{\infty} a_n \frac{(1+\sqrt{1+8\pi(\Delta+n)\alpha\delta})^{1-k}}{2^{1-k}\sqrt{1+8\pi(\Delta+n)\alpha\delta}} e^{-(1/2\alpha)(\sqrt{1+8\pi(\Delta+n)\alpha\delta}-1)}$$

- deformed spectrum and matrix elements

# Example

$$\vartheta_3(0,\delta) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi n^2 \delta} = \delta^{-1/2}\vartheta_3(0, 1/\delta)$$

This is also true of

$$\vartheta_3^{\alpha}(0,\delta) \equiv \sum_{n=-\infty}^{\infty} \frac{(1+\sqrt{1+4\pi n^2\alpha\delta})^{1/2}}{2^{1/2}\sqrt{1+4\pi n^2\alpha\delta}}\, e^{-(1/2\alpha)(\sqrt{1+4\pi n^2\alpha\delta}-1)}$$

Can be generalized to Jacobi forms, e.g. $\vartheta_3(z,\delta)$.

# Mellin transform

Associates a modular form to a Dirichlet series: if $F^0 = \sum_{n=0}^{\infty} a_n q^{\Delta+n}$ with $q = e^{-2\pi\delta}$

$$R^0(s) = \int_0^{\infty} \delta^{s-1} F^0(\delta) d\delta = (2\pi)^{-s}\Gamma(s) \sum_{n=0}^{\infty} \frac{a_n}{(\Delta+n)^s}$$

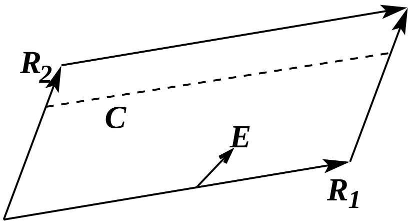
where $R^0(s)$ is analytic in $\mathrm{Re}\, s > k$ and $R^0(k-s) = R^0(s)$. Defining $R^\alpha(s) = \int_0^\infty \delta^{s-1} F^\alpha(\delta) d\delta$ we find

$$\boxed{R^\alpha(s) = I^\alpha(k;s)\, R^0(s)}$$

where $I^\alpha(k;s)$ is an entire function of $s$ satisfying $I^\alpha(k-s;s) = I^\alpha(k;s)$.

- Mellin transform diagonalizes the TTbar flow
- $R^\alpha(s)$ inherits the reflection property and zeroes of $R^0(s)$.

# Torus: 1-point function, example of a real modular form




We can play the same game thinking about the 1-point functions on the torus $\mathbb{T}_2 = \mathbb{C}/(\mathbb{Z}R_1 + \mathbb{Z}R_2)$.

$$\langle \Phi \rangle^0 (R_1, R_2) = \int \rho^0(|R_1|, E) e^{-\mathrm{Re}\,(ER_2^*)} d^2E$$

$$= |R_1|^{-k} F^0(\delta = -iR_2/R_1) = |R_1|^{-k} \sum_{m,n=0}^{\infty} a_{m,n} q^{\Delta+m} q^{*\,\Delta+n}$$

$F^0$ is a real modular form satisfying $F^0(\delta^{-1}) = |\delta|^k F^0(\delta)$, $F^0(\delta - i) = F^0(\delta)$.

(Note the usual $\tau = i\delta$.)

TTbar evolution is simple at fixed $E$ in a *fixed* frame:

$$R_1^\lambda = R_1^0 + i\lambda N_2$$

formally leading to the PDE

$$\partial_\lambda \langle \Phi \rangle^\lambda (R_1, R_2) = -\left(\partial_{R_1} \wedge \partial_{R_2}\right) \langle \Phi \rangle^\lambda (R_1, R_2)$$

Using Laplace transforms as before

$$F^{\alpha}(\delta) =$$

$$\iint (1-\alpha\delta_1 s_1)^2+\alpha^2\delta_1^2 s_2^2]^{-k/2+1} e^{\alpha\delta_1\delta_1'|s|^2+\mathrm{Re}\,(s^*(\delta-\delta'))} F^0(\delta') d^2\delta' \frac{d^2 s}{(2\pi i)^2}$$

$$= \int_{\mathbb{H}} K^{\alpha}(\delta,\delta')(\delta_1'/\delta_1)^{k/4} F_2^0(\delta) \frac{d^2\delta'}{{\delta_1'}^2}$$

where

$$K^{\alpha}(\delta,\delta') = K^{\alpha}(\delta^{-1},\delta'^{-1}) = \overbrace{e^{-|\delta-\delta'|^2/4\alpha\delta\delta'}}^{\text{Selberg kernel}} \times \text{stuff}$$

which ensures $F^{\alpha}$ transforms the same way as $F^0$.

On the other hand, integrating over $\delta'$ gives

$$F_2^{\alpha}(\delta) = \sum_{n=0}^{\infty}\sum_{p\in\mathbb{Z}} b_{n,p} \; \frac{\left(1+\sqrt{1+8\pi(\Delta+n)\alpha\delta_1+(4\pi p\alpha\delta_1)^2}\right)^{1-k}}{\sqrt{1+8\pi(\Delta+n)\alpha\delta_1+(4\pi p\alpha\delta_1)^2}}$$

$$\times e^{-(1/2\alpha)\left(\sqrt{1+8\pi(\Delta+n)\alpha\delta_1+(4\pi p\alpha\delta_1)^2}-1\right)+2\pi i p\delta_2}$$

which exhibits the deformed matrix elements as well as Zamolodchikov deformed spectrum.

$F_2^{\alpha}(\delta)$ has the same modular properties as $F_2^0(\delta)$.

# Maass forms

Maass forms are smooth real functions of $\delta$ in $\mathbb{H}$: Re $\delta > 0$ which are SL$(2, \mathbb{Z})$ invariant, polynomially bounded as Re $\delta \to \infty$, and are eigenfunctions of the invariant Laplacian

$$\Delta_{\mathbb{H}} = -\delta_1^2 \left(\partial_{\delta_1}^2 + \partial_{\delta_2}^2\right)$$

Recall the PDE

$$\partial_\lambda Z^\lambda(R_1, R_2) = -\left(\partial_{R_1} \wedge \partial_{R_2}\right) Z^\lambda(R_1, R_2)$$

A scaling solution $Z^\lambda(R_1, R_2) = F^{\alpha=\lambda/(R_1 \wedge R_2)}(\delta)$ then satisfies

$$\partial_\alpha F = -\frac{1}{4}\Delta_{\mathbb{H}}\, F$$

So if $F$ is a Maass form with eigenvalue $\Lambda$,

$$F^\alpha(\delta) = e^{-\frac{1}{4}\Lambda\alpha}\, F^0(\delta)$$

- Maass forms are eigenfunctions of the TTbar deformation

# Remarks

- the above has assumed that $\lambda > 0$ and $\Delta > 0$, so that $F^0(q) \to 0$ as $q \to 0$, but this means $c < 0$ in a CFT
- for $\lambda > 0$ and $\Delta < 0$, as for a unitary CFT, the treatment is still valid in regions of moduli space away from $q = 0, 1$, bounded by Hagedorn-type transitions.
- for $\lambda < 0$ solution near $q = 0$ is not continuously connected to that near $q = 1$: modular invariance is "broken"
- it is possible to choose the contours so as to give a convergent modular invariant expression, but it is no longer equal to a sum over a discrete spectrum

# Discussion

- the nice properties of the TTbar deformation of CFTs extend to more general mathematical objects
- this deformation is unique in some sense
- what is the significance for physics of Maass forms and Mellin transforms with respect to the modulus?