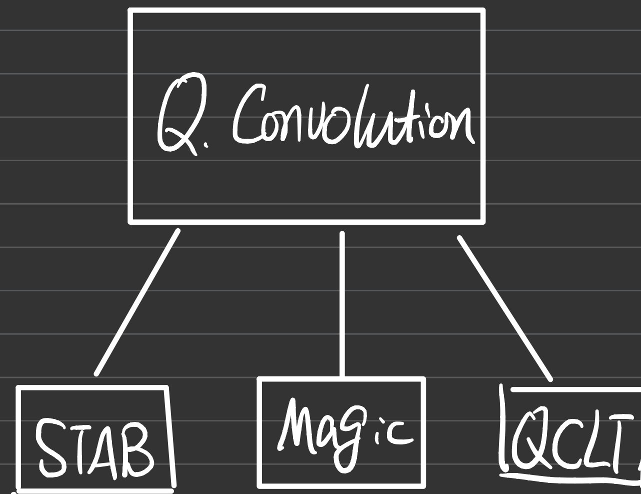
Magic from a Quantum Convolutional Approach

Joint work with Weichen Gu & Arthur Jaffe.

arxiv: 2302.07841, 2302.08423.



① Stabilizer States

1-qudit: $\mathcal{H}$, $\dim \mathcal{H} = d$

n-qudit: $\mathcal{H}^{\otimes n}$

Def [Pauli operators]

1-qudit pauli: $X|j\rangle = |j+1\rangle$

$Z|j\rangle = w_d^j |j\rangle$ $w_d = \exp(2\pi i/d)$ $\{|j\rangle\}_{j\in\mathbb{Z}_d}$: ONB

$X^a Z^b$: $a, b \in \mathbb{Z}_d$

n-qudit pauli: $\left\{ \bigotimes_{i=1}^{n} X^{a_i} Z^{b_i} = X^{\vec{a}} Z^{\vec{b}} \right\} = \mathcal{P}_n$

$\vec{a} = (a_1, \dots, a_n)$

$\vec{b} = (b_1, \dots, b_n)$

Def [pure stab. states]

common eigenstates of abelian subgroup $G = \langle g_1, \dots g_n \rangle$ $g_i \in P_n$, $[g_i, g_j] = 0$

pure stab state $|\psi\rangle = g_i |\psi\rangle \quad \forall i$

$$|\psi\rangle\langle\psi| = \frac{1}{d^n} \prod_{i=1}^{n} \left(\sum_{k_i \in \mathbb{Z}_d} g_i^{k_i} \right)$$

Ex : $n=2$, $d=2$ $\quad |Bell\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ stab. state

$$G = \langle X \otimes X, Z \otimes Z \rangle$$

Def [Minimal stab.-projection state, MSPS]

abelian $G = \langle g_1, \dots, g_r \rangle$ $\quad r \leq n$

$$\rho = \frac{1}{d^n} \prod_{i=1}^{r} \left(\sum_{k_i \in \mathbb{Z}_d} g_i^{k_i} \right)$$

Def [Clifford unitary]

$$Cl_n = \{ U : U P_n U^{\dagger} \subseteq P_n \}$$

$$U_{cl} : STAB \longrightarrow STAB$$

Gottesman-Knill Thm

$p(\vec{y})$ can be computed on a classical computer in poly(n) time.

$\Rightarrow$ no quantum advantage.

$\Rightarrow$ nonstabilizer states.

magic states : nonstabilizer states.

Ex : $|T\rangle$ state.

$$|T\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$

$$= T|+\rangle \qquad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Clifford + T $\Rightarrow$ universal Q. Computation.

$|T\rangle$ + Clifford + Pauli measurement $\Rightarrow$ T gate.

(magic state injection)

② Quantum Fourier Transform.

P_n = ONB for $B(\mathcal{H}^{\otimes n})$ w.r.t. $\langle A, B\rangle = \frac{1}{d^n} Tr[A^\dagger B]$

Def [characteristic function]

$$\chi_\rho(\vec{a}, \vec{b}) = Tr[\rho (X^{\vec{a}} Z^{\vec{b}})^\dagger] \Rightarrow \rho = \frac{1}{d^n} \sum \chi_\rho(\vec{a}, \vec{b}) X^{\vec{a}} Z^{\vec{b}}$$

$$\begin{tikzcd}
\boxed{\rho} \arrow[rr, bend left, "Q.F.T."] \arrow[d, phantom, "\Big\downarrow \text{ reduced to classical case}"] & & \boxed{\chi_\rho} \\
\boxed{f : \mathbb{Z}_d^n \to \mathbb{C}} \arrow[rr, "F.T."] & & \boxed{\hat{f} : \mathbb{Z}_d^n \to \mathbb{C}}
\end{tikzcd}$$

Ref: [1] Montanaro, Osborne "Quantum boolean function" 2010

[2] Garcia, Bu & Jaffe Resource theory of scrambling PNAS 2023

③ Quantum convolution on n-qudit system (Bu-Gu-Jaffe 2023)

Def [Discrete Beam Splitter]

2-qudit unitary $U_{s,t}$ $\quad s, t \in \mathbb{Z}_d \quad s^2 + t^2 \equiv 1 \mod d$

$$U_{s,t} : |i\rangle \otimes |j\rangle \longrightarrow |si + tj\rangle \otimes |ti - sj\rangle \quad \forall i, j.$$

2n-qudit unitary

$$\mathcal{V}_{s,t} = U_{s,t}^{\otimes n} = U_{s,t}^{1,n+1} \otimes U_{s,t}^{2,n+2} \otimes \cdots \otimes U_{s,t}^{n,2n}$$

ρ, σ : n-qudit state

$$\rho \boxtimes \sigma = \mathrm{Tr}_{II}\left[\mathcal{V}_{s,t}\, \rho \otimes \sigma\, \mathcal{V}_{s,t}^{\dagger}\right]$$

$$\mathrm{Tr}_{II} := \mathrm{Tr}_{n+1} \circ \mathrm{Tr}_{n+2} \circ \cdots \circ \mathrm{Tr}_{2n}$$

Prop 1.

$$\chi_{\rho \boxtimes \sigma}(\vec{a}, \vec{b}) = \chi_\rho(s\vec{a}, s\vec{b})\, \chi_\sigma(t\vec{a}, t\vec{b})$$

Prop 2.

$\rho, \sigma \in MSPS \Rightarrow \rho \boxtimes \sigma \in MSPS$.

$$\underbrace{\rho \boxtimes \rho \boxtimes \cdots \boxtimes \rho}_{N} = \boxtimes^N \rho.$$

Prop 3.

$$\boxtimes^N \rho \xrightarrow{N \to \infty} \rho_{STAB} \quad \text{(QCLT)}$$

informal.

Classical case.

X : random variable. $\mathbb{E}[X] = 0$ $\quad Var(X) = \sigma$

f_X : probability density function

Y : r.v. - - - - - - - f_Y

$$Z = \frac{X+Y}{\sqrt{2}} \Rightarrow f_Z(a) = f_X * f_Y(\sqrt{2}a)$$

characteristic function $\varphi_X(t) = \mathbb{E}_X\, e^{iXt}$

Prop 1. $\varphi_Z(t) = \varphi_X(\frac{1}{\sqrt{2}}t)\, \varphi_Y(\frac{1}{\sqrt{2}}t)$.

Prop. 2. $X, Y \in$ Gaussian $\Rightarrow Z \in$ Gaussian

Prop 3. $\frac{X_1 + \cdots + X_N}{\sqrt{N}} \xrightarrow{N \to \infty}$ Gaussian (CLT)

X_i: i.i.d

Q: what's the limit ρ_{STAB} ?

Q: the rate of convergence?

$$Z_N = \frac{X_1+\cdots+X_N}{\sqrt{N}}$$

prop $H(Z_N) \leq H(Z_{N+1})$

Lieb CMP 1978 $N=2^n$

Naor. et al JAMS 2004

Def [Mean state]

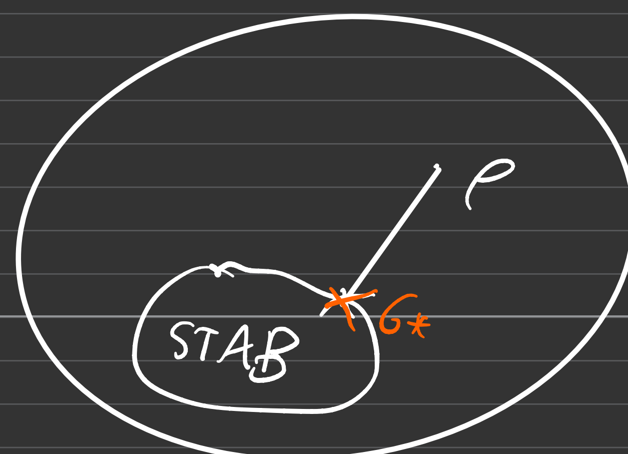
(mean state).

$$\rho = \frac{1}{d^n}\sum \Xi_\rho(\vec{a},\vec{b})\, X^{\vec{a}} Z^{\vec{b}} \Rightarrow M(\rho) = \frac{1}{d^n}\sum \Xi_{M(\rho)}(\vec{a},\vec{b})\, X^{\vec{a}} Z^{\vec{b}}$$

$$\Xi_{M(\rho)}(\vec{a},\vec{b}) = \begin{cases} \Xi_\rho(\vec{a},\vec{b}) & \text{if } |\Xi_\rho(\vec{a},\vec{b})| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Relative entropy $D(\rho\|\sigma) = Tr[\rho\log\rho] - Tr[\rho\log\sigma]$

$$\min_{\sigma\in STAB} D(\rho\|\sigma) = D(\rho\|M(\rho))$$



Def [magic gap]

$MG(\rho) =$ largest absolute value of Ξ_ρ

$-$ second-largest absolute value of Ξ_ρ

① $MG(\rho) \in [0,1]$ $\quad MG(\rho) = 0$ iff $\rho = \mathcal{M}(\rho)$

② $MG(U_{cl} \rho U_{cl}^{\dagger}) = MG(\rho)$. $\quad \forall$ Clifford unitary U_{cl}

Thm (QCLT, formal).

If ρ is zero-mean ($\chi_{\mathcal{M}(\rho)}$ is taking 0 or 1).

then $\| \boxtimes^N \rho - \mathcal{M}(\rho) \|_2 \leq (1 - MG(\rho))^N \| \rho - \mathcal{M}(\rho) \|_2$.

Prop. (Monotonicity of entropy under Q. Convolution).

$$S(\boxtimes^N \rho) \leq S(\boxtimes^{N+1} \rho)$$

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho]$$