Constructing algebraic quantum field theory

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The talk is based on joint work with Detlev Buchholz and with Romeo Brunetti, Michael Dütsch and Kasia Rejzner.

Introduction

Quantum Field Theory yields an almost complete description of fundamental physics (up to gravity), but still suffers from severe mathematical problems.

Approaches from 2 sides:

- Construction of models
- Characterization by axioms

The construction of physically relevant models in 4 dimensional spacetime is up to now possible only in terms of uncontrolled approximations (formal perturbation theory, lattice models,. . .)

On the other hand, the axiomatic approach does not specify concrete models, but yields only general features (spin-statistics connection, PCT theorem, *etc*).

Existing approaches *e.g.*

- Construction of the interacting Hamiltonian (functional analysis) (Glimm, Jaffe, . . .):
- Euclidean path integral (measure theory) (. . .)
- Conformal field theories in 2 dimensions (BPZ,. . .)
- Construction of models with factorizing S-matrices (Lechner)

New approach: Add algebraic relations valid in formal perturbation theory to the axiomatic structure.

Input: Reformulation of perturbation theory

- Generalization to generic Lorentzian spacetimes (Radzikowski, Brunetti, F, Hollands, Wald,. . .)
- Renormalization by conditions on S-matrices (Epstein-Glaser, Stora, Brunetti, Dütsch, F, . . .)

Result: Construction of a functor from a category of spacetimes to the category of C*-algebras satisfying fundamental physical properties, in particular causality in two aspects:

Independence of causally disjoint regions and dependence in causally dependent regions.

Framework: Smooth real functions ϕ on some globally hyperbolic Lorentzian manifold M, together with a Lagrangian density

$$L(\phi, d\phi) = (\frac{1}{2}g^{-1}(d\phi, d\phi) - V(\phi))d\mu_g \ , \ \phi \in \mathcal{C}^\infty(M) \ .$$

Algebras of observables generated by local operations, understood as S-matrices induced by a local interaction,

$$S(F) = Te^{iF} \ , \ T \text{ time ordering operator}$$

$$F[\phi] = \int \sum f_n(x)\phi(x)^n + b^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x)$$

local functional on smooth functions ϕ, with test densities f_n, $b^{\mu\nu}$, $b^{\mu\nu}$ sufficiently small.

Relations

Causal factorization

$$S(F+G) = S(F)S(G)\ ,\ \text{if supp}F \cap J_-(\text{supp}G) = \emptyset$$

J_- causal past of a region.

Unitary field equation

$$S(F) = S(F^\psi + \delta L(\psi))$$

ψ compactly supported, $F^\psi[\phi] = F[\phi+\psi]$,
$\delta L(\psi) = \int L(\phi+\psi, d(\phi+\psi)) - L(\phi, d\phi)$

(in perturbation theory equivalent to the Schwinger-Dyson equation)

Relative S-matrices

$$S_G(F) \doteq S(G)^{-1}S(F+G)$$

Interpretation: S-matrices under an additional interaction G, since S_G satisfies the analogous unitary field equation

$$S_G(F) = S_G(F^{\psi} + \delta L(\psi) + G^{\psi} - G)$$

Stable causal factorization: Causal factorization for relative S-matrices implied by

$$S(F+G+H) = S(F+G)\,S(G)^{-1}\,S(G+H)$$

$$\text{if } \mathrm{supp}F \cap J_-^G(\mathrm{supp}H) = \emptyset$$

(J_-^G causal past with respect to the interaction G)

Algebras of local observables

Definition

Let N be a globally hyperbolic subregion of the spacetime M.

The algebra $\mathfrak{A}(N, L)$ of local observables in N for a theory with Lagrangian L

is the C*-algebra freely generated by unitaries $S(F)$, F local functional with supp$F \subset N$ modulo the relations

- Stable causal factorization
- Unitary field equation
- $S(F_c) = e^{ic}$ for constant functionals $F_c[\phi] = c$, $c \in \mathbb{R}$.

Buchholz, F 2020:

Theorem

Let L be of second order in ϕ.

Then the subalgebra $\mathfrak{A}_1(N,L)$ generated by $S(F)$ with $F[\phi]=\int f(x)\phi(x)$, f smooth density on N,

is isomorphic to the Weyl algebra, i.e. the C-algebra generated by unitaries $W(f)$ with the Weyl relation*

$$W(f)W(g)=e^{-i/2\langle f,(\Delta_R-\Delta_A)g\rangle}W(f+g)$$

$\Delta_{R/A}$ retarded/advanced Green operator of the field equation associated to L.

Note that the canonical commutation relations are not imposed, but follow from causality and the unitary field equation.

Construction of a model of locally covariant QFT, as defined in [Brunetti, F, Verch 2003] :

$\mathfrak{A}$ functor from the category of manifolds equipped with Lagrangians with normally hyperbolic Euler-Lagrange derivatives to the category of unital C*-algebras.

$$\mathfrak{A} : (M, L) \mapsto \mathfrak{A}(M, L)$$

$$\mathrm{Hom}\left((M, L), (M', L')\right) = \{\chi : M \to M' \text{ embedding with } \chi^* L' = L\}$$

$$\mathfrak{A}\chi : \begin{cases} \mathfrak{A}(M, L) & \to & \mathfrak{A}(M', L') \\ S(F) & \mapsto & S(\chi_* F) \end{cases}$$

with $\chi_* F[\phi] = F[\phi \circ \chi]$.

Renormalization group

The Weyl algebra $\mathfrak{A}_1(M, L)$ is simple and satisfies the time slice axiom

$$\mathfrak{A}_1(N, L) = \mathfrak{A}_1(M, L) \ , \ \ N \supset \Sigma \text{ Cauchy surface of } M,$$

but the full algebra $\mathfrak{A}(M, L)$ admits a large class of automorphisms which act trivially on the Weyl algebra and map local algebras into themselves. They are of the form

$$\beta_Z(S(F)) = S(Z(F))$$

where Z maps local functionals to local functionals, acts trivially on linear and constant functionals, preserves the support and satisfies the additivity relation

$$Z(F+G+H) = Z(F+G) - Z(G) + Z(G+H)\ ,\ \mathrm{supp}F \cap \mathrm{supp}H = \emptyset$$

(this preserves the stable causality condition) and the relation

$$Z(F^{\psi} + \delta L(\psi)) = Z(F)^{\psi} + \delta L(\psi)$$

(this preserves the unitary field equation).

The group of invertible transformations Z may be considered as the

nonperturbative analogue of the renormalization group
in the the sense of Stückelberg-Petermann (1953).

This group characterizes the ambiguity in the process of renormalization. It is related, but not identical to the Wilsonian concept of the renormalization group
(see Brunetti, Dütsch, F 2009).

Unitary Anomalous Master Ward Identity

We now consider symmetries of the configuration space $\mathcal{C}^\infty(M,\mathbb{R})$:

$$(x \mapsto \phi(x)) \longrightarrow (x \mapsto (\phi \circ \chi(x))A(x)) \equiv \phi \cdot (\chi, A)$$

with a compactly supported diffeomorphism χ and a smooth real valued function $A : x \mapsto A(x)$ with $A \equiv 1$ outside of a compact region.

Then $g = (\chi, A)$ induces a transformation g_* of functionals $g_* F(\phi) = F(\phi \cdot g)$ and of the Lagrangian, and

$$\alpha_g(S(F)) = S(g_* F)$$

defines an isomorphism $\mathfrak{A}(M, L) \to \mathfrak{A}(M, g_* L)$.

On the other hand, we can embed the algebra $\mathfrak{A}(M, g_* L)$ into $\mathfrak{A}(M, L)$ by considering $\delta_g L \doteq \int g_* L - L$ as an interaction,

$$S(F) \mapsto S(\delta_g L)^{-1} S(\delta_g L + F) \ .$$

Composing both maps

$$S(F) \mapsto S(\delta_g L)^{-1} S(\delta_g L + g_* F)$$

we obtain an automorphism of $\mathfrak{A}(M, L)$.

For the free Lagrangian, this automorphism acts trivially on $S(F)$ for linear functionals F.

We postulate the following relation: there exists a map ζ from the group of symmetries $g = (\chi, A)$ to renormalization group elements ζ_g such that for all local functionals F

$$S(\delta_g L + g_* F) = S(\zeta_g F)$$

(unitary anomalous Master Ward identity)

The map ζ has to satisfy the cocycle relation

$$\zeta_{gh} = \zeta_h \zeta_g^h$$

where the action of h on a renormalization group element Z is defined by

$$Z^h = h_L^{-1} Z h_L \ , \ h_L F = h_* F + \delta_h L \ .$$

Theorem

The unitary Anomalous Master Ward Identity (UAMWI) holds in formal perturbation theory.

It is equivalent to the Anomalous Master Ward Identity of Brennecke-Dütsch (2008)

and to the renormalized Quantum Master Equation of the Batalin-Vilkovisky formalism (F, Rejzner 2013).

It is closely related to the Wess-Zumino consistency relations.

(Brunetti, Dütsch, F, Rejzner (2021,2022))

The UAMWI for a cocycle ζ defines an ideal I_ζ of $\mathfrak{A}(M, L)$, with quotient $\mathfrak{A}(M, L, \zeta)$ and with local subalgebras $\mathfrak{A}(N, L, \zeta)$.

Properties of the Haag-Kastler net $\mathfrak{A}_{M,L,\zeta} : N \mapsto \mathfrak{A}(N, L, \zeta)$ (BDFR 2022):

- $\mathfrak{A}_{M,L,\zeta}$ satisfies the time slice axiom.
- Anomalous Noether Theorem:
  Symmetries of the Lagrangian give rise to unitaries which implement locally the symmetry, up to renormalization group transformations ("quantum symmetries").
- For vanishing ζ classical symmetries coincide with quantum symmetries.

- Symmetries of the Lagrangian induce a flow of theories which can be described in two equivalent ways:
  As an action of the symmetry on the cocyle ζ
  or as a flow of the Lagrangian, together with a transformation of observables (known as "running coupling constants")

Proof that the time-slice axiom holds (sketch):

Let L be a stationary quadratic Lagrangian and let χ be a compactly supported diffeomorphism which acts in some region $\mathcal{O}_1$ as a time translation $-\tau < 0$. Then with $\mathcal{O}_2 = \mathcal{O}_1 \cap \mathcal{O}_1 - \tau$

$$\text{supp}(\delta_\chi L) \cap \mathcal{O}_2 = \emptyset$$

Let $\text{supp} F \subset \mathcal{O}_2$. For trivial anomaly the UAMWI yields

$$S(F) = S(\delta_\chi L + \chi_* F)$$

in particular $S(\delta_\chi L) = S(0) = 1$.
We decompose $\delta_\chi L = H_+ + H_-$ such that $\text{supp} H_\pm \cap J_\mp(\mathcal{O}_2) = \emptyset$.
Then by the stable causality relation

$$S(F) = \underbrace{S(H_+ + H_-)}_{=1} S(H_-)^{-1} S(H_- + \chi_* F) = S(H_-)^{-1} S(\chi_* F) S(H_-)$$

Thus within $\mathcal{O}_2$ the time translation is implemented by unitaries which are localized outside of the future of the region $\mathcal{O}_2$.

By an iteration of the argument and an analogous procedure for the time reversed situation the time slice property follows.

The argument remains valid also for nontrivial anomaly. For an interacting, not necessarily stationary Lagrangian the statement follows from the validity of the interaction picture.

(The last step was originally derived within formal perturbation theory (Chilian, F 2009))

Summary

The algebraic construction yields a definite algebraic quantum field theory which satisfies the axioms of locality, covariance and time-slice. It is fixed by two ingredients: a classical Lagrangian L and a cocycle ζ characterizing the anomalies. The cocycle relation corresponds to the Wess-Zumino consistency conditions of perturbation theory.

We restricted ourselves here to scalar fields. Fermionic fields can be treated similarly after adding auxiliary Grassmann parameters in a consistent way (Brunetti, Dütsch, F, Rejzner 21).

Perturbation theory delivers a nontrivial representation of a dense subalgebra in terms of formal power series of Hilbert space operators. In particular, the scaling anomaly for massless scalar theories and the axial anomaly for massless Fermi fields are recovered.

For the subalgebra generated by $S(F)$ with functionals F of second order in ϕ, the Fock space representation can be extended. This construction goes beyond formal perturbation theory, since changes of the causal structure are included. (Buchholz, F 2020)

Open problems:

Gauge theories are not yet included (work in progress).

As a C*-algebra, $\mathfrak{A}(M, L, \zeta)$ has a full state space and a faithful Hilbert space representation. But the structure of the state space, in particular its physical interpretation remains to be determined.

In our construction, ζ can be freely chosen. But according to Stora's Main Theorem of Renormalization, the equivalence class of the cocycle ζ should be fixed by the standard stability requirements, as *e.g.* existence of a vacuum and of KMS states, particle interpretation *etc*.

Conjecture: There is a distinguished equivalence class of cocycles which characterizes the spectrum condition.