# Non-positive sequences in analytic number theory and the Landau-Siegel zero

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# Twin-prime conjecture

A number of problems in analytic number theory can be reduced to showing that some related (finite or infinite) sequences of real numbers are *not* positive. For example, let $\rho(n)$ denote the characteristic function of primes, that is,

$$\rho(n) = \begin{cases} 1 & \text{if} \quad n \quad \text{is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

**Example 1.** Let

$$a_1(n) = 1 - \rho(n) - \rho(n+2).$$

The twin prime conjecture is equivalent to that the infinite sequence

$$\{a_1(n)\} = \{a_1(1),\, a_1(2),\, a_1(3), \ldots\}$$

contains infinitely many negative terms.

# Twin-prime conjecture

**Example 2.** A weak form of the twin prime conjecture can be obtained along the following line: Fix a finite sequence of non-negative integers

$$0 = n_1 < n_2 < \cdots < n_k \tag{1}$$

(some admissible conditions are required), and let

$$a_1^*(n) = 1 - \rho(n + n_1) - \rho(n + n_2) - \cdots - \rho(n + n_k).$$

If the sequence

$$\{a_1^*(n)\} = \{a_1^*(1),\, a_1^*(2),\, a_1^*(3), ...\}$$

contains infinitely many negative terms, then there are infinitely many pairs of primes $(p, q)$ such that

$$0 < q - p \le n_k.$$

This was first proved true for a sequence (1) with $n_k < 7 \times 10^7$. Such a numerical constant has been sharped to 246.

# The Goldbach conjecture

**Example 3.** Let $N \geq 4$ be even, and let

$$a_2(n) = 1 - \rho(n) - \rho(N - n)$$

for $n < N$. The Goldbach conjecture is equivalent to, for every $N$, that the finite sequence

$$\{a_2(n) : \ 1 < n < N\}$$

contains a negative term.

# The $\Lambda^2$-method

The above examples can be classified into two types.

**Type I.** To prove that a finite sequence $\{a(n)\}$ contains (at least) one negative term.

**Type II.** To prove that an infinite sequence $\{a(n)\}$ contains infinitely many negative terms.

There is an efficient method in analytic number theory to handle the Type II problems that can be described as follows. We construct a non-negative sequence $\{b(n)\}$ and evaluate the sum

$$\sum a(n)b(n), \tag{2}$$

where the summation is taken over an interval. In practice, we often choose

$$\sum = \sum_{x \le n < 2x} \quad \text{with} \quad x \to \infty.$$

# The $\Lambda^2$-method

If we can find appropriate coefficients $b(n) \geq 0$ and show that the sum (2) is negative, then there is a $n \in [x, 2x)$ such that $a(n) < 0$, as we need. In practice, the sum (2) is negative means, when it is evaluated, that the coefficient of the main term in the resulting expression is negative.

In the argument, the constraint $b(n) \geq 0$ severely limits the choice of $b(n)$. Based on the idea of Selberg's $\Lambda^2$-sieve method, we can first choose a sequence $c(n)$ of real numbers, and then take

$$b(n) = c(n)^2. \tag{3}$$

Thus the problem is reduced to verifying that

$$\sum a(n)c(n)^2 < 0$$

for some $c(n) \in \mathbf{R}$. The choice of $c(n)$ may depends on the parameter $x$.

# The $\Lambda^2$-method

Equivalently, if there exist $c_1(n),\ c_2(n) \in \mathbf{R}$ such that

$$\Big(\sum a(n)c_1(n)c_2(n)\Big)^2 > \Big(\sum a(n)c_1(n)^2\Big)\Big(\sum a(n)c_2(n)^2\Big),$$

then, by the Cauchy-Schwarz inequality, there is a $n \in [x, 2x)$ such that $a(n) < 0$.

The above method is successfully applied to the weak form of the twin prime conjecture (see Example 2). However, there might be certain ways to relax the constraints and get sharper estimates. We may ask:

1. Can we start with a more general form for $b(n)$ other than (3)?
2. Can we further replace the constraint $b(n) \geq 0$ by someone else?

# A simple inequality

For any $c_1(n)$, $c_2(n)$, $d_1(n)$, $d_2(n)$ we have

$$c_1(n)d_1(n)+c_2(n)d_2(n)=\big(c_1(n)+c_2(n)\big)d_1(n)-\big(d_1(n)-d_2(n)\big)c_2(n).$$

This implies that

$$\big|c_1(n)d_1(n)+c_2(n)d_2(n)\big|\le\big|c_1(n)+c_2(n)\big||d_1(n)|+\big|d_1(n)-d_2(n)\big||c_2(n)|.$$

Assume that

$$a(n)\ge 0 \quad \text{if} \quad x\le n<2x. \tag{4}$$

Then

$$\begin{aligned}&\Big|\sum a(n)\big(c_1(n)d_1(n)+c_2(n)d_2(n)\big)\Big|\\ &\qquad\le\sum a(n)\big|c_1(n)+c_2(n)\big||d_1(n)|\\ &\qquad\quad+\sum a(n)\big|d_1(n)-d_2(n)\big||c_2(n)|.\end{aligned} \tag{5}$$

# A simple inequality

On assuming (4), an upper bound for the right-hand side of (5) can be obtained via the Cauchy-Schwarz inequality. If (5) fails to hold for some $c_1(n)$, $c_2(n)$, $d_1(n)$, $d_2(n)$, a contradiction can be derived.

In some sieve problems, we may encounter the situation that the sums

$$\sum a(n)\big(c_1(n)+c_2(n)\big)^2 \quad \text{and} \quad \sum a(n)\big(d_1(n)-d_2(n)\big)^2$$

are close to zero, but the left-hand side of (5) is not. Thus a contradiction may be derived from (4). This method is equivalent to showing that the sum (2) is negative with

$$\begin{aligned} b(n) = & \big|c_1(n)+c_2(n)\big|\big|d_1(n)\big| + \big|d_1(n)-d_2(n)\big|\big|c_2(n)\big| \\ & - \big|c_1(n)d_1(n)+c_2(n)d_2(n)\big|. \end{aligned}$$

# Transfer to the Landau-Siegel zero

Let $\chi$ be a real primitive character to the modulus $D$. It is conjectured that the Dirichlet $L$-function

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

has *no* real zero very close to 1 (such a zero, if exists, is called the Landau-Siegel zero of $L(s,\chi)$). In our approach to this problem we first introduce a set $\Psi_1$ of primitive characters, and study the distribution of zeros of $L(s,\psi)$, with $\psi \in \Psi_1$, in a region $\Omega$. it can be shown, for every $\psi \in \Psi_1$ and for every zero $\rho$ of $L(s,\psi)$ in $\Omega$, that

$$\mathcal{C}^*(\rho,\psi) \geq 0 \tag{6}$$

## Transfer to the Landau-Siegel zero

where $\mathcal{C}^*(\rho, \psi)$ is approximately equal to, for certain quantities $\beta_1$, $\beta_2$ and $\beta_3$,

$$iZ(\rho, \psi)^{-1} \frac{L(\rho + \beta_1, \psi)L(\rho + \beta_2, \psi)L(\rho + \beta_3, \psi)}{L'(\rho, \psi)}$$

with

$$Z(s, \psi) = \frac{L(s, \psi)}{L(1 - s, \bar{\psi})}.$$

It should be stressed that the set $\Psi_1$ is formally defined. Without assuming the existence of the Landau-Siegel zero, one is even unable to determine whether $\Psi_1$ is empty or not. Thus, it is only possible to evaluate sums of the form

$$\sum_{\psi \in \Psi_1} \sum_{\rho} \mathcal{C}^*(\rho, \psi) |B(\rho, \psi)|^2 \omega(\rho) \tag{7}$$

under the assumption that the Landau-Siegel zero exists, where $\rho$ runs through the zeros of $L(s, \psi)$ in $\Omega$, and where $\omega$ is a positive, smooth weight.

# Contradiction deduced from the existence of the Landau-Siegel zero

In what follows we assume the existence of the Landau-Siegel zero, with the aim of deducing a contradiction. If we can prove that the sum (7) is negative for some $B(s, \psi)$, then a contradiction is immediately obtained by (6). As we are unable to do this directly, we argue with a relation similar to (5).

We construct functions $B(s, \psi)$, $J_1(s, \psi)$ and $J_2(s, \psi)$ of the forms

$$B(s, \psi) = \sum \frac{b(n)\chi\psi(n)}{n^s},$$

$$J_1(s, \psi) = \sum \frac{\varkappa_1(n)\chi\psi(n)}{n^s}, \quad J_2(s, \psi) = \sum \frac{\varkappa_2(n)\chi\psi(n)}{n^s}$$

where $b(n)$, $\varkappa_1(n)$ and $\varkappa_2(n)$ are supported on finite intervals.

# Contradiction deduced from the existence of the Landau-Siegel zero

Write

$$Z(s,\chi\psi) = \frac{L(s,\chi\psi)}{L(1-s,\chi\bar{\psi})}.$$

Assume $\psi \in \Psi_1$, $L(\rho,\psi) = 0$ and $\rho \in \Omega$. We have

$$\begin{aligned}
&B(\rho,\psi)\overline{J_1(\rho,\psi)} + \iota\overline{B(\rho,\psi)}J_2(\rho,\psi) \\
&\quad= \big(B(\rho,\psi) + \iota Z(\rho,\chi\psi)\overline{B(\rho,\psi)}\big)\overline{J_1(\rho,\psi)} \\
&\qquad - \iota\overline{B(\rho,\psi)}\big(Z(\rho,\chi\psi)\big)\overline{J_1(\rho,\psi)} - J_2(\rho,\psi)\big)
\end{aligned}$$

where $\iota$ is a constant. This gives

$$\begin{aligned}
&\big|B(\rho,\psi)\overline{J_1(\rho,\psi)} + \iota\overline{B(\rho,\psi)}J_2(\rho,\psi)\big| \\
&\quad\le \big|B(\rho,\psi) + \iota Z(\rho,\chi\psi)\overline{B(\rho,\psi)}\big|\,|J_1(\rho,\psi)| \\
&\qquad + |\iota B(\rho,\psi)|\,\big|Z(\rho,\chi\psi))\overline{J_1(\rho,\psi)} - J_2(\rho,\psi)\big|.
\end{aligned}$$

# Contradiction deduced from the existence of the Landau-Siegel zero

Hence, by (6),

$$\Big|\sum_{\psi\in\Psi_1}\sum_{\rho}\mathcal{C}^*(\rho,\psi)\big(B(\rho,\psi)\overline{J_1(\rho,\psi)}+\iota\overline{B(\rho,\psi)}J_2(\rho,\psi)\big)\omega(\rho)\Big|$$
$$\leq\sum_{\psi\in\Psi_1}\sum_{\rho}\mathcal{C}^*(\rho,\psi)\big|B(\rho,\psi)+\iota Z(\rho,\chi\psi)\overline{B(\rho,\psi)}\big||J_1(\rho,\psi)|\omega(\rho)$$
$$+|\iota|\sum_{\psi\in\Psi_1}\sum_{\rho}\mathcal{C}^*(\rho,\psi)|B(\rho,\psi)|\big|Z(\rho,\chi\psi))\overline{J_1(\rho,\psi)}-J_2(\rho,\psi)\big|\omega(\rho).$$

This can lead to a contradiction if $B(s,\psi)$, $J_1(s,\psi)$, $J_2(s,\psi)$ and $\iota$ are appropriately chosen.

Thank you.