A Bridge Between the Baum-Connes Conjecture and the Langlands Program

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The fields we will consider:

- p: prime number
- F: a non-archimedean local field, i.e. a finite extension of
  - $\mathbb{Q}_p := \left\{x = \sum_{m=M}^{+\infty} a_m p^m \,:\, a_m \in \{0, 1, \ldots, p-1\}\right\}$.
  - or of the field of formal Laurent series $\mathbb{F}_p((T))$ over the finite field $\mathbb{F}_p$, with p elements;
- $\mathfrak{o}_F$ ring of integers of F, and $\mathfrak{p}_F$ its maximal ideal.

Example

The p-adic norm:

$$\left|p^m \frac{a}{b}\right|_p := p^{-m}, \quad \text{if } a, b \in \mathbb{Z} \text{ not divisible by } p \text{ and } m \in \mathbb{Z}_{\geq 0}.$$

We have $|xy|_p = |x|_p \cdot |y|_p$ but $|x + y|_p \leq \max(|x|_p, |y|_p)$.

$$\mathfrak{o}_{\mathbb{Q}_p} = \mathbb{Z}_p = \{x \in \mathbb{Q}_p \,:\, |x|_p \leq 1\} \quad \text{and} \quad \mathfrak{p}_{\mathbb{Z}_p} = \{x \in \mathbb{Z}_p \,:\, |x|_p < 1\}.$$

The groups we will consider:

- **G** connected reductive algebraic group defined over F.
- $G = \mathbf{G}(F)$: *p-adic reductive group*

Examples

$\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$, $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_m(F)$, exceptional groups $\mathrm{E}_n(F)$, $n = 6, 7, 8$, $\mathrm{G}_2(F)$, $\mathrm{F}_4(F)$.

Topology

Every neighbourhood of the identity in G contains a compact open subgroup (equivalently, G is locally compact and totally disconnected).

Example

The subgroups $K_0 := \mathrm{GL}_n(\mathfrak{o}_F)$ and $K_m := 1 + \mathfrak{p}_F^m \mathrm{M}_n(\mathfrak{o}_F)$, $m \geq 1$, of $G = \mathrm{GL}_n(F)$ are compact open and give a fundamental system of neighbourhoods of the identity in G.

Representations of G

- (π, V): V a $\mathbb{C}$-vector space (in general of infinite dimension) and $\pi\colon G \to \mathrm{GL}(V)$ a group morphism
- (π, V) is smooth if for any $v \in V$, $G_v := \{g \in G : \pi(g)(v) = v\}$ is an open subgroup of G.

Let $\mathfrak{R}(G)$ denote the category of smooth representations of G.

Parabolic induction

- P: parabolic subgroup of G
- Levi decomposition: $P = LU$ with L Levi factor, U unipotent radical
- σ: smooth representation of L
- $\widetilde{\sigma}$: inflation of σ to P
- $\mathrm{Ind}_P^G(\widetilde{\sigma})$: induction of $\widetilde{\sigma}$ to G.

The parabolic induction functor $\mathrm{i}_{L,P}^G\colon \mathfrak{R}(L) \to \mathfrak{R}(G)$ is defined by

$$\mathrm{i}_{L,P}^G(\sigma) := \mathrm{Ind}_P^G(\widetilde{\sigma}).$$

Definition

A smooth representation π of G is supercuspidal if it is not a subquotient of a proper parabolically induced representation.

Characterization

π is supercuspidal if and only if all its matrix coefficients have compact support modulo the center of G. (A matrix coefficient of π is a function on G of the form $c_{\tilde{v},v}\colon g \mapsto \langle \tilde{v}, \pi(g)v \rangle$, for a fixed $(\tilde{v}, v) \in \widetilde{V} \times V$, where $\widetilde{V}$ is the space of the contragredient of (π, V).)

Definition/Notation

A character $\chi\colon G \to \mathbb{C}^\times$ is unramified if χ is trivial on every compact subgroup of G. Let $\mathfrak{X}(G)$ denote the group of unramified characters of G, and $\mathfrak{X}_{\mathrm{u}}(G) \subset \mathfrak{X}(G)$ the subgroup of unitary unramified characters of G.

Orbits of supercuspidal representations

Let L be a Levi subgroup of G and σ an irreducible supercuspidal smooth representation of L. We set

$$\mathcal{O} := \{\sigma \otimes \chi, \ \text{ with } \chi \in \mathfrak{X}_{\mathrm{nr}}(L)\}$$

and write $\mathfrak{s} := (L, \mathcal{O})_G = [L, \sigma]_G$ for the G-conjugacy class of the pair $(L, \mathcal{O})$. Let $\mathfrak{B}(G)$ denote the set of such classes $\mathfrak{s}$.

The Bernstein decomposition of the category of smooth representations

Let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every subquotient of π is equivalent to a subquotient of a parabolically induced representation $\mathrm{i}_{L,P}^{G}(\sigma')$, where $\sigma' \in \mathcal{O}$. The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ in a direct product:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

The Bernstein decomposition in concrete terms

Let (π, V) be a smooth representation of G. Then for each $\mathfrak{s} \in \mathfrak{B}(G)$ the space V has a unique G-subspace $V^{\mathfrak{s}}$ such that

(1) $V^{\mathfrak{s}}$ is an object of $\mathfrak{R}^{\mathfrak{s}}(G)$, and

(2) $V^{\mathfrak{s}}$ is maximal for property (1).

Moreover, $V = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} V^{\mathfrak{s}}$, and, if (π', V') is another smooth representation of G, we have $\mathrm{Hom}_G(V, V') = \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Hom}_G(V^{\mathfrak{s}}, V'^{\mathfrak{s}})$.

Application: the Bernstein decomposition of the Hecke algebra of G

Let $\mathcal{H}(G)$ be the vector space $C_c^\infty(G)$ of locally constant compactly supported functions on G, endowed with the convolution product. It provides a smooth representation of G, via left translation of functions. The spaces $\mathcal{H}(G)^{\mathfrak{s}}$ are then two-sided ideals of $\mathcal{H}(G)$.

Definition

Choose a left-invariant Haar measure on G and form the Hilbert space $L^2(G)$.

The left regular representation λ of $L^1(G)$ is given by $(\lambda(f))(h) := f * h$, where $f \in L^1(G)$ and $h \in L^2(G)$, and $*$ denotes convolution.

The reduced C^*-algebra of G is the C^*-algebra $C^*_{\mathrm{r}}(G)$ generated by the image of λ.

Its spectrum may be identified to the tempered dual $\mathrm{Irr}_{\mathrm{t}}(G)$ of G.

The Bernstein decomposition of the reduced C^*-algebra of G

We have

$$C^*_{\mathrm{r}}(G) = \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} C^*_{\mathrm{r}}(G)^{\mathfrak{s}},$$

where $C^*_{\mathrm{r}}(G)^{\mathfrak{s}}$ is the closure of $\mathcal{H}(G)^{\mathfrak{s}}$ in $C^*_{\mathrm{r}}(G)$.

The spectrum of $C^*_{\mathrm{r}}(G)^{\mathfrak{s}}$ may be identified to $\mathrm{Irr}_{\mathrm{t}}(G) \cap \mathrm{Irr}^{\mathfrak{s}}(G)$, where $\mathrm{Irr}_{\mathrm{t}}(G)$ is the tempered dual of G and $\mathrm{Irr}^{\mathfrak{s}}(G)$ is the set of irreducible objects of the category $\mathfrak{R}^{\mathfrak{s}}(G)$.

Notation

We set

$$W^{\mathfrak{s}} := \{n \in \mathrm{N}_G(L) \,:\, {}^n\sigma \sim \sigma \otimes \chi \text{ for some } \chi \in \mathfrak{X}(L)\}\,/L,$$

and $T^{\mathfrak{s}}_{\mathrm{u}} := \mathfrak{X}_{\mathrm{u}}(L) \cdot \sigma$, where σ is chosen to be unitary.

Conjecture [A-Baum-Plymen (2011)]

Suppose that G is F-split. For each $\mathfrak{s} \in \mathfrak{B}(G)$, we have

$$K^j_{W^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \simeq K_j(C^*_{\mathrm{r}}(G)^{\mathfrak{s}}) \quad \text{for } j = 0, 1,$$

where $K^j_{W_{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}})$ is the classical topological equivariant K-theory for the finite (extended Weyl) group $W^{\mathfrak{s}}$ acting on the compact torus $T^{\mathfrak{s}}_{\mathrm{u}}$.

Remark

The conjecture holds true when $\mathcal{H}(G)^{\mathfrak{s}}$ is Morita equivalent to an affine Hecke algebra, which is notably the case when G is a classical group or the exceptional group $\mathrm{G}_2(F)$. Note however that $\mathcal{H}(G)^{\mathfrak{s}}$ is not always Morita equivalent to an affine Hecke algebra (counter-example when $G = \mathrm{SL}_n(F)$).

Q: What about arbitrary p-adic groups?

A: Add a 2-cocycle twist!

Conjecture [A-Baum-Plymen-Solleveld, 2017]

Let G be an arbitrary p-adic group. For each $\mathfrak{s} \in \mathfrak{B}(G)$, there exists a 2-cocycle $\natural^{\mathfrak{s}}\colon W^{\mathfrak{s}} \times W^{\mathfrak{s}} \to \mathbb{C}^\times$ such that

$$K^j_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \simeq K_j(C^*_{\mathrm{r}}(G)^{\mathfrak{s}}) \quad \text{for } j = 0, 1,$$

where

$$K^j_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) := p_{\natural^{\mathfrak{s}}} K^j_{\widetilde{W^{\mathfrak{s}}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \simeq K_j(C_0(T^{\mathfrak{s}}_{\mathrm{u}}) \rtimes \mathbb{C}[W^{\mathfrak{s}}, \natural^{\mathfrak{s}}]),$$

with $C_0(T^{\mathfrak{s}}_{\mathrm{u}}) := \{f \in C(T^{\mathfrak{s}}_{\mathrm{u}} \sqcup \{\infty\}) : f(\infty) = 0\}$ and $p_{\natural^{\mathfrak{s}}} \in \mathbb{C}[W^{\mathfrak{s}}]$ a minimal idempotent such that $p_{\natural^{\mathfrak{s}}}\mathbb{C}[\widetilde{W^{\mathfrak{s}}}] \simeq \mathbb{C}[W^{\mathfrak{s}}, \natural^{\mathfrak{s}}]$, for a central extension $\widetilde{W^{\mathfrak{s}}}$ of $W^{\mathfrak{s}}$.

The conjecture holds modulo torsion [Solleveld, 2022]:

We have

$$K^j_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T^{\mathfrak{s}}_{\mathrm{u}}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq K_j(C^*_{\mathrm{r}}(G)^{\mathfrak{s}}) \otimes_{\mathbb{Z}} \mathbb{C} \quad \text{for } j = 0, 1.$$

The Baum–Connes Conjecture (proved by V. Lafforgue)

Let $\mathcal{I}(G)$ be the Bruhat-Tits building of G, it is a universal space for proper G-actions. Let $K_*^G(\mathcal{I}(G))$ denote its G-equivariant K-homology. The canonical assembly map

$$K_*^G(\mathcal{I}(G)) \to K_*(C_{\mathrm{r}}^*(G))$$

is an isomorphism.

Consequence

The ABPS K-theory conjecture asserts the existence of a bijection

$$K_*^G(\mathcal{I}(G)) \to \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K^j_{W^{\mathfrak{s}},\natural^{\mathfrak{s}}}(T_{\mathrm{u}}^{\mathfrak{s}}).$$

Notation

- Γ: (finite) group acting on a topological space X
- Γ_x: stabilizer in Γ of $x \in X$.

Let $\natural = (\natural_x)_{x \in X}$ be a collection of 2-cocycles

$$\natural_x \colon \Gamma_x \times \Gamma_x \to \mathbb{C}^\times,$$

such that $\natural_{\gamma x}$ and $\gamma_* \natural_x$ define the same class in $H^2(\Gamma_{\gamma x}, \mathbb{C}^\times)$, where $\gamma_* \colon \Gamma_x \to \Gamma_{\gamma x}$ sends α to $\gamma \alpha \gamma^{-1}$.

A topological space

Let $\mathbb{C}[\Gamma_x, \natural_x]$ be the group algebra of Γ_x twisted by $\natural_x$. We set

$$\widetilde{X}_\natural := \{(x, \tau) \,:\, x \in X,\, \tau \in \mathrm{Irr}\, \mathbb{C}[\Gamma_x, \natural_x]\},$$

and topologize $\widetilde{X}_\natural$ by decreeing that a subset of $\widetilde{X}_\natural$ open if its projection to the first coordinate is open in X.

Required condition:

For every $(\gamma, x) \in \Gamma \times X$, there is an algebra isomorphism

$$\phi_{\gamma,x} \colon \mathbb{C}[\Gamma_x, \natural_x] \to \mathbb{C}[\Gamma_{\gamma x}, \natural_{\gamma x}]$$

such that

(a) if $\gamma x = x$, then $\phi_{\gamma,x}$ is the conjugation by an element of $\mathbb{C}[\Gamma_x, \natural_x]^\times$;

(b) $\phi_{\gamma',\gamma x} \circ \phi_{\gamma,x} = \phi_{\gamma'\gamma,x}$ for all $\gamma', \gamma \in \Gamma$ and $x \in X$.

Definition

Define a Γ-action on $\widetilde{X}_\natural$ by $\gamma \cdot (x, \tau) := (\gamma x, \tau \circ \phi_{\gamma,x}^{-1})$.
The *twisted extended quotient* of X by Γ with respect to $\natural$ is

$$(X /\!/ \Gamma)_\natural := \widetilde{X}_\natural / \Gamma.$$

In the case when the 2-cocycles $\natural_x$ are trivial, write simply $X /\!/ \Gamma$ for $(X /\!/ \Gamma)_\natural$ and refer to it as the *extended quotient* of X by Γ.

The ABPS Conjecture (2015)

For each $\mathfrak{s} \in \mathfrak{B}(G)$, there is a collection of 2-cocycles

$$\natural_t \colon W_t^{\mathfrak{s}} \times W_t^{\mathfrak{s}} \to \mathbb{C}^\times \quad (t \in T^{\mathfrak{s}})$$

and a bijection (with good properties)

$$\eta^{\mathfrak{s}} \colon \mathrm{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} (T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_\natural$$

which restricts to a bijection

$$\eta_{\mathrm{t}}^{\mathfrak{s}} \colon \mathrm{Irr}_{\mathrm{t}}(G) \cap \mathrm{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} (T_{\mathrm{u}}^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_\natural.$$

Theorem [Brodzki-Plymen 2000: $\mathrm{GL}_n(F)$; Moussaoui 2015: split classical groups, ABPS 2017: principal series of split p-adic groups; 2019: inner forms of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$; Solleveld 2021: in general]

The conjecture holds true. Moreover, the 2-cocycles $\natural_t$ can be taken to be trivial for principal series of split p-adic groups [ABPS], for $\mathrm{G}_2(F)$ [A-Xu], and for pure inner forms of split classical groups [A-Moussaoui-Solleveld].

Notation (Galois side)

- $W_F \subset \mathrm{Gal}(F_{\mathrm{sep}}/F)$: Weil group of F
- $G^\vee$: complex reductive group with root datum dual to that of G

Definition (assuming that G is an inner form of an F-split group)

A Langlands parameter – or L-parameter – is a continuous morphism $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee =: {}^L G$ such

- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is morphism of algebraic groups,
- $\varphi(w)$ is a semisimple element of $G^\vee$, for any $w \in W_F$.

$\Phi(G) :=$ set of $G^\vee$-conjugacy classes of L-parameters for G

The Local Langlands Correspondence (crude form)

It predicts a surjective map with finite fibers, called L-packets, (with good properties),

$$\begin{array}{ccc} \mathrm{Irr}(G) & \longrightarrow & \Phi(G) \\ \pi & \mapsto & \varphi_\pi \end{array}.$$

Notation

- $G^\vee_{\mathrm{der}}$ derived group of $G^\vee$
- $G^\vee_{\mathrm{sc}}$ simply connected cover of $G^\vee_{\mathrm{der}}$

Groups attached to an L-parameter φ:

- $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi) := \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi(W'_F))$, where $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$
- $S_\varphi := \pi_0\left(\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)\right)$ component group of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)$

Definition

An enhanced L-parameter is a pair (φ, ρ) where φ is an L-parameter and ρ an irreducible representation of S_φ.
The representation ρ is called an enhancement of φ.

Action of $G^\vee$ on the set of enhanced L-parameters:

$g \cdot (\varphi, \rho) := (g\varphi g^{-1}, {}^g\rho)$, for $g \in G^\vee$, where ${}^g\rho\colon h \mapsto \rho(g^{-1}hg)$.
$\Phi_{\mathrm{e}}(G)$: set of $G^\vee$-conjucagy classes of enhanced L-parameters.

The Local Langlands Correspondence (refined form)

It predicts a bijection (with good properties),

$$\begin{array}{c}\mathrm{Irr}(G) \xrightarrow{\mathfrak{L}^G} \Phi_{\mathrm{e}}(G) \\ \pi \mapsto (\varphi_\pi, \rho_\pi).\end{array}.$$

Q: How does the Bernstein decomposition of $\mathrm{Irr}(G)$ reflect on $\Phi_{\mathrm{e}}(G)$?

Notation/Proposition

- $\mathcal{G}_\varphi$: inverse image in $G^\vee_{\mathrm{sc}}$ of the group $\mathrm{Z}_{G^\vee_{\mathrm{ad}}}(\varphi(W_F))$
- $u_\varphi := \varphi(1, \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)) \in \mathcal{G}_\varphi^\circ$ (a unipotent element)
- $A_{\mathcal{G}_\varphi}(u_\varphi) := \pi_0(\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi))$

We have $S_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi)$.

Definition [A-Moussaoui-Solleveld (2018)]

An enhanced L-parameter $(\varphi, \varrho) \in \Phi_{\mathrm{e}}(G)$ is cuspidal if

- φ is discrete: $\varphi(W_F')$ is not contained in any proper Levi subgroup of $G^\vee$,
- (u_φ, ϱ) is a cuspidal pair in $\mathcal{G}_\varphi$: there is an $\mathcal{G}_\varphi^\circ$-equivariant cuspidal local system on the $\mathcal{G}_\varphi^\circ$-conjugacy class of u_φ.

$\Phi_{\mathrm{e}}^{\mathrm{cusp}}(G) := \{G^\vee\text{-conjugacy classes of cuspidal } (\varphi, \varrho)\} \subset \Phi_{\mathrm{e}}(G)$.

Theorem

We have

$$\mathrm{LLC}\colon \mathrm{Irr}^{\mathrm{cusp}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathrm{cusp}}(G),$$

when G is an inner form of $\mathrm{GL}_n(F)$ or $\mathrm{SL}_n(F)$ [A-Moussaoui-Solleveld, 2018], when $G = \mathrm{G}_2(F)$ [A-Xu, 2022], and when G is a pure inner form of a split classical group [A-Moussaoui-Solleveld, 2022].
(We expect it always holds.)

Mimic the p-adic side:

Let L be a Levi subgroup of G and $(\varphi_c, \varrho_c) \in \Phi_e^{cusp}(L)$.
Define $\mathfrak{s}^\vee := [{}^L L, (\varphi_c, \varrho_c)]_{G^\vee}$ to be the $G^\vee$-conjugacy class of the orbit of (φ_c, ϱ_c) under the action of $\mathfrak{X}_{nr}({}^L L)$, a group naturally isomorphic to the group $\mathfrak{X}_{nr}(L)$. Let $\mathfrak{B}^\vee(G)$ be the set of such $\mathfrak{s}^\vee$.

More details:

$\mathfrak{X}_{nr}({}^L L) := (Z_{L^\vee \rtimes I_F})^\circ_{W_F}$, where $I_F \subset W_F$ is the inertia group of F and $Z_{L^\vee \rtimes I_F}$ is the center of $L^\vee \rtimes I_F$.

Theorem [A.-Moussaoui-Solleveld, 2018]

The set $\Phi_e(G)$ is partitioned as

$$\Phi_e(G) = \prod_{\mathfrak{s}^\vee \in \mathfrak{B}({}^L G)} \Phi_e^{\mathfrak{s}^\vee}(G),$$

where $\Phi_e^{\mathfrak{s}^\vee}(G)$ consists of the enhanced L-parameters whose cuspidal support lies in $\mathfrak{s}^\vee$.

Theorem [A-Moussaoui-Solleveld, 2018] (Galois version of ABPS)

For each $\mathfrak{s}^\vee = [{}^L L, \lambda]_{G^\vee} \in \mathfrak{B}({}^L G)$, there is a collection of 2-cocycles

$$\natural_t^{\mathfrak{s}^\vee} : W_t^{\mathfrak{s}^\vee} \times W_t^{\mathfrak{s}^\vee} \to \mathbb{C}^\times \quad (t \in T^{\mathfrak{s}^\vee})$$

and a bijection (with good properties)

$$\eta^{\mathfrak{s}^\vee} : \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \xleftrightarrow{1-1} (T^{\mathfrak{s}^\vee} /\!/ W^{\mathfrak{s}^\vee})_{\natural^{\mathfrak{s}^\vee}}$$

where $T^{\mathfrak{s}^\vee} := \mathfrak{X}({}^L L) \cdot \lambda$ and $W_t^{\mathfrak{s}^\vee}$ is the stabilizer of t in

$$W^{\mathfrak{s}^\vee} := \left\{ w \in W(L^\vee) : {}^w\lambda \sim \lambda \otimes \chi^\vee \text{ for some } \chi^\vee \in \mathfrak{X}({}^L L) \right\}$$

with $W(L^\vee) = \mathrm{N}_{G^\vee}(L^\vee)/L^\vee$. Moreover, $\eta^{\mathfrak{s}^\vee}$ restricts to a bijection

$$\eta_{\mathrm{t}}^{\mathfrak{s}^\vee} : \Phi_{\mathrm{e}}^{\mathrm{bd}}(G) \cap \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) \xleftrightarrow{1-1} (T_{\mathrm{u}}^{\mathfrak{s}^\vee} /\!/ W^{\mathfrak{s}^\vee})_{\natural^{\mathfrak{s}^\vee}},$$

where $T_{\mathrm{u}}^{\mathfrak{s}^\vee} := \mathfrak{X}_{\mathrm{u}}({}^L L) \cdot \lambda$ and $\Phi_{\mathrm{e}}^{\mathrm{bd}}(G)$ consists of the $G^\vee$-conjugacy classes of (φ, ρ) with φ bounded.

Summary:

For any p-adic group G, any $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$ and any $\mathfrak{s}^\vee = [{}^L L, \lambda]_{G^\vee} \in \mathfrak{B}^\vee(G)$, we have bijections:

$$\mathrm{Irr}^{\mathfrak{s}}(G) \overset{1-1}{\longleftrightarrow} (T^{\mathfrak{s}} /\!/ W^{\mathfrak{s}})_{\natural^{\mathfrak{s}}} \quad \text{and} \quad (T^{\mathfrak{s}^\vee} /\!/ W^{\mathfrak{s}^\vee})_{\natural^{\mathfrak{s}^\vee}} \overset{1-1}{\longleftrightarrow} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G).$$

Hypothesis

For any $\mathfrak{s} \in \mathfrak{B}(G)$, there exists a map

$$\begin{array}{rccc} \mathfrak{L}^{\mathfrak{s}}: & \mathrm{Irr}^{\mathfrak{s}}(L) & \longrightarrow & \Phi_{\mathrm{e}}^{\mathrm{cusp}}(L) \\ & \sigma & \mapsto & \lambda = (\varphi_\sigma, \varrho_\sigma) \end{array}$$

which satisfies the following functoriality property.

Property

(1) For any $\chi \in \mathfrak{X}_{\mathrm{nr}}(L)$, we have

$$(\varphi_{\chi\otimes\sigma}, \varrho_{\chi\otimes\sigma}) = \chi^\vee \cdot (\varphi_\sigma, \varrho_\sigma),$$

where $\chi \mapsto \chi^\vee$ is the canonical isomorphism $\mathfrak{X}_{\mathrm{nr}}(L) \xrightarrow{\sim} \mathfrak{X}_{\mathrm{nr}}({}^L L)$.

(2) For any $w \in W(L) := \mathrm{N}_G(L)/L$, we have

$${}^{w^\vee}(\varphi_\sigma, \varrho_\sigma) \simeq (\varphi_{{}^w\sigma}, \varrho_{{}^w\sigma}),$$

where $w \mapsto w^\vee$ is the canonical isomorphism $W(L) \xrightarrow{\sim} W(L^\vee)$.

(3) The collections of 2-cocycles $\natural^{\mathfrak{s}}$ and $\natural^{\mathfrak{s}^\vee}$ satisfy the following

$$\natural^{\mathfrak{s}^\vee}_{\chi^\vee} = \natural^{\mathfrak{s}}_\chi \quad \text{for any } \sigma \in \mathfrak{s} \text{ and any } \chi \in T^{\mathfrak{s}} \simeq \mathfrak{X}_{\mathrm{nr}}(L)/\mathfrak{X}_{\mathrm{nr}}(L,\sigma),$$

where $\mathfrak{X}_{\mathrm{nr}}(L,\sigma) := \{\chi \in \mathfrak{X}_{\mathrm{nr}}(L) \,:\, \chi \otimes \sigma \simeq \sigma\}$.

Conclusion

Then there is a canonical bijection

$$(T^{\mathfrak{s}}/\!/W^{\mathfrak{s}})_{\natural^{\mathfrak{s}}} \xleftrightarrow{1-1} (T^{\mathfrak{s}^\vee}/\!/W^{\mathfrak{s}^\vee})_{\natural^{\mathfrak{s}^\vee}},$$

and, as a consequence:

$$(*) \qquad \mathrm{Irr}(G) = \bigcup_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G) \xleftrightarrow{1-1} \bigcup_{\mathfrak{s}^\vee\in\mathfrak{B}^\vee(G)} \Phi_{\mathrm{e}}^{\mathfrak{s}^\vee}(G) = \Phi_{\mathrm{e}}(G).$$

$$(*^{\mathrm{t}}) \qquad \mathrm{Irr}^{\mathrm{t}}(G) \xleftrightarrow{1-1} \Phi_{\mathrm{e}}^{\mathrm{bd}}(G).$$

Theorem [A-Moussaoui-Solleveld, 2022]

Let G be a pure inner inner form of an F-split classical group, with F a p-adic field. Then all the cocycles are trivial, the restriction to $\mathrm{Irr}^{\mathrm{cusp}}(L)$ of the LLC defined by Arthur satisfies the Hypothesis, and the bijection $(*)$ (resp. $(*^{\mathrm{t}})$) coincides with the LLC (resp. the tempered LLC) constructed by Arthur.

Consequence

When the Hypothesis and the ABPS K-theory conjecture hold (for instance when G is $\mathrm{G}_2(F)$, or when F is a p-adic field and G is a pure inner inner form of an F-split classical group), we obtain

$$K_*^G(\mathcal{I}(G)) \xrightarrow{\text{BC}} K_*(C^*_{\mathrm{red}}(G)) \xleftrightarrow{1-1} K^*(\mathrm{Irr}^{\mathrm{t}}(G)) \xleftrightarrow{1-1} \text{“}K^*(\Phi_{\mathrm{e}}^{\mathrm{bd}}(G))\text{”},$$

where

$$\text{“}K^*(\Phi_{\mathrm{e}}^{\mathrm{bd}}(G))\text{”} := \bigoplus_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} K^*_{W^{\mathfrak{s}^\vee}, {}^L\natural_{\mathfrak{s}^\vee}}(T_{\mathrm{u}}^{\mathfrak{s}^\vee}).$$

Thank you very much for your attention!