Determining distributions of groups from their moments

Distributions of numbers canbe recognized by their moments

kth moment: M_k = average of x^k

random veriable

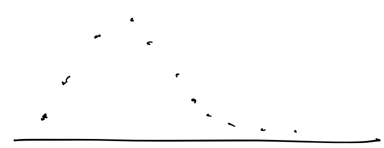
$\int_X x^k \, d\mu$ or $\mathbb{E}[X^k]$

distribution

Gaussian distribution

$\mathbb{E}[(X - \text{mean})^k] = \begin{cases} 0 & k \text{ odd} \\ \sigma^k (k-1)!! & k \text{ even} \end{cases}$

Poisson distribution



$\mathbb{E}[X(X-1)\cdots(X-(k-1))] = \lambda^k$

λ parameter

Knowing these $\iff \mathbb{E}[X^k]$

<u>Thm</u> (Uniqueness of the moment problem)
When the moments don't grow too quickly, there is at most 1 distribution with these moments.

$M_k = e^k$ not too quickly $\qquad M_k = e^{k^2}$ is too quick

- Moments often more acessible
  $\Rightarrow$ makes this useful

<u>Random groups</u> ← random variable valued in {Groups}

or just think of a measure on {Groups}

<u>Examples</u>

Number Theory

Topology

$[K:\mathbb{Q}]<\infty$ | Random K in some family | Random M | 3-manifold M (3 dim'l manifold compact)

e.g. $K=\mathbb{Q}(i), \mathbb{Q}(\sqrt[3]{2})$ | $\rightsquigarrow$ | $\rightsquigarrow$

Cl_K class group, finite abelian group | Cl_K random group | $\longleftrightarrow$ | $H_1(M,\mathbb{Z})=\pi_1(M)^{ab}$

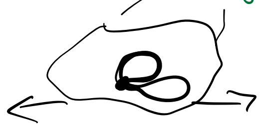
measures the failure of unique factorization in K into primes

Cl_K/pCl_K | $H_1(M,\mathbb{F}_p)$

$Cl_{\mathbb{Q}}=1$

Kummer could show Fermat's Last Theorem case $x^p+y^p=z^p$ when $p \nmid |Cl_{\mathbb{Q}(e^{2\pi i/p})}|$

$\pi_1(M)$ random group

$Gal(K^{un}/K)=\pi_1^{ét}(\operatorname{Spec}\mathcal{O}_K)$  $\pi_1(M)$ fundamental group

History Heath-Brown '94 2-Selmer groups of $y^2=x^3-Dx$ (elliptic curves)

Fouvry-Klüners '06 $2Cl_K/4Cl_K$ as K varies over $\mathbb{Q}(\sqrt{D})$

Both studying distributions on $\{$finite dim'l $\mathbb{F}_2$-vector spaces$\}=\{\mathbb{F}_2^0,\mathbb{F}_2^1,\mathbb{F}_2^2,\ldots\}$

Treat V as $|V|$

- Found moments of $|V|$, i.e. $\mathbb{E}[|V|^k]$
- there was a conjectural distribution w/ moments known that matched $M_k\approx 2^{k^2}$
- Proved new thms that moments determine dist for their cases

$|V|^k=|\operatorname{Hom}(V,\mathbb{F}_2^k)|$ $\quad V=\mathbb{F}_2^d \quad |V|^k=2^{dk}$

Modern theory of moments of random (pro)-finite groups

- moments indexed by finite groups
- moments are real numbers
- G^{th} moment is $\mathbb{E}[\#\operatorname{Hom}(X,G)]$

this theory works for random groups w/ many

of X
random group
of μ
measure on {Groups}

$$\int_X \#\mathrm{Hom}(X,G)\, d\mu$$

finite gps

Thm (Wang - W. '21) If X & Y are random finite abelian groups & for each finite abelian group

$$\mathbb{E}(\#\mathrm{Hom}(X,A)) = \mathbb{E}(\#(\mathrm{Hom}\,Y,A))$$

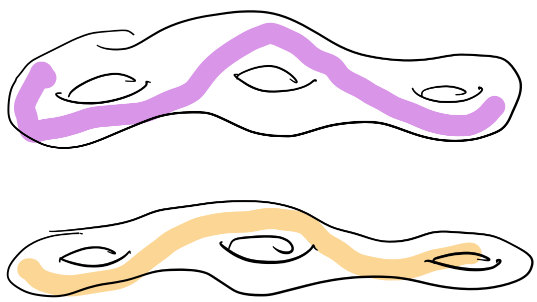
& these don't grow too quickly
then X & Y have the same distribution.

Application: Liu, W., Zureick-Brown '19 to distributions of class groups in function field analog ($\mathbb{Q} \rightsquigarrow \mathbb{F}_q(t)$) rational functions over a finite field
as $q \to \infty$

Jt work w/ Sawin
What if you don't know an explicit distribution w/ moments you find?

We construct a distribution from moments (of random gps) or show none exists (moments not growing too quickly).

Application Dunfield-Thurston model of a random Heegaard splitting



2
genus g
handlebodies

← solid 3-dim'l manifolds w/ boundary

Glue surfaces together
⇓
3-manifold w/o boundary

How to glue? Different ways to glue given by mapping class group of genus g

- Random walk in mapping class group of length $\ell \to \infty$
- Let $g \to \infty$ $\boxed{\mathcal{M}}$
- Can compute moments $\mathbb{E}(\#\mathrm{Hom}(\pi_1(\mathcal{M}), G))$ for each G
& use our theory to write down distribution.

Cor (Sawin–W. '22) Let S be a finite set of primes.

$$\mathrm{Prob}(\pi_1(\mathcal{M}) \text{ has no non-trivial } S\text{-gp qts}) = \prod_{p \in S} \prod_{j \geq 1} (1+p^{-j})^{-1} \prod_{\substack{N \text{ non-ab} \\ \text{finite simple} \\ S\text{-groups}}} e^{-\frac{|H_2(N,\mathbb{Z})|}{|\mathrm{Out}(N)|}}.$$

finite group whose order is a product of primes in S

surjective

$$\mathbb{E}(\#\mathrm{Sur}(X,G)) \longleftrightarrow \mathbb{E}(\#\mathrm{Hom}(X,G))$$

$$\#\mathrm{Sur}(X,G) = \sum_{H \subset G} \mu(H,G)\,\#\mathrm{Hom}(X,H) \qquad \#\mathrm{Hom}(X,G) = \sum_{H \subset G} \#\mathrm{Sur}(X,H)$$

$$\mathbb{E}(X(X-1)\cdots(X-(k-1))) \qquad \mathbb{E}(X^k)$$