MD - James Gundry 1502.03034. CMP 2016

MD - Roger Penrose 2203.08567. Ann. Phys 2023

MD 2304.08574

Equivalence principle, de-Sitter space, twistors

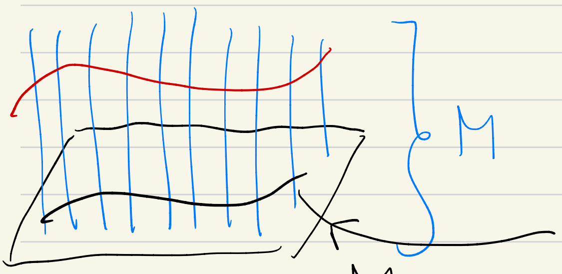
$$m\ddot{\vec{x}} = -\underset{\sim}{\nabla} V \ ; \quad V : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \qquad (1)$$

- Eisenhart (1928) Ann. Maths.

(M, G) Lorentzian mfd, dim 5.

$$G = 2\,du\,dt + 2\frac{V}{m}dt^2 - d\vec{x}^2.$$

$(u, t, \vec{x})$ coordinates on M



on $N = M/\mathbb{R}^*$ (space of orbits of $\frac{\partial}{\partial u}$)

$$L = \dot{u}\dot{t} + \frac{V}{m}\dot{t}^2 - |\dot{\vec{x}}|^2, \quad L = 0$$

Euler-Lagrange eq. $\longrightarrow$ (1).

- Cartan : (1) = geodesics of a non-metric connection.

$x^a = (\vec{x}, t)$; $\ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0$

non-zero components $\Gamma^i_{44} = \delta^{ij} \frac{\partial V}{\partial x^j}$

$a = 1, \dots, 4$, $i = 1, \dots, 3$

1 - parameter family of Lorentzian metrics

$g(\varepsilon)$; $\begin{cases} \lim_{\varepsilon \to 0} g(\varepsilon) \text{ blows up} \\ \lim_{\varepsilon \to 0} \nabla_{(\varepsilon)} \text{ finite.} \end{cases}$

e.g. Hyper ~~K~~ähler metric (Ricci-flat)

$$g = (1 + 2c^{-2} V) d\vec{x}^2 + \frac{c^2}{(1 + 2c^{-2} V)} (d\tau + 2c^{-3} A)^2$$

$V : \mathbb{R}^3 \to \mathbb{R}$, $A \in \Lambda^1(\mathbb{R}^3)$; $\boxed{*_{\mathbb{R}^3} dV = dA}$

$\varepsilon \to 0$, $\Gamma^i_{\tau\tau} = \delta^{ij} \partial_j V$

Quantum Mechanics

$$-\frac{\hbar^2}{2m}\Delta_{\mathbb{R}^3}\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad (2)$$

time dependent Schrödinger eq.

Eisenhart : $G = 2\,du\,dt + 2\frac{V}{m}dt^2 - (d\vec{x})^2$.

Δ_G = wave operator on (M, G)

$$\underbrace{\Delta_G \phi = 0, \quad \phi = e^{-\frac{im}{\hbar}u}\,\Psi(\vec{x}, t)}_{(2)}$$

Applications MD + Roger Penrose

$V = -m\,\vec{g}\cdot\vec{x}$ uniform gravitational field.

G flat. Flat coordinates $(T, U, \vec{X})$

$T = t$, $U = u - t\,\vec{g}\cdot\vec{x} + \frac{1}{6}|\vec{g}|^2 t^3$

$\vec{X} = \vec{x} - \frac{1}{2}\vec{g}\,t^2$

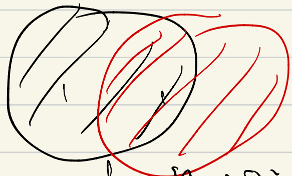
$$\boxed{-\frac{\hbar^2}{2m}\Delta_{\mathbb{R}^3}\Psi = i\hbar\frac{\partial\Psi}{\partial T}}$$

Free particle

$$\psi(\vec{x},t) = e^{-\frac{im}{\hbar}\left(\frac{t^3|\vec{g}|^2}{6} - t\,\vec{g}\cdot\vec{x}\right)}\,\Psi(\vec{X},T)$$

non-linear phase: ambiguity in decomposition into positive/negative frequency.

Non-relativistic counterpart of the Unruh effect.



superposition of many states.

• Non-relativistic limit of de-Sitter space

$$ds^2 = c^2 f\,dt^2 - f^{-1}dr^2 - r^2 \underbrace{h_{S^2}}_{\text{round metric on } S^2}$$

$$f = 1 - \frac{\Lambda r^2}{3}.$$

Take a limit; $c \to \infty$, $\Lambda \to 0$, $\omega^2 = \frac{\Lambda c^2}{3}$ finite.

$$\Gamma^{i}_{44} = f^{ij}\partial_j V \,; \quad \underbrace{V = -\frac{\omega^2 r^2}{2}.}$$

$\bar{G}$ conformally flat

$$G = \Omega^2 \left(dN\,dT - \underbrace{d\vec{X}}\right)^2$$

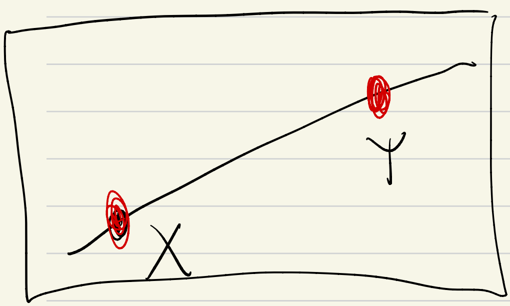
$$T = \frac{1}{\omega}\tanh(\omega t), \quad R = \frac{1}{\cosh(\omega t)}\, v$$

$$V = u - \frac{1}{2}\omega t^2 \tanh(\omega t)$$

isotropic inverted oscillator $\to$ free particle.

Non-relativistic de-Sitter twistor space

$\mathbb{CP}^3$ $\quad \alpha = 1, \dots, 4$



$$P^{\alpha\beta} = X^\alpha Y^\beta - Y^\alpha X^\beta$$

$$\sim k P^{\alpha\beta}$$

$$k \in \mathbb{C}^*$$

$P \wedge P = 0$ $\quad$ Klein quadric $Q \subset \mathbb{CP}^5$

Q = space-time

$\mathbb{CP}^3$ lines $\longleftrightarrow$ points

conf. structure.

$L_R, L_P \longleftrightarrow R, P$ null separated

~~null separated~~

intersect at a point

<u>point</u> $I \in \mathbb{CP}^5 \setminus Q$

$\exists!$ plane containing $\cancel{Q}, R, I$.

intersection of polar plane of I with $Q \longrightarrow$ two points A, B.

distance

$$d(P,R) = \frac{1}{2} \ln |P, R, A, B|.$$

Symmetry of $\mathbb{CP}^3$, Q $\quad SL(4,\mathbb{C})$,

— " — preserving I $\quad SO(5,\mathbb{C})$

[de-Sitter group).

İnönü–Wigner contraction

$\Lambda \to 0$, $c \to \infty$, $c^2 \Lambda$ const.

10D Lie Group . Newton Hooke group.

$\mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O} \oplus \mathcal{O}(2)$

normal bundle jump.

Thank you.