# Limit Multiplicities, Trace Formulas and Von Neumann Dimensions

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# Contents

**Prime number theorem**:

$$\lim_{n\to\infty} \frac{\#\{\text{prime numbers} \leq n\}}{\frac{n}{\log n}} = 1$$

**Limit multiplicity**:

$$\lim_{n\to\infty} \frac{\text{multiplicity}}{\text{von Neumann dimension}} = 1$$

1. An Example:

   $SL(2, \mathbb{Z})$ and Discrete Series of $SL(2, \mathbb{R})$
2. From Discrete Series to Bounded Subsets of Irrep$(G)$.
3. The Trace & the Arthur-Selberg Trace Formula
4. The Results on Limits Multiplicities

# The Multiplicity Problem: finite groups

- $G$: a finite group;
- $\Gamma \subset G$: a subgroup of $G$;
- the quasi-regular representation $R\colon G \curvearrowright L^2(\Gamma\backslash G)$,
$$(R(g)\phi)(x) = \phi(xg),\ g \in G.$$
- $L^2(\Gamma\backslash G) \overset{G\text{-module}}{=} \bigoplus_{\pi\in \mathrm{Irrep}(G)} m_\pi \cdot \pi.$
- Question 1: the multiplicity $m_\pi = ?$
- Answer 1: $L^2(\Gamma\backslash G) = \mathrm{Ind}_\Gamma^G(1_\Gamma)$ (the induced rep).

$$\begin{aligned}\Rightarrow m_\pi &= \dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_\Gamma^G(1_\Gamma), \pi)\\ &= \dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(1_\Gamma, \mathrm{Res}_\Gamma^G(\pi)) \quad (\textit{Frobenius reciprocity})\\ &= \dim_{\mathbb{C}} \pi^\Gamma.\end{aligned}$$

Here $\pi^\Gamma := \{v \in H_\pi | \pi(\gamma)v = v, \forall \gamma \in \Gamma\}$

# The Multiplicity Problem: $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$

- $G = \mathrm{SL}(2,\mathbb{R})$;
- $\Gamma = \mathrm{SL}(2,\mathbb{Z})$;
- the quasi-regular representation $R$: $G \curvearrowright L^2(\Gamma\backslash G)$

$$(R(g)\phi)(x) = \phi(xg).$$

- Question 2: What is the decomposition of $R$?
- Answer 2: It is NOT a direct sum, $R \neq \oplus m_\pi \cdot \pi$

**Theorem (Selberg 1950s)**

$$L^2(\Gamma\backslash G) = \underbrace{L^2_{disc}(\Gamma\backslash G)}_{\text{discrete spectrum}} \bigoplus \underbrace{L^2_{cont}(\Gamma\backslash G)}_{\text{continuous spectrum}}$$

$$= (\oplus_\pi m_\pi \cdot \pi) \bigoplus \int_{(0,\infty)}^{\oplus} \pi_s d\nu(s).$$

**Theorem**

$L^2_{disc}(\Gamma\backslash G) = \oplus_\pi m_\Gamma(\pi)\cdot\pi$ *with each multiplicity* $m_\Gamma(\pi) < \infty$.

- Question 3: $m_\Gamma(\pi) = ?$
- Answer 3: Unknown for most $\pi$ (No Frobenius reciprocity)
- Irrep$(G)$ =? or the unitary dual $\widehat{G}$ =?

1. the discrete series $\{\pi_k^\pm | k \geq 2\}$;
2. the principal series $\{\pi_{it}^\pm | t \in \mathbb{R}\}$;
3. the complementary series $\{\sigma_s | s \in (0,1)\}$;
4. the limits of discrete series $\delta_1^+, \delta_1^-$;
5. the trivial rep $\mathbb{C}$.

# Visualize Irrep($SL(2,\mathbb{R})$) or $\widehat{SL(2,\mathbb{R})}$

$$\widehat{G} = \underbrace{\{\pi_k^{\pm} | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\{\pi_{it}^{\pm} | t \in \mathbb{R}\}}_{\text{principal series}} \bigsqcup \underbrace{\{\sigma_s | s \in (0,1)\} \bigsqcup \{\delta_1^{\pm}\} \bigsqcup \{\mathbb{C}\}}_{\text{the remaining irreps}}$$

$$\approx \underbrace{\sqcup_{1,2}\{k | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\sqcup_{1,2}\mathbb{R}}_{\text{principal series}} \bigsqcup \underbrace{(0,1) \bigsqcup \{\pm 1\} \bigsqcup \{1\}}_{\text{the remaining irreps}}$$

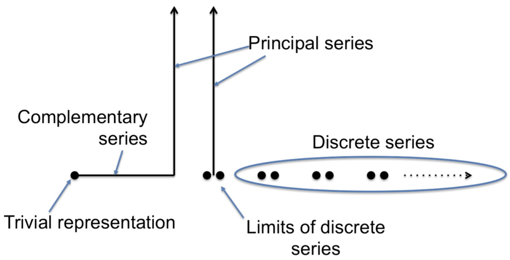


Figure: The unitary dual of $SL(2,\mathbb{R})$ by P. Hochs

# The Multiplicity for $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ and Discrete Series

$L^2_{\text{disc}}(\Gamma\backslash G) = \oplus_\pi m_\Gamma(\pi)\cdot\pi$ with each $m_\Gamma(\pi) < \infty$.

- A few $m_\Gamma(\pi)$ are known! They are related to cusp forms.

**Definition**

Let $k \in \mathbb{Z}$. A cusp form of weight $k$ with respect to $\Gamma$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

1. $f(z) = (cz+d)^{-k} \cdot f(\frac{az+b}{cz+d})$, $z \in \mathbb{H}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
2. $f$ vanishes at each *cusp* of $\Gamma \Leftrightarrow |f(x+iy)| \leq C \cdot y^{-k/2}$.

Denote by $S_k(\Gamma)$ the space of cusp form of weight $k$.

**Theorem (Gelfand et al. 1960s)**

*For* $\pi_k$, $m_\Gamma(\pi_k) = \dim S_k(\Gamma)$= *dim of cusp forms of weight* $k$.

# The Limit Multiplicity Problem

- $m_\Gamma(\pi_k) = \dim S_k(\Gamma)$ .
- the principal congruence subgroups
  $\Gamma(n) := \left\{ g \in \mathrm{SL}(2,\mathbb{Z}) | g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n \right\}.$
- $m_{\Gamma(n)}(\pi_k) = \dim S_k(\Gamma(n)) =$
  $$(k - 1 - \tfrac{6}{n}) \cdot \tfrac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2}),$$
  (by Riemann-Roch theorem for modular curves).
- $\Rightarrow \lim_{n\to\infty} m_{\Gamma(n)}(\pi) = \infty$.
- Question 4: Find some "nice" $f : \mathbb{Z} \to \mathbb{R}$ such that
  $$\lim_{n\to\infty} \frac{m_{\Gamma(n)}(\pi)}{f(n)} = 1,$$
  which is the limit multiplicity problem.
- Answer 4: $f$ can be given by von Neumann dimensions!

# The Von Neumann Dimensions

- $L\colon \Gamma \curvearrowright l^2(\Gamma)$, the left regular rep of a discrete group $\Gamma$;
- the group von Neumann algebra of $\Gamma$,
$$L\Gamma :=\overline{L(\Gamma)}^{\text{w.o.t}} \subset B(l^2(\Gamma)).$$
- "the trace" on $L\Gamma$: $\operatorname{tr}(x) = \langle x\delta_e, \delta_e \rangle_{l^2(\Gamma)}$.
- Given a $L\Gamma$-module $H$, there is an isomorphism
$$H \overset{L\Gamma\text{-module}}{\cong} p(l^2(\Gamma) \otimes l^2(\mathbb{N})),$$
for some projection $p \in L\Gamma' \cap B(l^2(\Gamma) \otimes l^2(\mathbb{N}))$.
- the von Neumann dimension $\dim_{L\Gamma} H := (\operatorname{tr} \otimes \operatorname{Tr})(p)$.

## Lemma

1. $\dim_{L\Gamma} l^2(\Gamma) = 1$;
2. $\dim_{L\Gamma}(H_1 \oplus H_2) = \dim_{L\Gamma} H_1 + \dim_{L\Gamma} H_2$;
3. *If $Z(L\Gamma) = \mathbb{C}$ ($L\Gamma$ is a factor),*
$$\dim_{L\Gamma} H_1 = \dim_{L\Gamma} H_2 \Leftrightarrow H_1 \cong H_2 \text{ as } L\Gamma\text{-modules.}$$
4. ...

# The Von Neumann Dimensions

- a Lie group $G$, a lattice $\Gamma \subset G$: $\mathrm{vol}(\Gamma \backslash G) < \infty$,
- a discrete series $(\pi, H)$ of $G$ := an irrep $\leq L^2(G)$.

**Theorem (Atiyah & Schmid, 1970s)**

$\dim_{L\Gamma} H = \mathrm{vol}(\Gamma \backslash G) \cdot d(\pi)$.

- $d(\pi)$: $=$ the formal dimension of $\pi$:

$$d(\pi) \cdot \langle c^{\pi}_{u,v}, c^{\pi}_{x,y} \rangle_{L^2(G)} = \langle u, x \rangle_H \cdot \overline{\langle v, y \rangle_H},$$

for all $u, v, x, y \in H$, where $c^{\pi}_{u,v}(g) = \langle \pi(g)u, v \rangle \in L^2(G)$.

**Example**

1. Take $\Gamma(n) \subset G = \mathrm{SL}(2, \mathbb{R})$.
2. Take the discrete series $(\pi_k, H_k)$ of $G$.
3. Gauss-Bonnet $\Rightarrow \mathrm{vol}(\Gamma(n) \backslash G)$.
4. Harish-Chandra $\Rightarrow d(\pi_k)$.
5. $\dim_{L\Gamma(n)}(H_k) = (k-1) \cdot \frac{n^3}{24} \cdot \prod_{p|n}(1 - p^{-2})$.

# The Limit Multiplicity for $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ and Discrete Series

**Let us collect the data!**

- A family of lattices $\{\Gamma(n)\}_{n\geq 1}$ in $G = SL(2,\mathbb{R})$,
- $(\pi_k, H_k)$: the discrete series of $G$,
- For the multiplicity,

$$m_{\Gamma(n)}(\pi_k) = (k - 1 - \tfrac{6}{n}) \cdot \tfrac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2}).$$

- For the von Neumann dimension,

$$\dim_{L\Gamma(n)}(H_k) = (k-1) \cdot \tfrac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2}).$$

- Take the quotient and then the limit:

$$\lim_{n\to\infty} \frac{m_{\Gamma(n)}(\pi_k)}{\dim_{L\Gamma(n)}(H_k)} = \lim_{n\to\infty} \frac{(k - 1 - \frac{6}{n}) \cdot \frac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2})}{(k-1) \cdot \frac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2})}$$

$$= \lim_{n\to\infty} \frac{k - 1 - \frac{6}{n}}{k-1} = 1$$

# The Limit Multiplicity in General

- For $\{\Gamma(n)\}_{n\geq 1}$ in $G = \mathrm{SL}(2,\mathbb{R})$, we have
$$\lim_{n\to\infty} \frac{m_{\Gamma(n)}(\pi_k)}{\dim_{L\Gamma(n)}(H_k)} = 1.$$
- Question 5: Is this true "in general"?
- **Some difficulties** for the generalization:
  1. Most $m_\Gamma(\pi)$ are unknown (different from the d.s. of $\mathrm{SL}(2,\mathbb{R})$)
  2. Some Lie groups have no discrete series:

     (Harish-Chandra) $G$ has d.s iff $\mathrm{rank}\, G = \mathrm{rank}\, K$
  3. If $(\pi, H)$ is NOT a d.s,

     $H$ is NOT a $L\Gamma$-module. $\Rightarrow$ No $\dim_{L\Gamma} H$.
- discrete series $\to$ "bounded subsets of the unitary dual"

# Bounded Subsets of the Unitary Dual

**the embedding**: $\mathrm{Irrep}(SL(2,\mathbb{R})) \hookrightarrow \mathbb{R}^2$.

$$\widehat{G} = \underbrace{\{\pi_k^{\pm} | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\{\pi_{it}^{\pm} | t \in \mathbb{R}\}}_{\text{principal series}} \bigsqcup \underbrace{\{\sigma_s | s \in (0,1)\} \bigsqcup \{\delta_1^{\pm}\} \bigsqcup \{\mathbb{C}\}}_{\text{the remaining irreps}}$$

$$\approx \underbrace{\sqcup_{1,2}\{k | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\sqcup_{1,2}\mathbb{R}}_{\text{principal series}} \bigsqcup \underbrace{(0,1) \bigsqcup \{\pm 1\} \bigsqcup \{1\}}_{\text{the remaining irreps}}$$

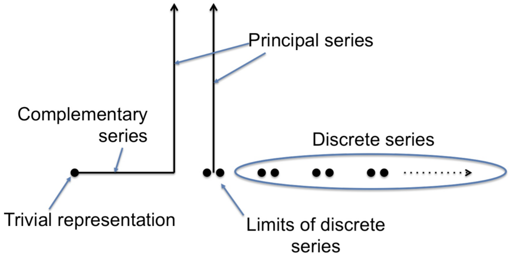


Figure: The unitary dual of $SL(2,\mathbb{R})$ by P. Hochs

# Bounded Subsets of the Unitary Dual of a Lie group

- $G$ : a semisimple real Lie group,
- **the embedding**: $\widehat{G} \hookrightarrow \bigsqcup_{\text{finite}} \mathbb{R}^{\text{rank } G}$ (as a set).
- $X \subset \widehat{G}$ is bounded : if it is bounded in $\bigsqcup_{\text{finite}} \mathbb{R}^{\text{rank } G}$.
- $\Leftrightarrow$ relatively compact in the Fell topology.

**Definition**

For a bounded $X \subset \widehat{G}$, $m_\Gamma(X) := \sum_{\pi \in X} m_\Gamma(\pi)$.

Question 6: Is $m_\Gamma(X)$ finite?

**Theorem (Borel & Garland 1980s)**

*For a bounded $X$, only finitely many $\pi \in X$ occur in $L^2_{disc}(\Gamma \backslash G)$*

$\implies m_\Gamma(X) := \sum_{\pi \in X} m_\Gamma(\pi)$, a finite sum of finite numbers.
$\implies$ Answer 6: $m_\Gamma(X)$ is finite and well-defined!

**Theorem (Plancherel Theorem)**

*There is a measure $\nu_G$ on $\widehat{G}$ (Plancherel measure) such that*

$$L^2(G) \overset{G-G-\text{bimod}}{\cong} \int_{\widehat{G}}^{\oplus} H_\pi \otimes H_{\overline{\pi}} \, d\nu_G(\pi),$$

*where the isomorphism given by the Fourier transform:*

$$f \mapsto \widehat{f}(\pi) = \int_G f(g)\pi(g^{-1})dg.$$

1. $X \subset \widehat{G}$ bounded $\Rightarrow \nu(X) < \infty$.
2. $\pi$ is a d.s. iff it is an atom: $\nu(\{\pi\}) > 0$. $\nu(\{\pi\}) = d(\pi)$.
3. $\mathrm{supp}(\nu_G)$ = tempered irreps $:= \{\pi | c^{\pi}_{u,v} \in L^{2+\varepsilon}(G), \forall \varepsilon > 0\}$.
4. $\widehat{G} = \widehat{G}_{\mathrm{temp}} \bigsqcup \widehat{G}_{\mathrm{untemp}}$.
5. $\widehat{\mathrm{SL}(2,\mathbb{R})}_{\mathrm{temp}} = \{$ discrete seires, principal series $\}$.
6. Wassermann, Plymen, Clare-Crisp-Higson:
   decompositions of $C^*_{\mathrm{red}}(G)$ (the reduced $C^*$-algebra)

# Plancherel Measure on the Unitary Dual

- $X$ = a bounded subset of $\widehat{G}$,
$$H_X := \int_X^{\oplus} H_\pi d\nu(\pi)$$
- $\Rightarrow H_X$ is a module over $G$, $\Gamma$ and also $L\Gamma$.

**Theorem (Y, 2022)**

*Given a lattice* $\Gamma \subset G$,
$$\dim_{L\Gamma} H_X = \mathrm{vol}(\Gamma \backslash G) \cdot \nu(X).$$

- (Kyed, Petersen & Vaes) $\dim_{LG} H$
$\longleftarrow$ a faithful normal tracial weight on $LG$.
- $X = X_{\text{temp}} \bigsqcup X_{\text{untemp}}$, only $X_{\text{temp}}$ contributes to $\nu(X)$.
- reduces to the Atiyah-Schmid Thm if $X = \{\pi\}$ = a d.s.

**Let us collect the data again!**

- Recall $m_\Gamma(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(H_\pi, L^2_{\mathrm{disc}}(\Gamma\backslash G))$,
  the multiplicity $m_\Gamma(X) := \sum_{\pi \in X} m_\Gamma(\pi)$.
- Recall $H_X = \int_X^{\oplus} H_\pi d\nu(\pi)$,
  the von Neumann dimension $\dim_{L\Gamma} H_X$.

**Theorem (Y, 23)**

*Let $G$ be a semisimple real Lie group and $X$ is a bounded subset of $\widehat{G}$. We have*

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(X)}{\dim_{L\Gamma_n}(H_X)} = 1$$

*when $G$ and $\{\Gamma_n\}_{n\geq 1}$ satisfy either one of the following conditions:*

1. *cocompact lattices such that* $\cap_n \Gamma_n = \{1\}$, $\Gamma_n \lhd \Gamma_1$, $[\Gamma_1 : \Gamma_n] < \infty$.
2. $G = \mathrm{SL}(n, \mathbb{R})$ *and* $\Gamma_n = \ker\{\mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, \mathbb{Z}/n\mathbb{Z})\}$.

- $$\underbrace{\Gamma \curvearrowright L^2(G)}_{\text{restricted left action}} \overset{\Gamma\text{-module}}{\cong} \underbrace{l^2(\Gamma)}_{\text{left regular rep}} \otimes \underbrace{L^2(\Gamma\backslash G)}_{\text{id}}.$$

$$\text{the commutant of } L\Gamma = \underbrace{R\Gamma}_{\text{the right group vN-alg}} \otimes B(L^2(\Gamma\backslash G))$$

$$\text{the trace} = \underbrace{\text{tr}}_{\text{"the trace"}} \otimes \text{Tr}$$

- $R_\Gamma : L^2(\Gamma\backslash G) \curvearrowleft G \Rightarrow R_\Gamma : L^2(\Gamma\backslash G) \curvearrowleft C^\infty_{cpt}(G)$

$$(R_\Gamma(\phi)f)(x) := \int_G \phi(g)f(xg)dg, \; \phi \in C^\infty_{cpt}(G)$$

- If $L^2(\Gamma\backslash G) = \oplus m_\Gamma(\pi)\cdot\pi$,

$$R_\Gamma(\phi) = \oplus m_\Gamma(\pi)\cdot\pi(\phi).$$

- $\Rightarrow$Take the trace of both side.

**If $\Gamma\backslash G$ is compact,**

1. $R_\Gamma(\phi)$ is a trace-class operator.

   $$\mathrm{Tr}(R_\Gamma(\phi)) = \textbf{the Selberg trace formula}.$$

2. $R_\Gamma(\phi) \in$ the commutant $= R\Gamma \otimes B(L^2(\Gamma\backslash G))$.
3. $\implies \sigma_\Gamma(R_\Gamma(\phi)) := (\mathrm{tr} \otimes \mathrm{Tr})(\mathrm{id} \otimes R_\Gamma(\phi))$.

**If $\Gamma\backslash G$ is NOT compact,**

1. $R_\Gamma(\phi)$ is not in the trace class.
2. the projection $P_{\mathrm{cusp}} : L^2(\Gamma\backslash G) \to L^2_{\mathrm{cusp}}(\Gamma\backslash G)$.
3. $P_{\mathrm{cusp}} R_\Gamma(\phi) P_{\mathrm{cusp}}$ is in the trace class,

   $$\mathrm{Tr}(P_{\mathrm{cusp}} R_\Gamma(\phi) P_{\mathrm{cusp}}) = \textbf{the Arthur trace formula}.$$

4. $\implies \sigma_\Gamma(R_\Gamma(\phi)) := (\mathrm{tr} \otimes \mathrm{Tr})(\mathrm{id} \otimes P_{\mathrm{cusp}} R_\Gamma(\phi) P_{\mathrm{cusp}})$.

**Another right action of $G$**

- $\Gamma \curvearrowright L^2(G) \overset{G-G\text{-bimodule}}{\cong} \int_{\widehat{G}}^{\oplus} H_{\pi^*} \otimes H_\pi d\nu(\pi) \curvearrowleft^R C^\infty_{\mathrm{cpt}}(G)$.
- $R(\phi) \in$ the commutant, if $\phi \in C^\infty_{\mathrm{cpt}}(G)$.

$$\Longrightarrow \sigma_\Gamma(R(\phi)) = \mathrm{vol}(\Gamma \backslash G)\phi(1).$$

**Lemma**

*Given a tower of lattices $\Gamma_1 \supset \Gamma_2 \supset \dots$ in $G$.*

$$\text{If } \lim_{n\to\infty} \frac{\sigma_{\Gamma_n}(R_{\Gamma_n}(\phi))}{\sigma_{\Gamma_n}(R(\phi))} = 1, \Longrightarrow \text{ then } \lim_{n\to\infty} \frac{m_{\Gamma_n}(X)}{\dim_{L\Gamma_n}(H_X)} = 1.$$

Its proof is mainly based on

1. Sauvageot's density result: the Fourier transforms of $C^\infty_{\mathrm{cpt}}(G)$ are dense in integrable functions on $\widehat{G}_{\mathrm{temp}}$ (1997).
2. Finis, Lapid & Müller's result on Arthur's trace formula (2011).

# End

Questions?

Thank you!