My Adventures in Quantumland

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MATHEMATICAL PICTURE LANGUAGE SEMINAR

Outline

- ***What is a Quantum Computer?***
  - -- **Qubits**
  - -- **density matrices**
- ***The Taming of Shrews***
  - -- **tensor products**
  - -- **Quantum Entanglements**
  - -- **Completely positive linear maps**
- ***Old and New Results***

Units of Information – Bits vs Qubits

- A ***bit*** (binary integer) is the base of conventional computer memory.

- 1-bit is read as either a zero or a one with probability in the real interval [0, 1].

- A 3-bit corresponds to an element in
  $\{0, 1\} \times \{0, 1\} \times \{0, 1\} = 2^3$, as vertices of a cube;

  but very WRONG to have a cube for probability!

- When n = 40, we get 2^{40} = tera

• To get a setting of a possible non-commutative generalization, we associate each 1-bit with a rank-1 diagonal 2 X 2 projection matrix, i.e.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

• Each 3-bit corresponds to a rank-1 diagonal 8 X 8 projection matrix, as the tensor product of three 2 X 2 matrices where each is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Thus there are eight 3-bits located in an 8-dimensional space.

• Apparently, nobody in computer science mentioned of diagonal matrices and tensor products.

- A ***qubit*** (quantum bit) is a unit of quantum computer memory.
- Mathematically, each 1-qubit is regarded as an element in

$$S^2 \simeq \left\{ \frac{1}{2} \begin{pmatrix} 1-x & y+iz \\ y-iz & 1+x \end{pmatrix} \quad with\; x^2+y^2+z^2=1 \right\}$$

= {all 2 X 2 rank-1 projection matrices}
= {all vector states acting on $\mathbf{C}^2$ }
= {one-dimensional complex linear subspaces of $\mathbf{C}^2$ }
= {*special* two-dimensional real subspaces of $\mathbf{R}^4$}

- Physically, a 1-qubit is a superposition of the spherical surface (called the Bloch sphere), because an “electron” can move freely to any direction from the origin of $\mathbf{R}^3$.

- Thus, S^2 need not be a material surface.

- There are uncountably many 1-qubits, to make S^2 symmetry with continuity for approximation, while 1-bits are just the north pole and the south pole.

What on earth does S^2 mean?

- Think of geography and physics and philosophy, instead of set theory and computers.
- It means of measure theory (as length / area / volume) and continuity and dimension and analog. Always uncountably infinite points (beyond the capacity of any conventional computer memory).
- It goes along with human memory, which could be transcendental and sensible and sensational and sentimental.

- However, S^2 is a mathematical simple object. The combinatorial effect of S^2 symmetry is not comparable to 2^n when $n > 30$. So, a 1-qubit computer cannot replace the conventional computer, as used in digital photo and music.
- But, we should look further in n-qubit computers, with physical meanings and mathematical ideas.

Def. An ***n-qubit*** (= a vector state) is regarded as a 1-dimensional complex linear subspace of the dim 2^n Hilbert space, which can also be identified as a rank-1 projection in the form as a 2^n X 2^n complex matrix.

Density Matrices

- In the formal setting of non-commutative probability, the random position of an n-qubit can be regarded as a density matrix, to be defined as a convex combination of rank-1 projections in M_{2^n}.

Def: A *density matrix* is a positive semidefinite matrix of trace 1.

- Thus, each 2 x 2 density matrix is expressed as

$$\frac{1}{2}\begin{bmatrix} 1-x & y+iz \\ y-iz & 1+x \end{bmatrix}$$

with $x^2 + y^2 + z^2 \leq 1$; so all density matrices fill up the whole solid sphere with S^2 as boundary.

- Nevertheless, for the case n >2, there is no easy geometrical picture for the collection of all n x n density matrices.

Recap of the *simplest*
Quantum Computer

- The setting of 1-qubit computer is a solid sphere in R^3

--- just like the solid Earth in space.

- To send out quantum information ----- to *communicate* between two 1-qubit computers, we consider a feasible *affine transform* (preserving the 3-dimensional convex structure) of the solid Earth.

AFFINE TRANSFORMS induces LINEAR MAPS

- $M_n = M_n^+ - M_n^+ + i M_n^+ - i M_n^+$
- $M_n^+ = R^+ \times$ { density matrices}

➢ {affine transforms on density matrices}

≈ {trace-preserving positive linear maps}.

➢ {affine transforms on density matrices, fixing the scalar matrix }

≈ {unital trace-preserving positive linear maps}.

➢ {feasible affine transforms on density matrices}

≈ {trace-preserving completely positive linear maps}

with deep unknown features of matrix analysis.

Shrew = Quantum Entanglements

of Positive Semi-Definite Matrices

Who's Afraid of

Quantum Entanglements?

Tensor - product setup for the Taming of the Shrew

- Consider a Hilbert space

$$H = H_1 \otimes H_2 .$$

- Some natural / simple / easy phenomena on H could be entangled in H_1 and H_2 separately.

- We wish to control the whole situation, bypassing / conquering /ignoring the entanglements.

Math Settings

- $L^2(X\mathbf{x}Y) = L^2(X) \otimes L^2(Y).$
- Often consider of finite-dimensional Hilbert spaces as $\mathbf{C}^n$ with a positive integer n.

➢ Thus $\mathbf{C}^n \otimes \mathbf{C}^k = \mathbf{C}^{nk}$.

$$M_n = \text{ linear maps from } \mathbf{C}^n \text{ to } \mathbf{C}^n$$

$$M_n \otimes M_k = M_{nk} = M_n(M_k) = M_k(M_n) .$$

--- no need to mention of anything as the universal property.

➢ In such an easy mathematical setting, who is afraid of quantum entanglements and local-global effects with respect to $\bigotimes$

Math Settings $M_n \otimes M_k = M_{nk}$ (with n>1, k>1)

➢ *{ the sums of $A_j \otimes B_j$ with A_j in M_n^+, B_j in M_k^+}* *is only a proper subset of* $(M_n \otimes M_k)^+ = M_{nk}^+$.

Reason: M_n^+ = { positive linear combinations of rank-1 projections }

•There are many rank-1 projections in M_{nk} which are not tensor product of rank-1 projections.

➢ Along this line, completely positive linear maps can go through the quantum entanglements, while positive linear maps cannot.

Quantum Entanglements provide exciting features for positive linear maps

Structure Theory

Notation: Each linear map $\varphi : M_n \rightarrow M_k$ can be extended to a linear map

$$\varphi \otimes id_p : M_n \otimes M_p \longrightarrow M_k \otimes M_p .$$

Def: φ is said to be ***p-positive*** when $\Phi \otimes \mathrm{id}_p$ is a positive linear map.

Def: φ is said to be ***completely positive*** when φ is a p-positive linear map for each positive integer p.

Structure Theory

Thm (Choi) : All p-positive linear maps from M_n to M_k are completely positive when $n \leq p$ or $k \leq p$.

- Nevertheless, various p provide distinct classes of p-positive linear maps as elaborated in the following:

Example (Choi): The linear map $\varphi : M_n \to M_n$ defined as $\varphi(A) = (n\text{-}1)(trace\ A)I_n - A$ is (n-1)-positve but not n-positive.

Main Thm: (Choi, 1975) A linear map

$\varphi : M_n \rightarrow M_k$ is ***completely positive***

iff $[\varphi(E_{ij})]_{i,j}$ is positive

where $\{ E_{ij} \}$ are the matrix units

iff $\varphi(A) = \Sigma V_j^* A V_j$ for all $A \in M_n$

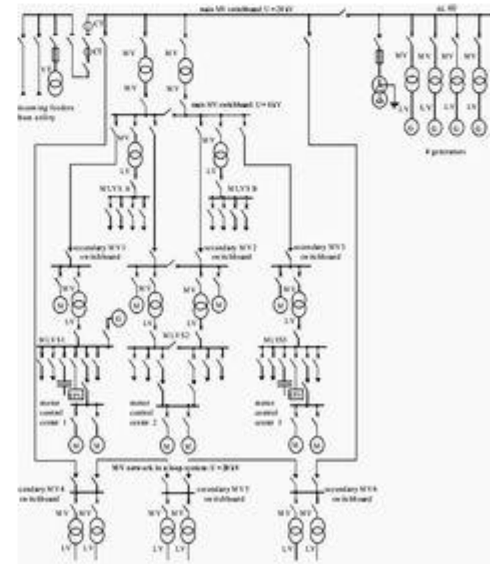
with n x k matrices V_j

- This 1975 paper (6 pages) has been cited in nearly 3000 research papers, as of 2023 September

Google Scholar

- More than 2000 citations in publications of Quantum Information.

CIRCUIT THEORY



- Each transformer defines a

  positive linear map $A \rightarrow V^{*}AV$.

  Thus several transformers in series

  define a completely positive linear map.

- Main concern in circuit theory: General linear maps of mathematical expressions in terms of $[\varphi(E_{ij})]_{i,j}$ are not implementable.

➢ Classical computer vs Quantum computer

❖ A classical computers produces 0-1 sequences while a quantum computers produces psd matrices. Thus only completely positive maps are usable to connect *Quantum computers*

The Main Thm (Choi 1975) revisited

Let $\varphi : M_n \rightarrow M_k$ be a linear map. TFAE:

(1) φ is p-positive for all positive integer p.

(2) $[\varphi(E_{ij})]_{i,j}$ is positive

(3) $\varphi(A) = \Sigma V_j^* A V_j$ for all $A \in M_n$ with n x k matrices V_j

- (1) means to be the hardest nature to conquer all incredible quantum entanglements in $(M_n \otimes M_p)^+$ of various p.

  (2) is intended for the simplest mathematical expression of a general linear map.

  (3) turns to be the only possible connection in circuit theory.

- Stinespring Theorem (1955) covers the case (1) ⇔ (3).
- Theorem1975 says much about (2) ⇔ (3) and (2) ⇔ (1), which is most needed in theory of quantum information.

Taming of Shrews

- NO way to describe so many incredible *entanglements* in $(M_n \otimes M_p)^+$ of various p.

- The most outstanding $T = \Sigma E_{ij} \otimes E_{ij} \in (M_n \otimes M_n)^+$, is *a* well behaved *entanglement* which serves as the representative for ALLwild entanglements.

- THEOREM says that to tame ALL shrews (= entanglements) is equivalent to tame a single LOVELY shrew (without worrying how nasty/dirty/undisciplined of other shrews).

CLOSE-UP of the LOVELY *Shrew*

Example $n = 3$, $T = \sum E_{ij} \otimes E_{ij} \in (M_3 \otimes M_3)^+ = M_9^+$

- T is the NATURAL assemblage of matrix units

$$\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]$$

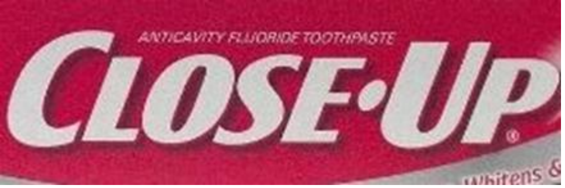
- Indeed, $T^2 = nT$, so $\frac{1}{n}T$ is a rank-1 projection, but T serves as the best witness to test all completely positive linear maps $M_3 \rightarrow M_3$.

Why Not Down to *n=2*?

- The simplest example of **quantum entanglement** is

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

as a positive 4 x 4 matrix, but not of the form as the sum of $A_j \otimes B_j$ with A_j in $M_2{}^+$ and B_j in $M_2{}^+$.



Purpose: Wish to classify all linear maps

$\varphi : M_2 \to M_2$ by means of the 4 x 4 Choi Matrix $C\varphi$

$$\begin{bmatrix} \varphi(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) & \varphi(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) \\ \varphi(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) & \varphi(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \end{bmatrix},$$

Challenge: *What sort of non-commutative geometry could be hidden/shown in the 4 x4 matrix* $C\varphi$ *?*

Newest Classification Theorem

(Joint work with C.K. Li, 2023, JQIC)

Consider all $\varphi : M_2 \rightarrow M_2$ *as* unital trace-preserving and hermitian- preserving linear maps.

Then the 4 real eigenvalues of the Choi Matrix C_φ determine *the linear map* φ up to unitary equivalence.

*I.e., **iff** C_φ and C_ψ have the same eigenvalues, then there exist unitaries U and W such that*

$\varphi(A) = U^*\Psi\,(W^*AW)U$ for all A in M_2.

The Most Important Example:

By means of Pauli Matrices

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$,

and 4 real numbers λ_j *with* $\sum \lambda_j = 1$.

Define $\varphi : M_2 \rightarrow M_2$

as $\varphi(A) = \lambda_1 A + \lambda_2 ZAZ + \lambda_3 XAX + \lambda_4 YAY$

- Then φ is a unital linear map preserving traces and hermitian matrices.
- *The Choi Matrix* $C\varphi$ has $\{2\lambda_j\}$ as four eigenvalues.

Newest Classification Theorem Restated

Each unital qubit channel φ

(unital trace preserving completely positive linear map $M_2 \rightarrow M_2$)

is unitarily equivalent to a **concrete** map of the form $A \rightarrow \lambda_1 A + \lambda_2 ZAZ + \lambda_3 XAX + \lambda_4 YAY,$

where X, Y and Z are Pauli Matrices;

$\{2\lambda_j\}$ *are eigenvalues of the Choi Matrix* $C\varphi$.

- *This provides the WHOLE picture of unital qubit channels.*

OPEN QUESTION

What would be next *Classification Theorems* ?

Want to study the case n=3.

- Need to understand the quantum entanglement of

$$\left[\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$