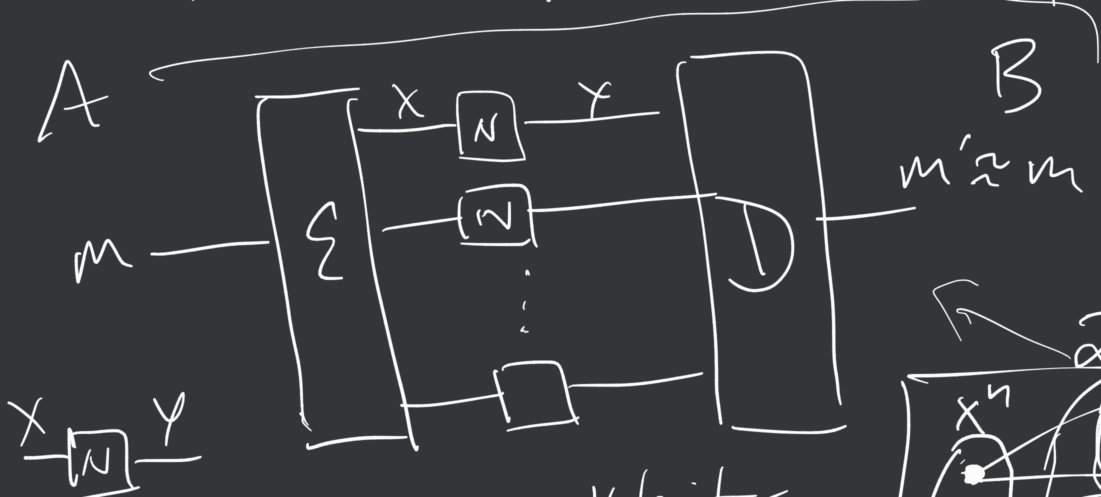
Mathematical Challenges in Quantum Information Theory

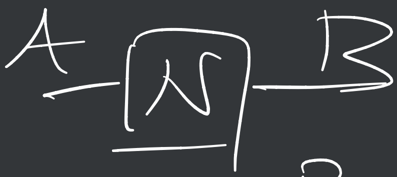


$$C = \frac{\#\text{bits}}{\#\text{channel}}$$

$$C(N) = \max_{X} I(X;Y)$$

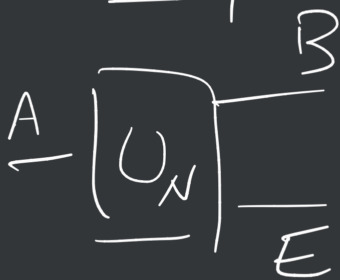
$$I(X;Y) = H(X) + H(Y) - H(XY)$$

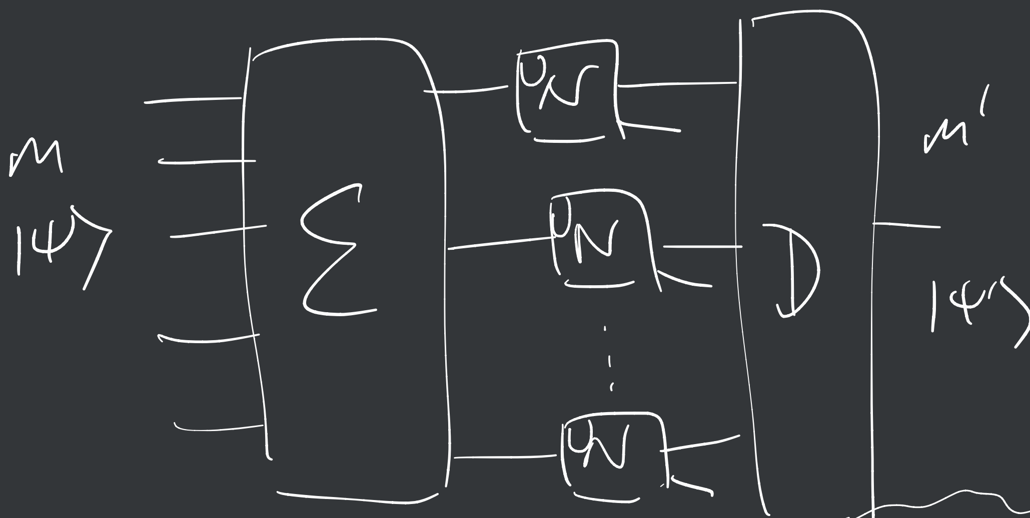
codes random $\sim X^n$



CPTP

- coding thm
- Additivity
- Nonadditivity
- summary





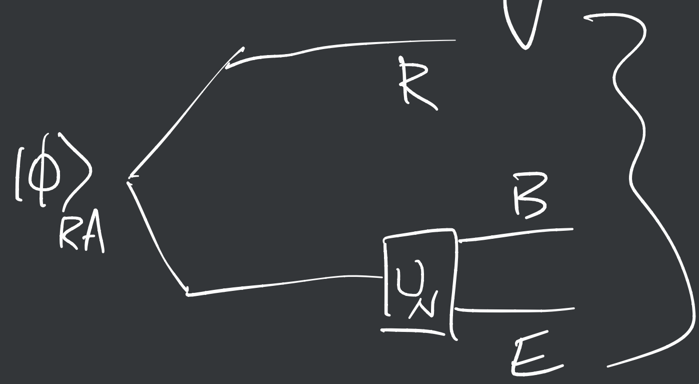
$$Q \leq P \leq C$$

$$C = \frac{\#\text{bits}}{\#\text{channel}}$$

$$Q = \frac{\#\text{qubits}}{\#\text{chan}}$$

$$P = \frac{\#\text{bits}}{\#\text{channel}}$$

Coding Theorems

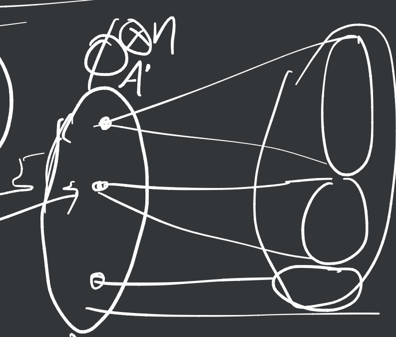


$S(\rho) = -\mathrm{Tr}\,\rho\log\rho$

$$Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N}) = \max_{|\phi\rangle_{RA}} \overbrace{\left(S(B)-S(E)\right)}^{I(R\rangle B)}$$

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$$

"regularization"



$$Q^{(1)}(\mathcal{N}^{\otimes k}) > k\,Q^{(1)}(\mathcal{N})$$

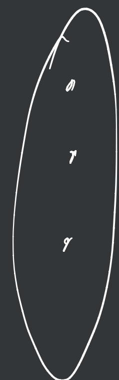
structured codes outperform random codes.

Classical

$$\sum_x p_x |x\rangle\langle x| \otimes \phi_x^A$$

$$\chi(N) = \max_{\{p_x,\, \phi_x^A\}} I(X;B)$$

$$C(N) = \lim_{n\to\infty} \frac{1}{n}\chi(N^{\otimes n})$$



$\exists$ examples

$$\chi(N\otimes N) > 2\chi(N)$$

Additivity

$$\mathcal{N}(\rho) = \mathrm{Tr}_E U\rho U^\dagger \qquad U: A \to BE$$

$$\mathcal{N}^c(\rho) = \mathrm{Tr}_B U\rho U^\dagger$$

Degradable: $\exists \mathcal{D},\ \mathcal{D}\circ\mathcal{N} = \mathcal{N}^c$

$$Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$$

basically because:

$$S(B_1) - S(E_1) + S(B_2) - S(E_2) \geq S(B_1B_2) - S(E_1E_2)$$

$$I(B_1;B_2) \geq I(E_1;E_2)$$

- dephasing $\rho \to (1-p)\rho + pZ\rho Z$
- erasure $\rho \to (1-p)\rho + p|e\rangle\langle e|$

Question 1:

$\forall\, |\phi\rangle_{VA}$ $*\ I(V;B) \geq I(V;E)$

satisfied by deg.

$\exists N$? satisfying $*$ that are not degradable.

$* \Rightarrow Q(N) = Q^{(1)}(N)$.

classically? ✓

challenge

challenge: $|V|$ unbounded.

Symmetric Assistance.

U_A $\quad |ij\rangle_A \longrightarrow \begin{cases} |ij\rangle + |ji\rangle_{BE} & i>j \\ |ii\rangle_{BE} & i=j \end{cases}$

$i \geq j \quad i,j = 0 \ldots \infty$

$Q(A) = 0$

$Q^{(1)}_{ss}(N) = Q^{(1)}(N \otimes A)$

$Q^{(1)}_{ss}(N \otimes M) = Q^{(1)}_{ss}(N) + Q^{(1)}_{ss}(M)$

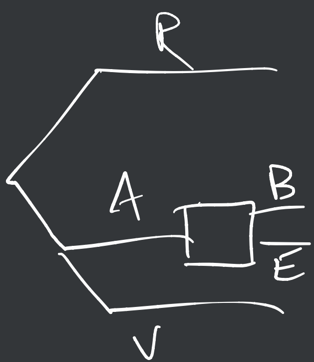
cap assisted by A

$Q_{ss}(N) = Q^{(1)}_{ss}(N)$

Symmetric assistance

$Q_{ss}(N) = Q^{(1)}(N)$ for deg. N

$$Q_{ss}(N) = \frac{1}{2} \sup_{|\phi\rangle_{RAV}} \left[I(R\rangle BV) - I(R\rangle EV) \right]$$



Q: - how do we eval with $|V|$, $|R|$?

- can we bound?

- can we relax the opt?

Easier (?) Question:

$$I^{cc}(\mathcal{N}) = \sup_{\phi_{VA}} \left[S(BV) - S(EV) \right]$$

Properties:

- additive
- $I^{cc}(\mathcal{N}) = I^{cc}(\mathcal{N}^c)$
- $I^{cc}(\mathcal{N}) = Q^{(1)}(\mathcal{N})$ for deg.

Q:

- bound $|V|$? special classes?
- u.b. & relaxations?
- other entropies.

Non-Additivity.

$$Q(N) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(N^{\otimes n})$$

$$Q^{(1)}(N^{\otimes k}) > k Q^{(1)}(N)$$

Examples:

$$Q^{(1)}(N \otimes M)$$
$$> Q^{(1)}(N) + Q^{(1)}(M)$$

when is $Q(\mathcal{N})=0$?

1) $Q(\mathcal{A})=0$ symmetric anti-degradable.

2) PPT channels.

$(\mathbb{I}\otimes\mathcal{N})(|\psi\rangle\langle\psi|)$

$\rho^{T}\geq 0$ $(\mathbb{I}\otimes T)(|\phi_0\rangle\langle\phi_0|)$

$Q(\mathcal{N})=0.$

$\exists \mathcal{N}_H$ PPT but $P(\mathcal{N}_H)>0$

* $\frac{1}{2}P(\mathcal{N}_H)\leq Q_{ss}(\mathcal{N}_H)$

$\boxed{Q(\mathcal{N}_H)=0,\ Q(\mathcal{A})=0}$

Nonadditivity

$$U_{\mathcal{N}_p} : \left.\begin{aligned} |0\rangle_A &\longrightarrow \frac{|00\rangle_{BE} + |11\rangle_{BE}}{\sqrt{2}} \\ |1\rangle_A &\longrightarrow |2\rangle_B |0\rangle_E \end{aligned}\right\} \text{deg.}$$

$$|2\rangle_A \longrightarrow |2\rangle_B |1\rangle_E .$$

$$Q^{(1)}(\mathcal{N}_p \otimes \mathcal{E}_{1/2}) > Q^{(1)}(\mathcal{N}_p) + 0$$

$$Q^{(1)}(\mathcal{N}_p \otimes \mathcal{A}_{1/2}) > Q^{(1)}(\mathcal{N}_p) + 0$$

$$Q^{(1)}(\mathcal{N}_p \otimes \mathcal{D}_p) > Q^{(1)}(\mathcal{N}_p) + 0$$

$$\boxed{Q^{(1)}\left(\mathcal{N}_p^{\otimes n}\right) = n\,Q^{(1)}\left(\mathcal{N}_p\right)}$$

<u>idea:</u> move weight from $|2\rangle$ to $|1\rangle$

spin alignment:

$$S\left(\sum_{I\subseteq[n]} \mu_I \, \phi_I \otimes \left(\frac{\mathbb{I}}{2}\right)^{I^c}\right)$$

minimized by $|0\rangle\langle 0|^{I}$

$n = 1, 2, 3,$

C, P, Q