Adventures in Phase Space: Quantum Signal Processing for Universal Control of Oscillators

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Disclosure: SMG is a consultant and equity holder in Quantum Circuits, Inc. and an equity holder in IBM, Inc.

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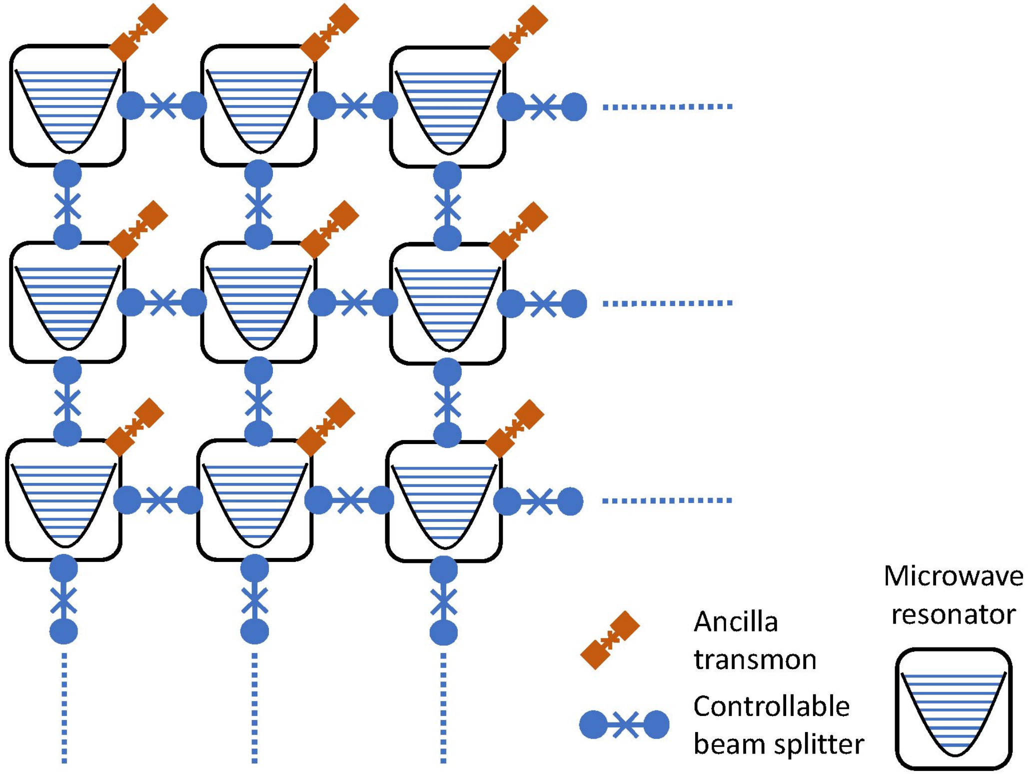
Instruction Set Architectures and Abstract Machine Models for Hybrid Oscillator-Qubit Processors

Isaac Chuang, Ella Crane, Alec Eickbusch, Steven M. Girvin, Richard Li, Yuan Liu, John Michael Martyn, Jasmine Sinanan, Shraddha Singh, Kevin Smith, Micheline B. Soley, Nathan Wiebe



C2QA manuscript in preparation
(127 pages, 473 references and counting …)

Take-home message:

- **Hardware native bosonic modes offer advantages for:**
  - Efficient quantum error correction for computation
  - Efficient quantum simulation of physical models containing bosons
- **Hybrid qubit/oscillator combinations can achieve universal control**
  - We need a simple (human-readable) instruction set architecture (ISA) in order to be able to develop algorithms and reason about circuit depth/complexity

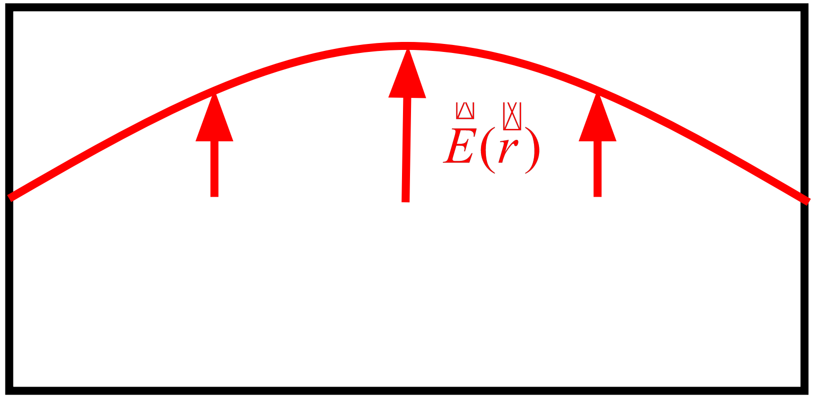
Goal 1: Learn how to create arbitrary quantum states in hybrid qubit/oscillator systems using quantum control theory.

Example for this talk: deterministic creation of a non-Gaussian state, the Schrödinger cat state

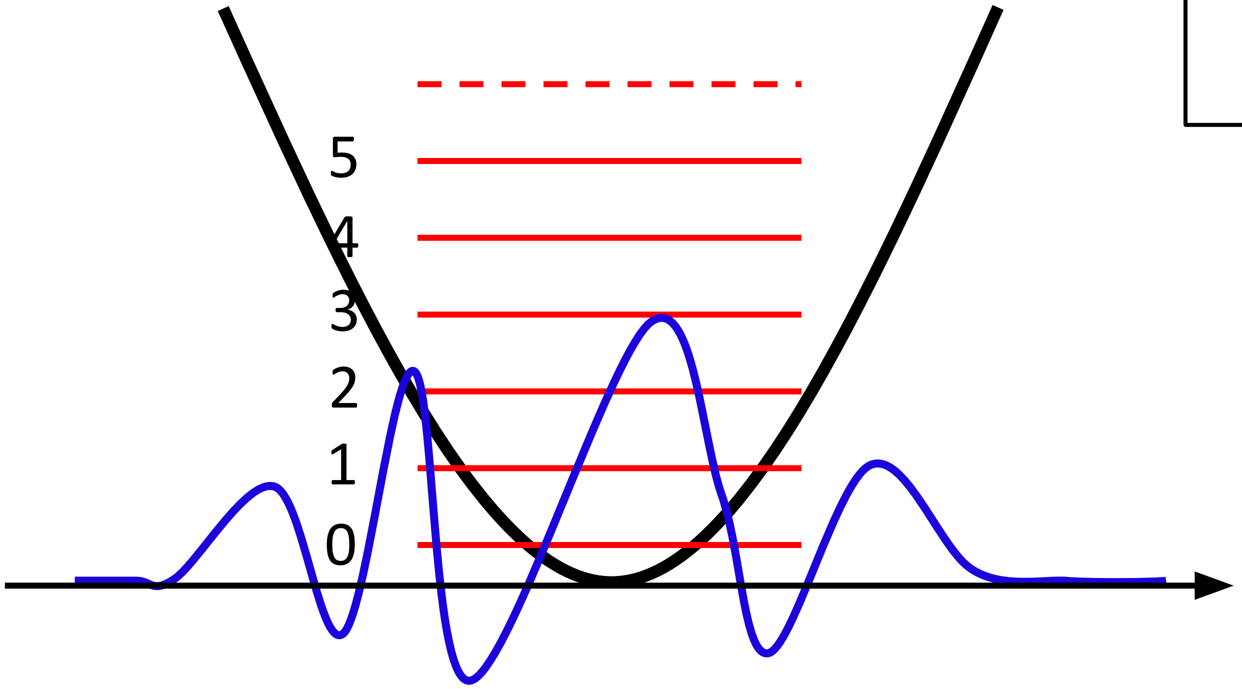
Goal 2: Learn how to do state and process tomography and benchmarking of continuous-variable (CV) systems.

Tomography is understood and works well experimentally, as we will see.

Randomized Benchmarking is less well understood because the group analogous to the Clifford gate set is non-compact.



$$H = \hbar\omega\, a^{\dagger} a = \hbar\omega\, \hat{n}$$



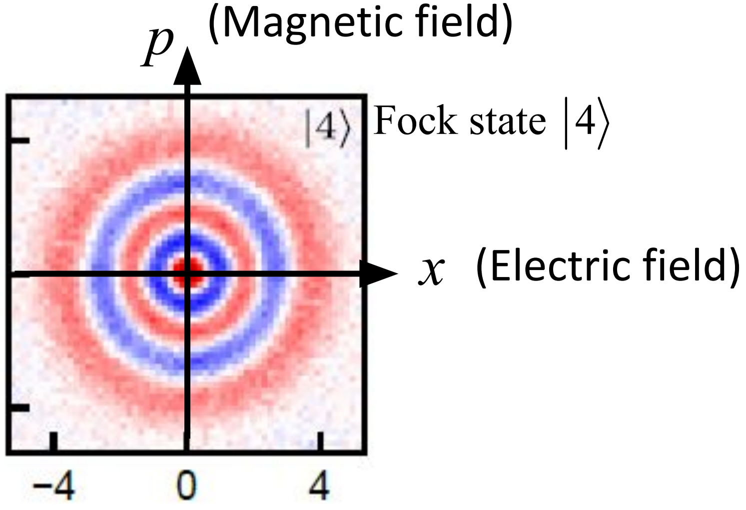
Microwave harmonic oscillator basics:

Fock Basis: $|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + \ldots$

Position Basis: $\psi(x) = \langle x|\psi\rangle = \sum_{n=0}^{\infty} a_n\, H_n(x) e^{-x^2}$

Representations using quasiprobability distributions in phase space:

- Wigner Function
- Characteristic Function



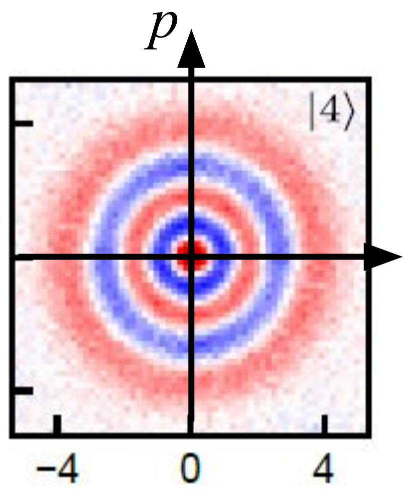
State Tomography for bosonic modes

Three equivalent (up to Fourier transforms) representations arranged in order of descending experimental difficulty

Density matrix: $\rho(x,x') = \langle x|\Psi\rangle\langle\Psi|x'\rangle$

Wigner function: $W(\alpha) = \langle\Psi|D^{\dagger}(-\alpha)\,\hat{P}\,D(-\alpha)|\Psi\rangle; \quad \hat{P} \equiv (-1)^{a^{\dagger}a}$

Characteristic function: $C(\alpha) = \langle\Psi|D(\alpha)|\Psi\rangle = \langle\Psi|e^{\left[\alpha a^{\dagger} - \alpha^{*} a\right]}|\Psi\rangle$



All three provide complete information about the quantum state.

Hierarchy of qubit state complexities

Category	Generators	Complexity
Product states in Z basis		classical
States generated from product states via Clifford gates	Clifford generators (H,S=Sqrt(Z),CNOT)	Quantum entangled but no 'magic.' Clifford operations easy to simulate classically via stabilizer evolution
States generated from product states via Non-Clifford Gates	T=Sqrt(S) gate	'Magic' prevents efficient classical simulation but allows violation of Bell inequalities and universal computation

Hierarchy of oscillator state complexities*

*Subtlety:
Complexity of CV states depends on what measurement resources are available.

Category	Generators	Complexity
Vacuum state		Easy to make if $\hbar\omega \gg k_B T$
Gaussian states generated from vacuum via Gaussian gates, or incoherent mixtures of same	H(x,p) quadratic form	Easy to simulate classically via evolution of mean and covariance matrix. Easy to sample from.
Non-Gaussian (e.g. superpositions of Gaussians, not mixtures of Gaussians)	H(x,p) higher than quadratic polynomial, e.g., x^3	Wigner function negativity prevents efficient classical simulation but allows violation of Bell-like inequalities and universal computation

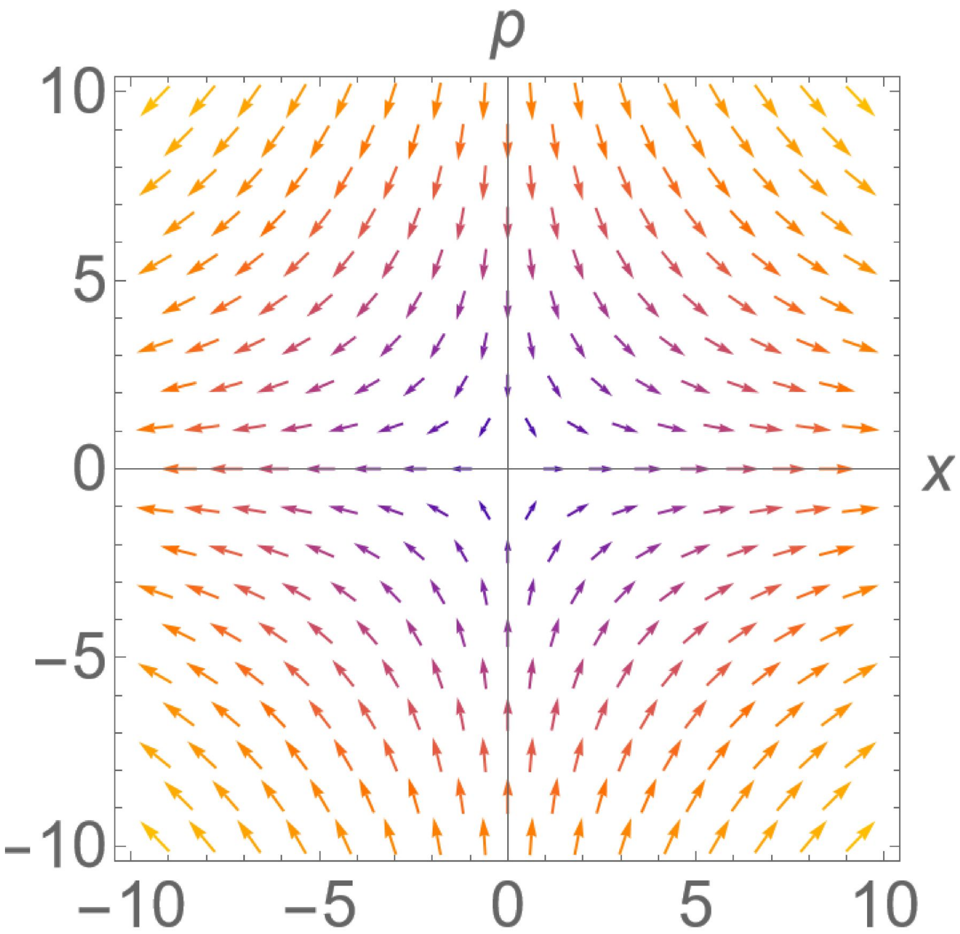
CV-DV comparison

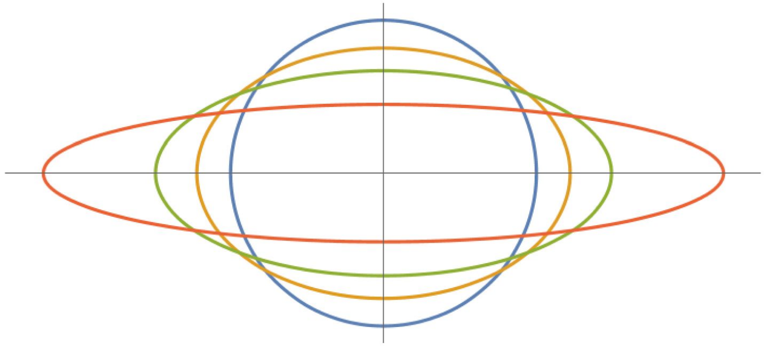
Aspect	Qubits (DV)		Bosonic Modes (CV)
Standard Basis	Z, $\{0,1\}$		$\hat{x}$, $\|x\rangle$
Conjugate Basis	X, $\|0\rangle \pm \|1\rangle$		$\hat{p}$, $\|p\rangle = \int dx\, e^{ixp}\|x\rangle$
Basis Transformation	Hadamard		Fourier Transform (i.e., $e^{i\frac{\pi}{2}\hat{n}}$)
Group	Pauli		Weyl-Heisenberg (e.g., $e^{i\alpha\hat{x}}$, $e^{i\beta\hat{p}}$)
	Clifford		Gaussian unitaries (e.g., $e^{i\theta\hat{n}}, e^{i\lambda\hat{x}^2}, e^{i\hat{x}_1\hat{p}_2}$)
Channel	Pauli/Clifford		Displacement/Gaussian
Measurement	Pauli basis Measurement of X, Z requires Clifford operations		Homodyne Measurement of $\hat{x}, \hat{p}$ requires Gaussian operations
	Non-Clifford $(\vec{b} \cdot \vec{\sigma})$		Non-Gaussian $(a^\dagger a, \|n\rangle\langle n\|, aa, \hat{x}^2, \hat{p}^3, \cos k\hat{x})$
Representation	State Tomography		Wigner (or Characteristic) Function Tomography
Benchmarking	Clifford operations are 2-design		Gaussian operations are not 2-design
Example Entangled State	Bell: $\|00\rangle + \|11\rangle$		EPR (TMS, SUM): $\sum_n \|n\rangle_1\|n\rangle_2$, $\int dx\, \|x\rangle_1\|x\rangle_2$
Universal Computation	a non-Clifford gate, $T : e^{i\pi/8Z}$ augments the Clifford group with magic (non-Cliffordness)		a non-Gaussian unitary, cubic gate: $e^{i\chi\hat{x}^3}$ augments Gaussian operations with Wigner negativity

Phase space picture is useful to understand unitary transformations of oscillator states

Classically, time evolution generates volume-preserving diffeomorphism on phase space—symplectic transformation of x,p that preserves the Poisson brackets.

Classical ‘flow’ in phase space provides useful intuition for building quantum unitaries (and leads to pretty pictures)





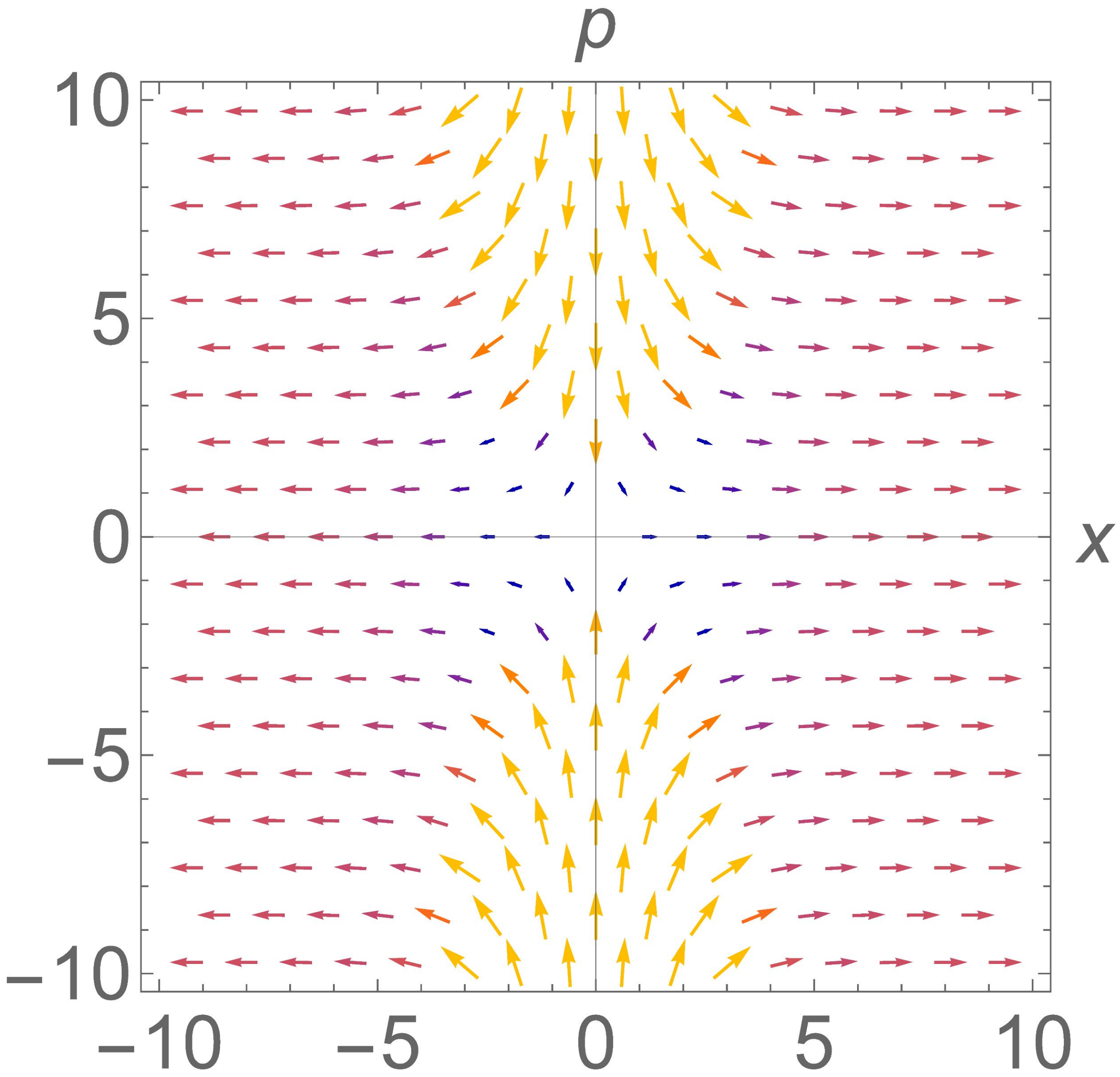
[Classical] squeezing

$$H = px$$

$$\dot{x} = \frac{\partial H}{\partial p} = x \qquad x(t) = x(0)e^{t}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -p \qquad p(t) = p(0)e^{-t}$$

$$x(t)\,p(t) = \text{ constant}$$



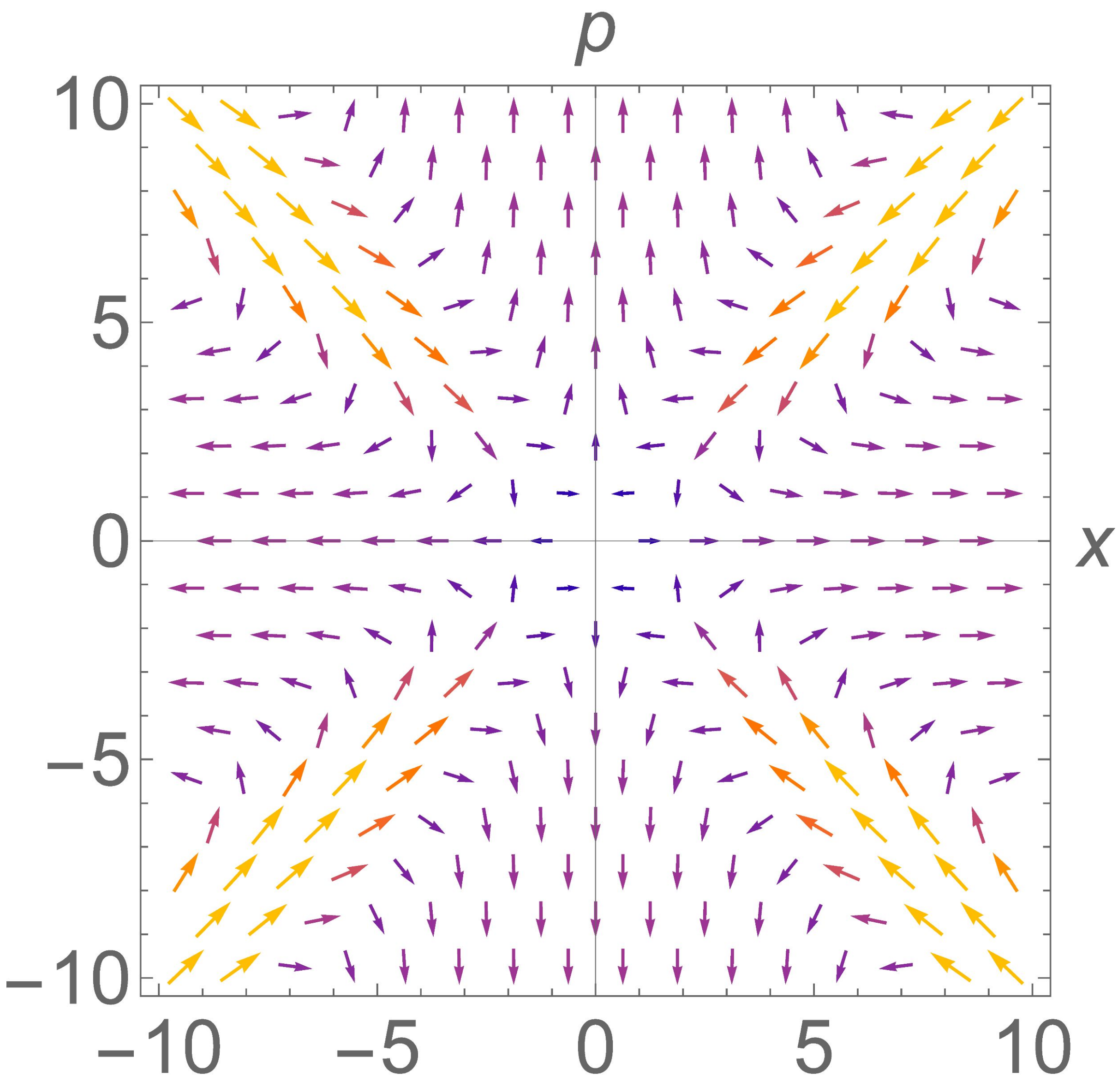
Cat Amplifier

$$H = \tanh(x)\, p$$

$$\dot{x} = \tanh(x)$$

$$\dot{p} = -\frac{1}{\cosh^2(x)}$$

$$|\psi_{\text{cat}}\rangle = \frac{1}{\sqrt{2}}\left[|\alpha\rangle + |-\alpha\rangle\right]$$



4-legged cat amplifier

$$H = -\tanh\left[\left(x+p\right)\right]\left(x-p\right) + \tanh\left[\left(x-p\right)\right]\left(x+p\right)$$

4-cat QEC code words

$$\left|\psi_{4\text{-cat}}\right\rangle = \frac{1}{2}\left[\left(\left|\alpha\right\rangle + \left|-\alpha\right\rangle\right) \pm \left(\left|i\alpha\right\rangle + \left|-i\alpha\right\rangle\right)\right]$$

Short course on quantum control theory

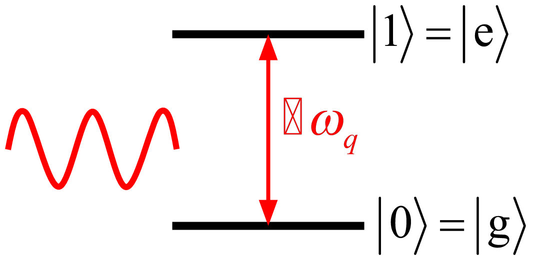
A two-level system (qubit) is controllable with classical drives

$$V(t)=\left[\epsilon_x(t)\cos(\omega_q t)+\epsilon_y(t)\sin(\omega_q t)\right]\sigma^x$$

Interaction picture ($H_0 \to 0$) rotating wave approximation:

$$\mathrm{V(t)} \approx \epsilon_x(t)X+\epsilon_y(t)Y$$

where X and Y are Pauli matrices in the rotating frame



$$H_0=-\frac{\hbar\omega_q}{2}\sigma^z$$

Theorem:
A quantum system is controllable iff the Lie algebra generated by the available controls spans the space of operators on the Hilbert space, i.e., fails to close* so is full-rank.
(*fine print for infinite dimensional spaces)

$[X,Y]=2iZ$

Hence a qubit with two classical controls $\epsilon_x(t),\epsilon_y(t)$ is fully controllable.

[Lie Algebra generates the full SU(2) Lie group.]

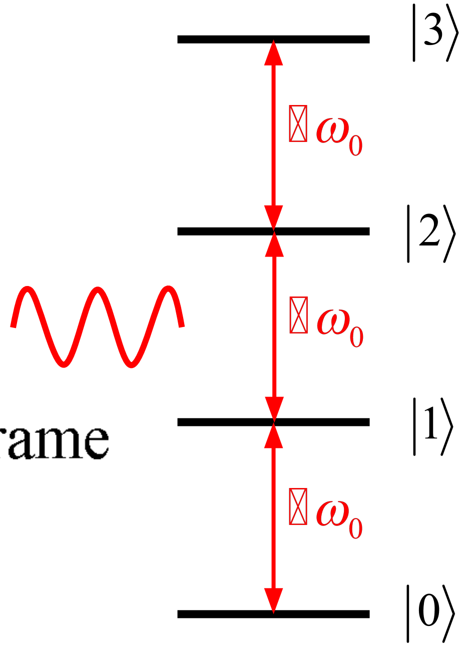
A quantum harmonic oscillator is not controllable with classical drives

$$V(t) = \left[\epsilon_x(t)\cos(\omega_0 t) + \epsilon_y(t)\sin(\omega_0 t)\right](a + a^\dagger)$$

Interaction picture ($H_0 \to 0$) rotating wave approximation:

$$\mathrm{V(t)} \sim \epsilon_x(t)\hat{x} + \epsilon_y(t)\hat{p}$$

where $\hat{x}$ and $\hat{p}$ are phase space coordinates in the rotating frame



$$H_0 = \hbar\omega_0 a^\dagger a$$

$$[\hat{x}, \hat{p}] = i\hbar$$

Operator algebra closes.

Classical drives can displace/boost the oscillator (e.g., $\hat{x} \to \hat{x} + \Delta_{\mathrm{classical}}$) to create coherent states, but cannot create, e.g., Fock states.

In order to fully control a harmonic oscillator we require an anharmonic object (e.g., a qubit) as an auxiliary controller.

Microwave resonator | Transmon qubit | Dipole Coupling

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + g\sigma^x[a + a^\dagger] + H_{\text{damping}}$$ [Rabi]

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + g[a\sigma^+ + a^\dagger\sigma^-] + H_{\text{damping}}$$ [Jaynes-Cummings]

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + \chi\sigma^z a^\dagger a + H_{\text{damping}}$$ [Dispersive]

Strong Dispersive Limit

Microwave resonator

Transmon qubit

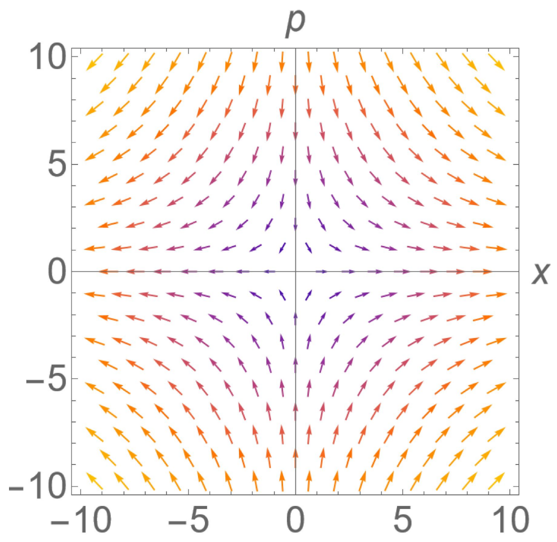
Dispersive Coupling

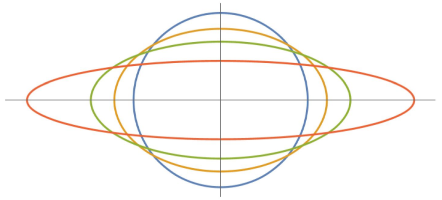
$$H = \omega_{\mathrm{r}} a^{\dagger} a + \frac{\omega_{\mathrm{q}}}{2}\sigma^{z} + \chi\sigma^{z} a^{\dagger} a + H_{\mathrm{damping}} \qquad \text{[Dispersive]}$$

With drives on the cavity and qubit we can generate ion-trap-like 'spin-dependent' forces. In RWA:

$V = \left[-\grave{o}_x(t)\hat{x} + \grave{o}_y(t)\hat{p}\right][Z] \leftarrow$ can also be X or Y

$[\hat{x}X, \hat{p}Y] = i(\hat{x}\hat{p} + \hat{p}\hat{x})Z$ (conditional squeezing!)





$$[\hat{x}X, \hat{p}Y] = i(\hat{x}\hat{p} + \hat{p}\hat{x})Z \text{ (conditional squeezing!)}$$

$$[\hat{x}X, [\hat{x}X, \hat{p}Y]] = (\hat{x}^2\hat{p} + 2\hat{x}\hat{p}\hat{x} + \hat{p}\hat{x}^2)Y$$

Operator algebra does not close $\Rightarrow$ universal control

With Trotter-Suzuki and Baker-Campbell-Hausdorf we can use the commutator algebra to generate universal (effective) Hamiltonian:

$$H = \vec{\Lambda}(\hat{x}, \hat{p}) \cdot \vec{\sigma} + h(\hat{x}, \hat{p}) \qquad \text{with} \quad \vec{\sigma} = (X, Y, Z)$$

Arbitrary polynomials of bounded order

ISA Example: (Echoed) Controlled-Displacement + Qubit Rotations

Conditional Displacement Gate

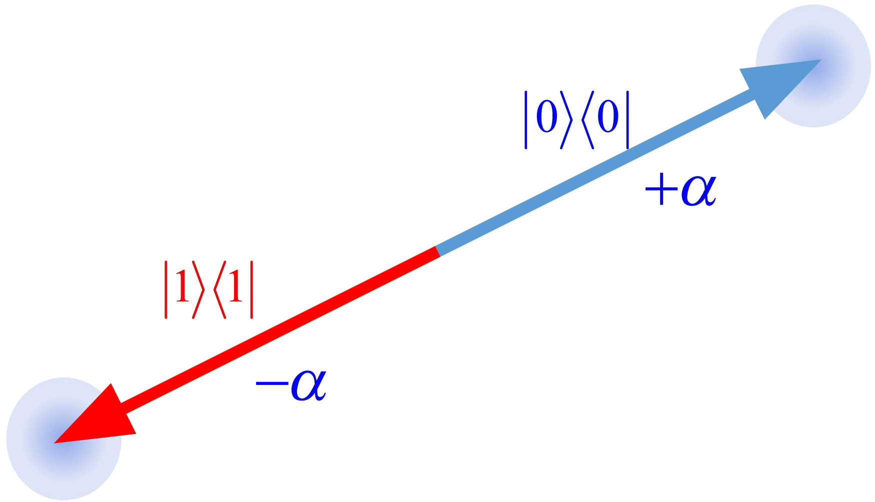
$$D = \exp\left[-it\left(-\epsilon_x \hat{x} + \epsilon_y \hat{p}\right)\right] Z$$

Fast Universal Control of an Oscillator with Weak Dispersive Coupling to a Qubit, A. Eickbusch et al. (Devoret Lab)

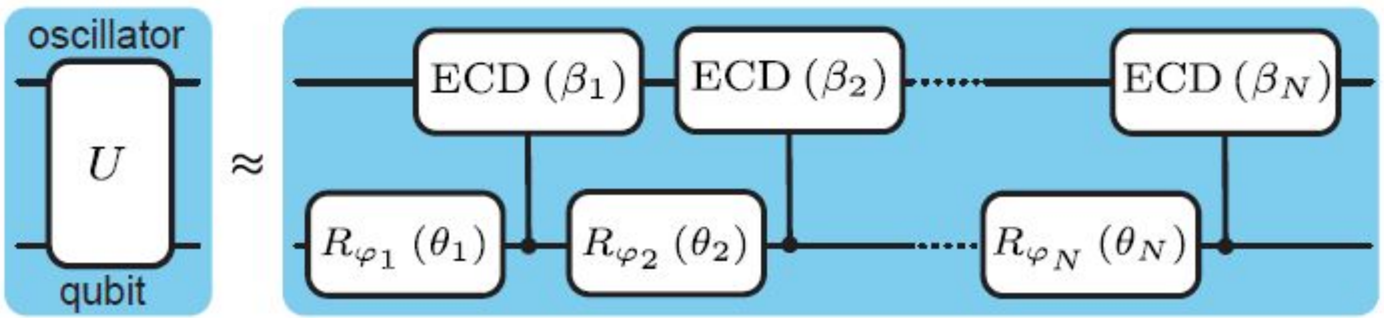
Nature Physics (2022)

Qubit Rotation Gate

$$R_\varphi(\theta) = e^{-i\frac{\theta}{2}\left[\cos\varphi\,\sigma^x + \sin\varphi\,\sigma^y\right]}$$

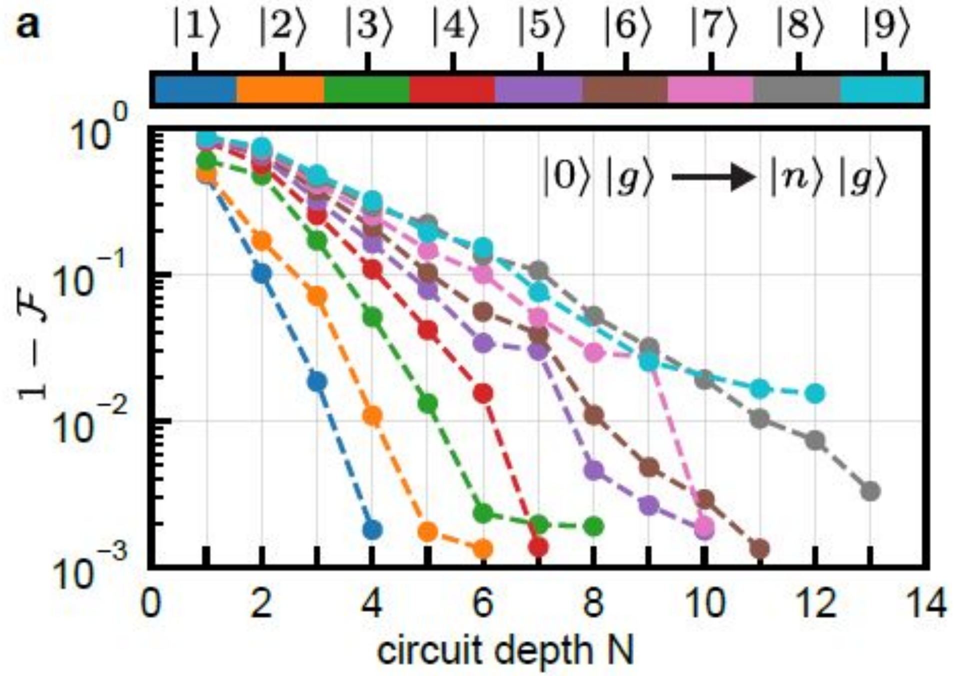


$\mathrm{ECD} = \sigma^x D_c$



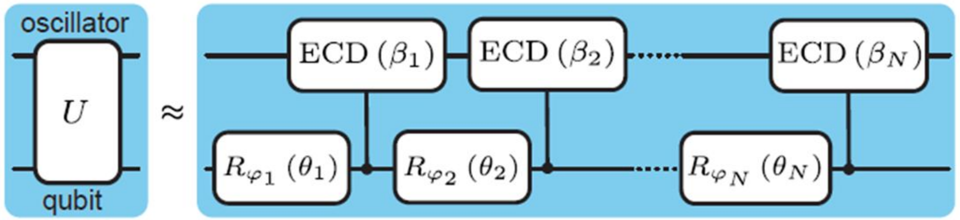
Circuit depth →

(Echoed) Controlled-Displacement ISA:



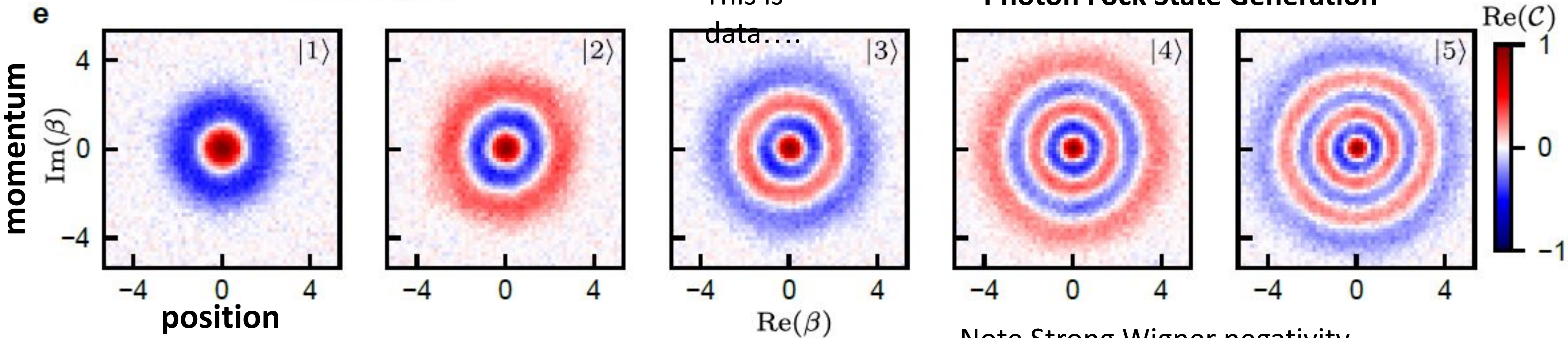
Eickbusch et al. (Devoret Lab) ***Nature Physics*** (2022)

Numerically optimized circuits to solve the disentangling problem and create correct unitary.



Phase Space 'Portraits' of Photon Fock State Generation

This is data….

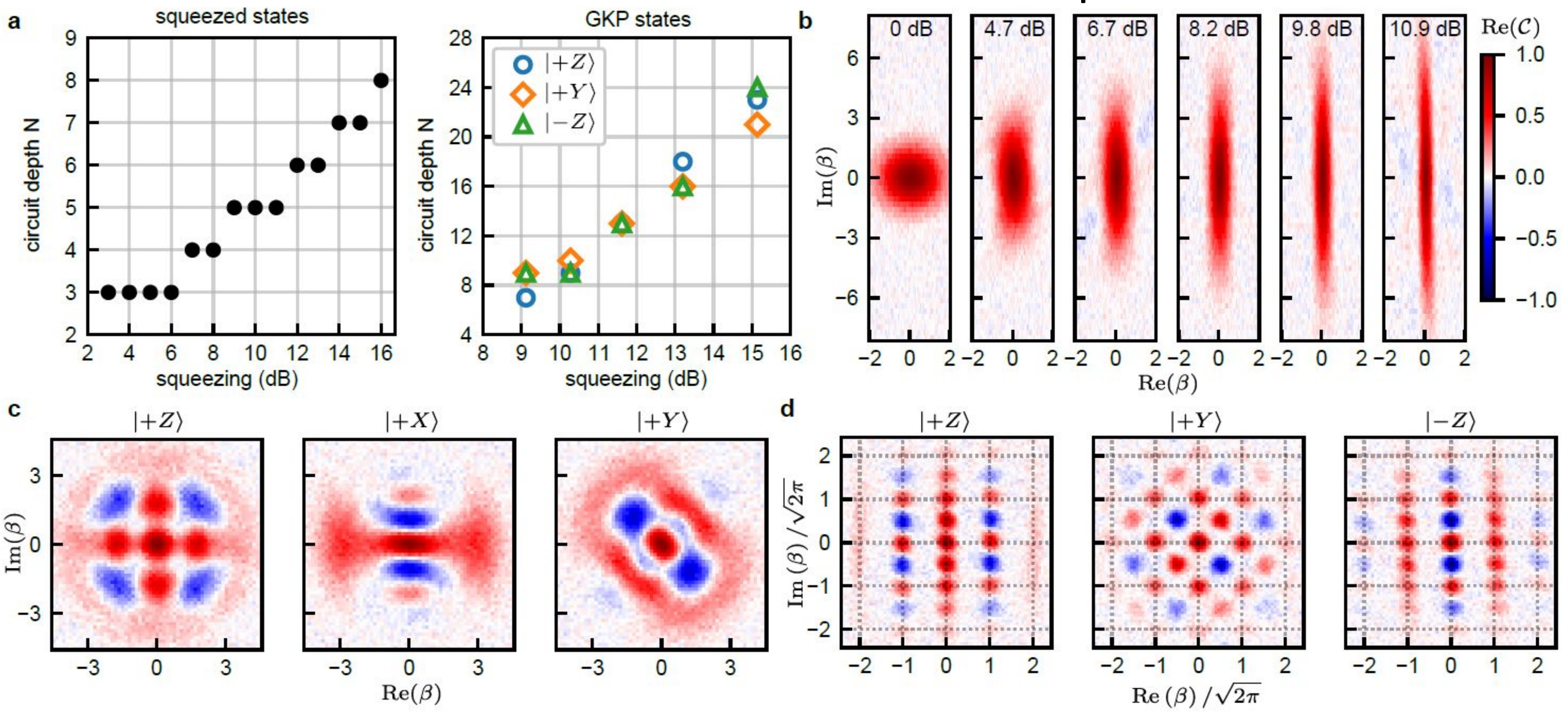


Note Strong Wigner negativity

(Echoed) Controlled-Displacement ISA:

Fast Universal Control of an Oscillator with Weak Dispersive Coupling to a Qubit, A. Eickbusch et al. (Devoret Lab) *Nature Physics* (2022)

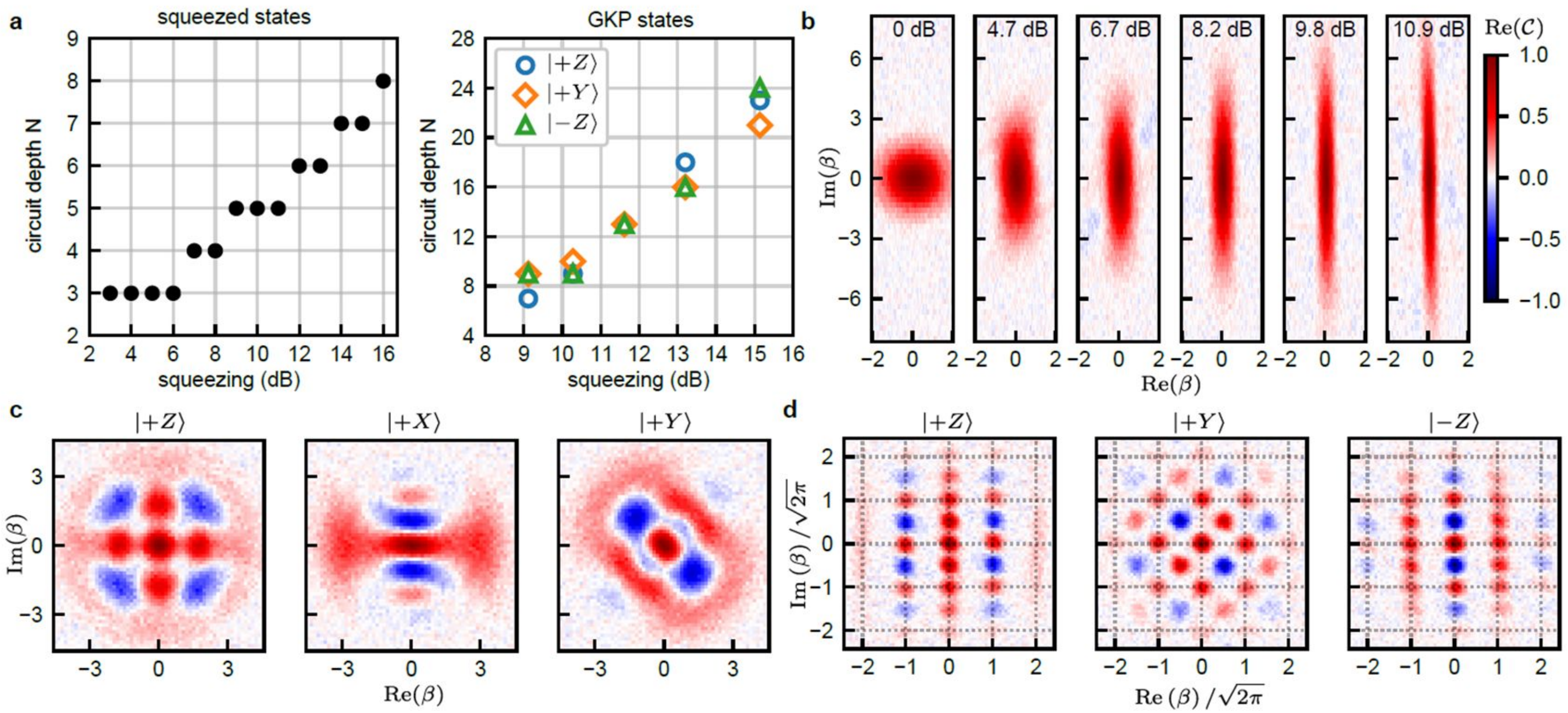
Squeezed states



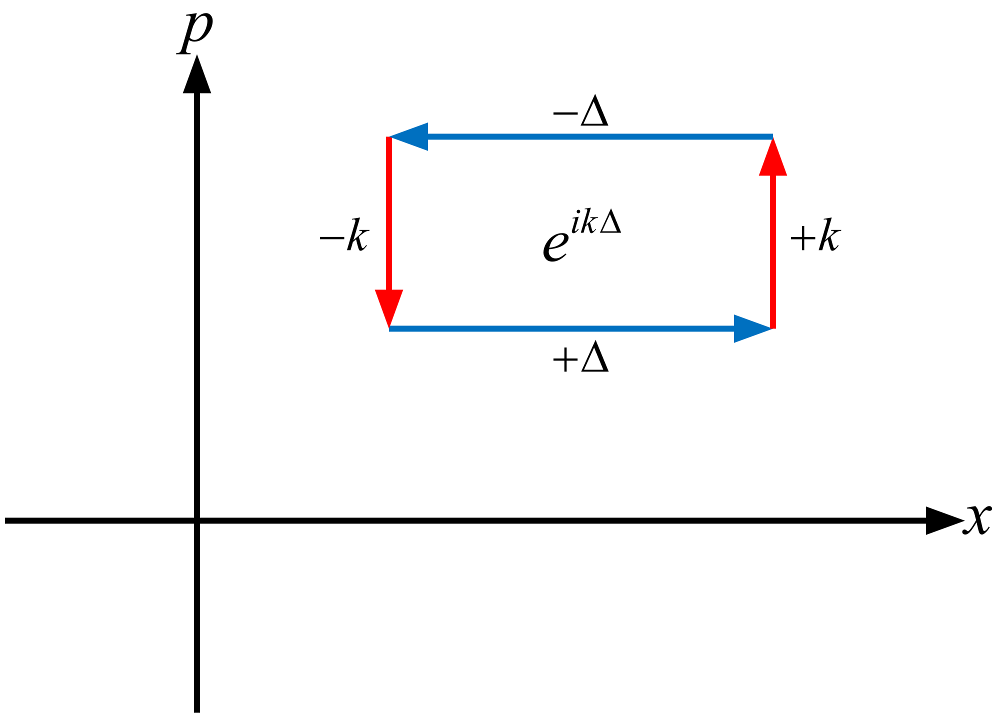
Binomial QEC code word states

GKP QEC code word states

Numerically optimized gate sequences are highly expressive (efficient) but <u>incomprehensible</u>.



Towards 'human readable' circuits for the gate set: {ECD + Qubit Rotations}



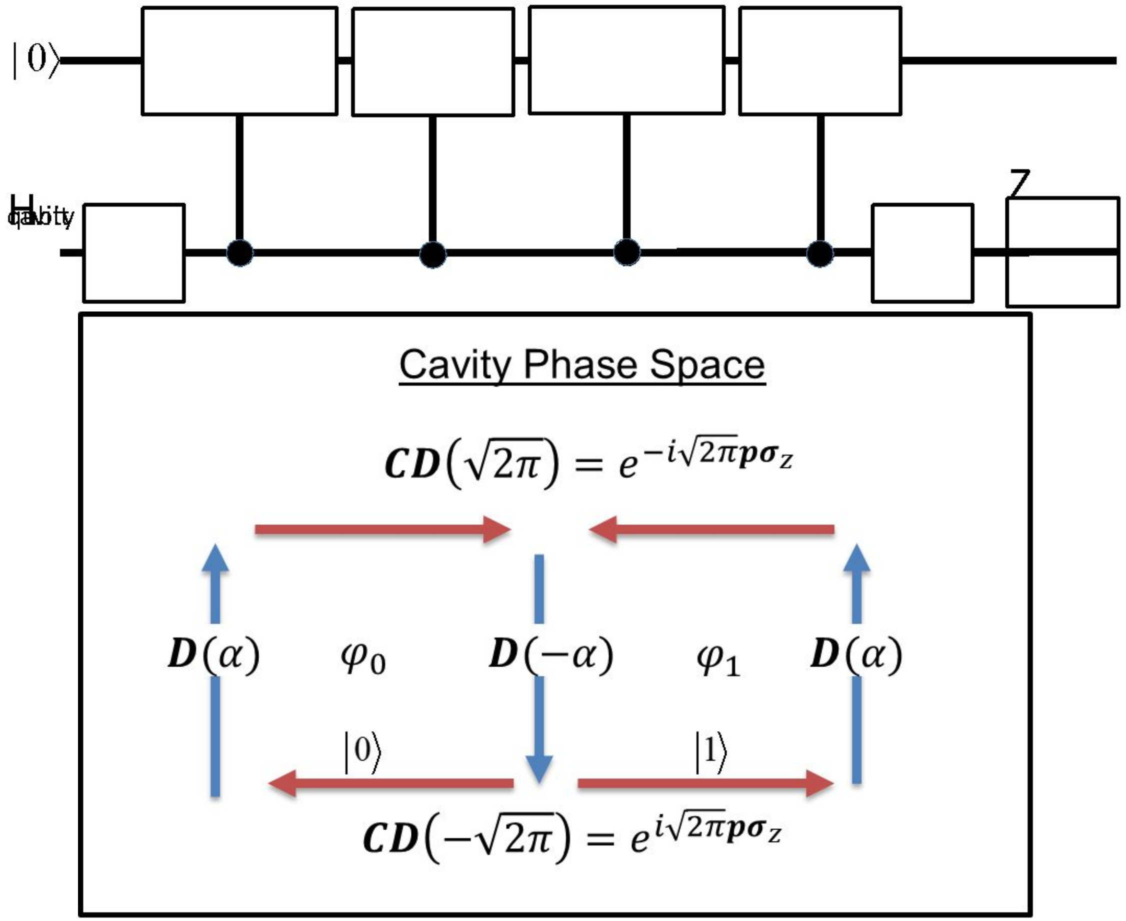
Non-Commutative Geometry of Phase Space [hbar = 1]

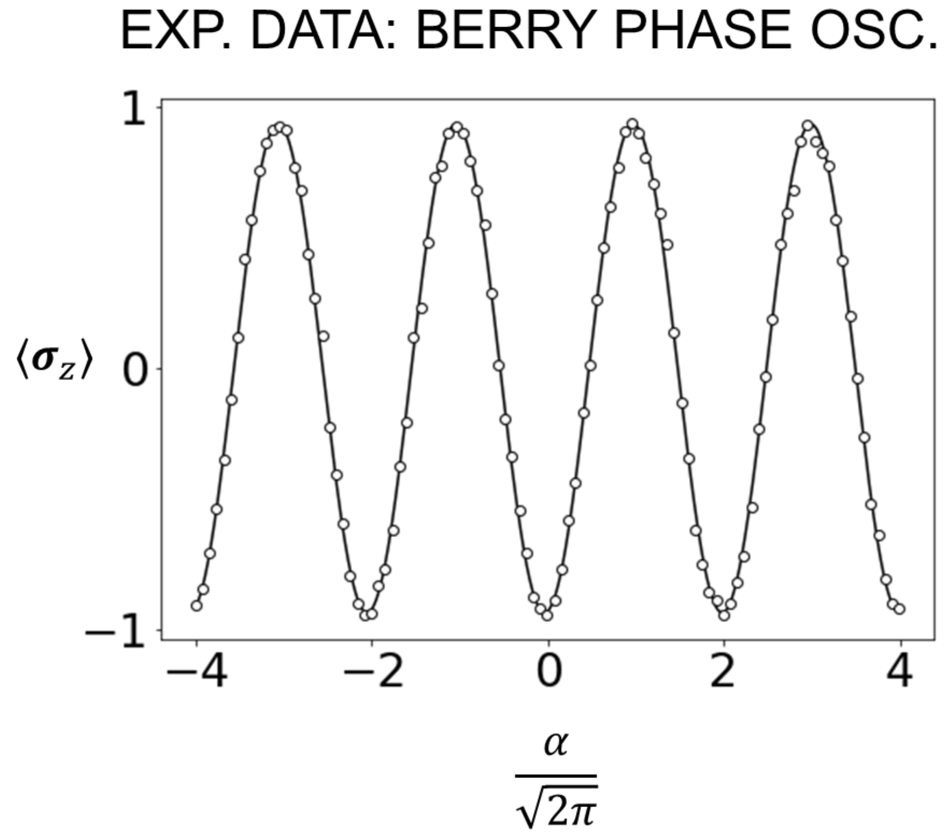
Geometric Phase = Enclosed Area

$$\psi(x) \rightarrow \psi(x-\Delta) \rightarrow e^{ikx}\psi(x-\Delta) \rightarrow e^{ik(x+\Delta)}\psi(x) \rightarrow e^{ik\Delta}\psi(x)$$

Experimental Calibration of Controlled Displacements Using Berry Phase (Devoret Group)

Phase kick-back on conditional displacements from non-commuting geometry





Campagne-Ibarcq, Eickbusch, Touzard, et al *Nature* **584**, 368 (2020)

geometric phase acquired over 1 cycle:

$$|0\rangle + |1\rangle \rightarrow e^{-i\varphi_0}|0\rangle + e^{i\varphi_1}|1\rangle$$

Numerically optimized gate sequences (either SNAP or ECD) are highly efficient but incomprehensible.

Could use Trotter-Suzuki + Baker Campbell Hausdorf (BCH) to synthesize less efficient but 'human readable' circuits.

For sufficiently small λ:

$$S_\lambda = e^{\lambda A} e^{\lambda B} e^{-\lambda A} e^{-\lambda B} \approx e^{\lambda^2 [A,B]}$$

Example:

$$A = i\hat{x}\sigma^x \qquad B = i\hat{p}\sigma^y$$

$$\left[A, B\right] = -i(\hat{x}\hat{p} + \hat{p}\hat{x})\sigma^z = \frac{1}{2}(a^{\dagger 2} - a^2)\sigma^z$$

Conditional Squeezing Gate:

$$S_\lambda \approx e^{\frac{\lambda^2}{2}(a^{\dagger 2} - a^2)\sigma^z}$$

Quantum Signal Processing

Tutorial (Chuang group) :
'Grand Unification of Quantum Algorithms,' *PRX Quantum* **2**, 040203 (2021).

Ideas descended from theory of robust control pulses in NMR.

Roughly speaking:

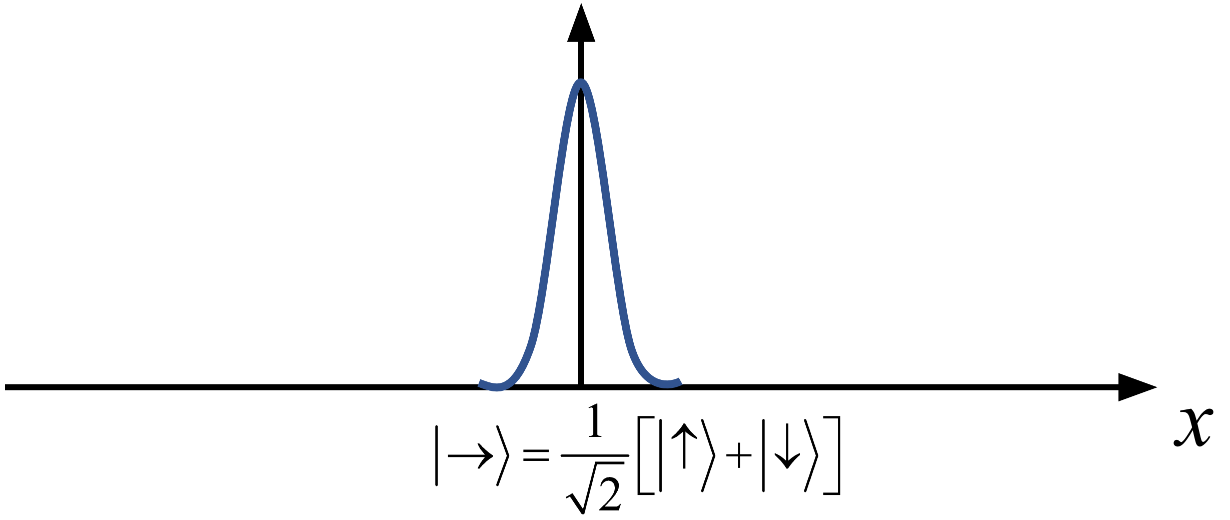
$$\text{Given} \quad e^{-i\theta\,\sigma^z} \quad \text{with } \theta \text{ unknown}$$

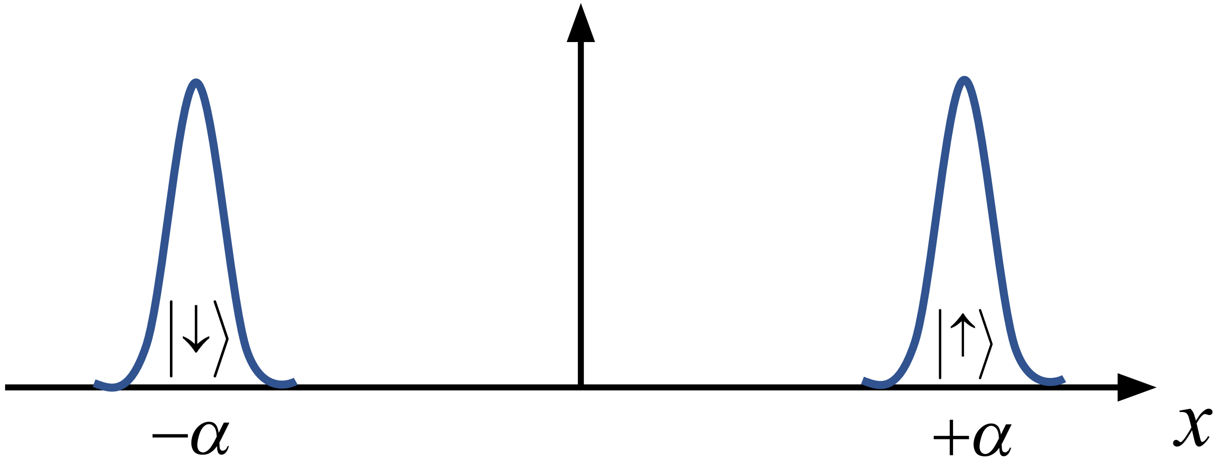
$$\text{Create from this} \quad e^{-if(\theta)\,\sigma^z}$$

We will extend these ideas from qubits to the control of hybrid qubit/oscillator systems and deterministically create a cat state.

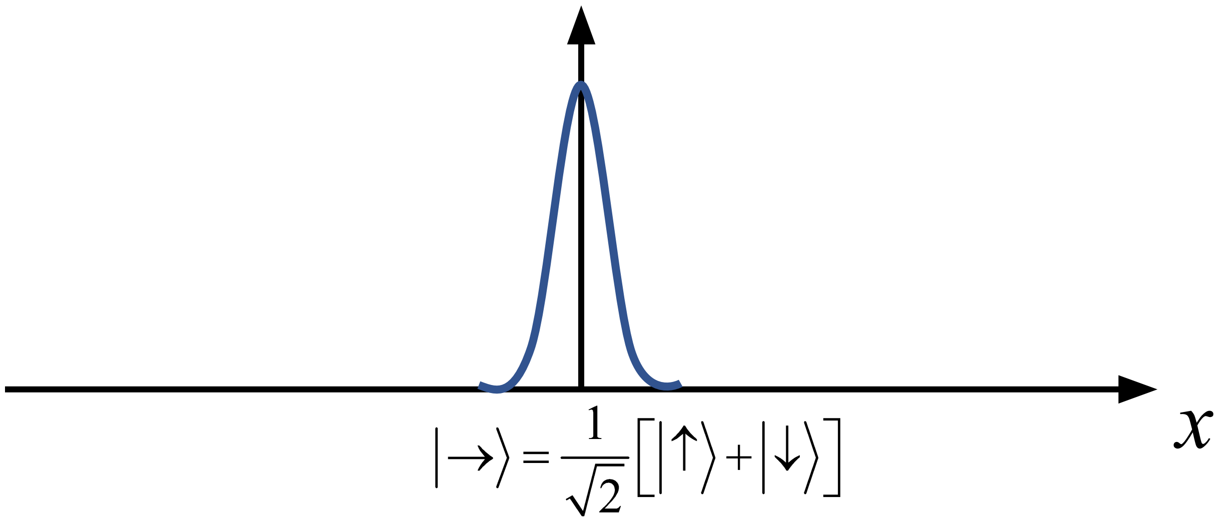
Application: creation of oscillator cat states via controlled displacements

$$e^{-i\hat{p}\,\alpha Z}\left|\rightarrow\right\rangle\left|0\right\rangle = \frac{1}{\sqrt{2}}\Big[\left|\uparrow\right\rangle\left|+\alpha\right\rangle + \left|\downarrow\right\rangle\left|-\alpha\right\rangle\Big]$$



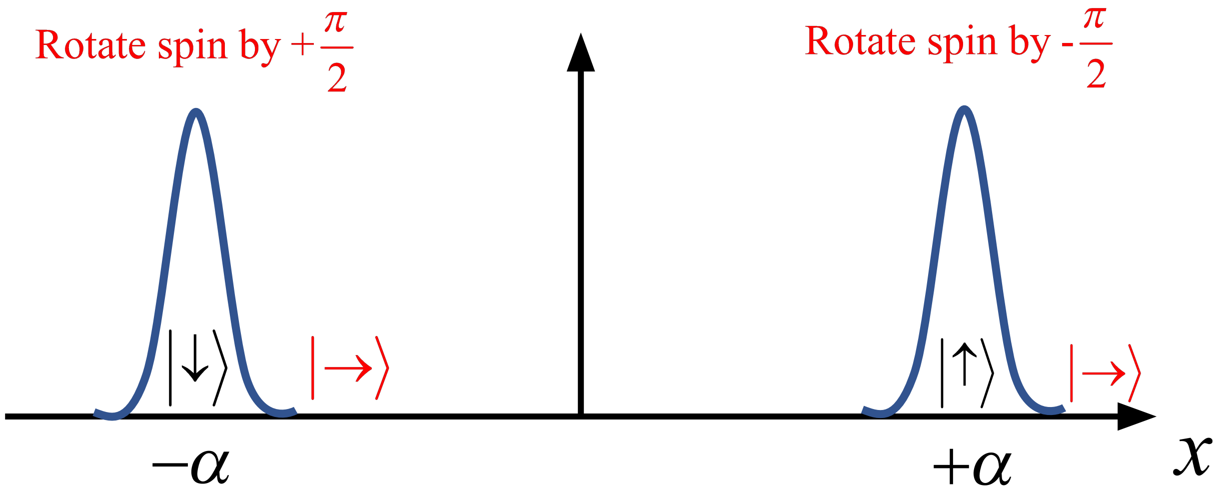


Application: creation of oscillator cat states via controlled displacements



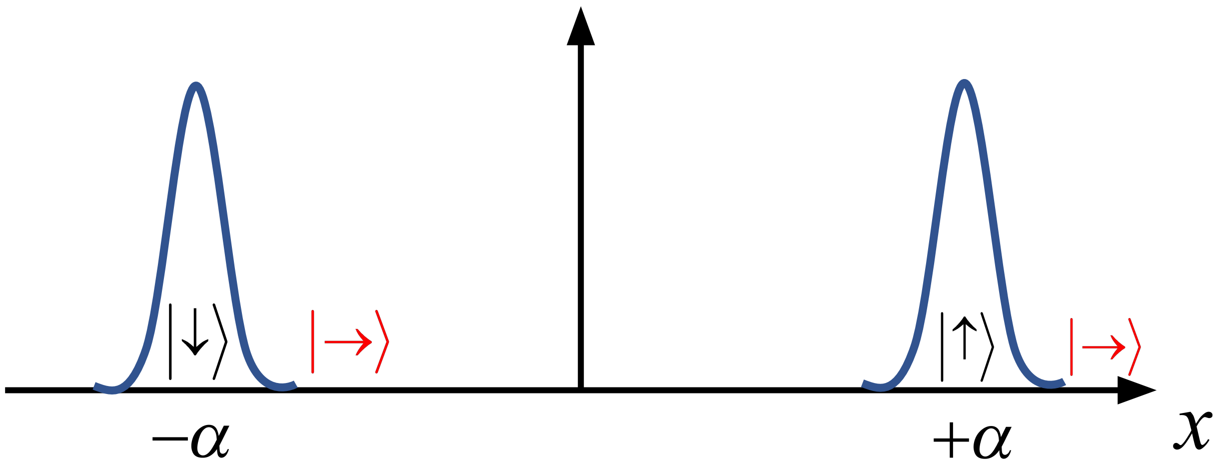
$$e^{-i\hat{p}\,\alpha\sigma^z}\,|\rightarrow\rangle|0\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle|+\alpha\rangle + |\downarrow\rangle|-\alpha\rangle\right]$$

Q: How do we disentangle the ancilla from the cavity?



Q: How do we disentangle the ancilla from the cavity?

A: Curiously, the answer is to apply a conditional momentum boost!

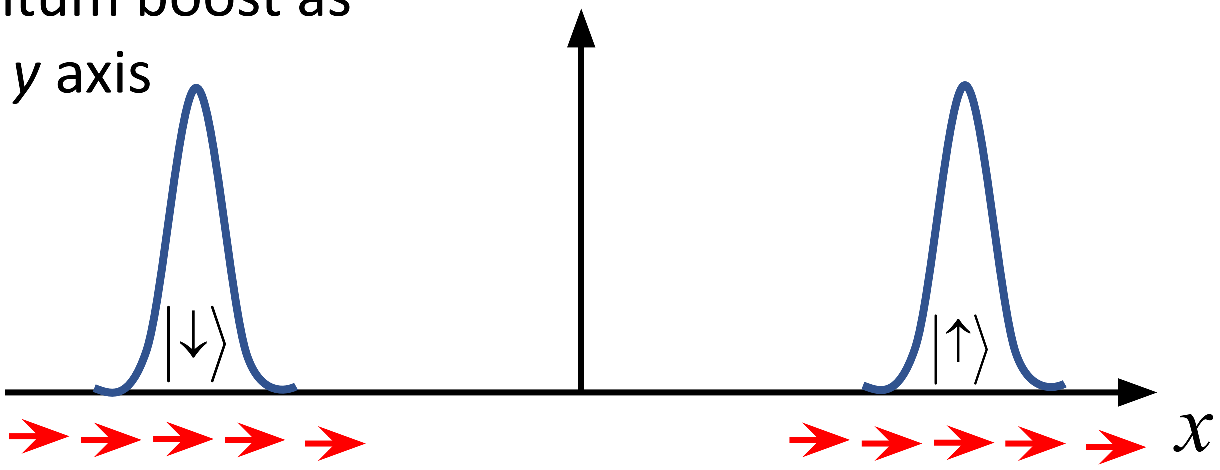


$$U = e^{ik\hat{x}\,\sigma^y}$$

Reinterpret conditional momentum boost as rotation of the qubit about the *y* axis

$$U = e^{-i\frac{\theta}{2}\sigma^y}$$

with $\theta = -2k\,\hat{x} = -\frac{\pi}{2\alpha}\hat{x}$



For large α, ancilla is nearly perfectly disentangled from the cavity.

Fidelity to ideal cat state $|\rightarrow\rangle \frac{1}{\sqrt{2}}\left[|+\alpha\rangle + |-\alpha\rangle\right]$ is $F \sim 1 - \frac{\pi^2}{64\alpha^2}$.

$$U = e^{ik\hat{x}\,\sigma^y}$$

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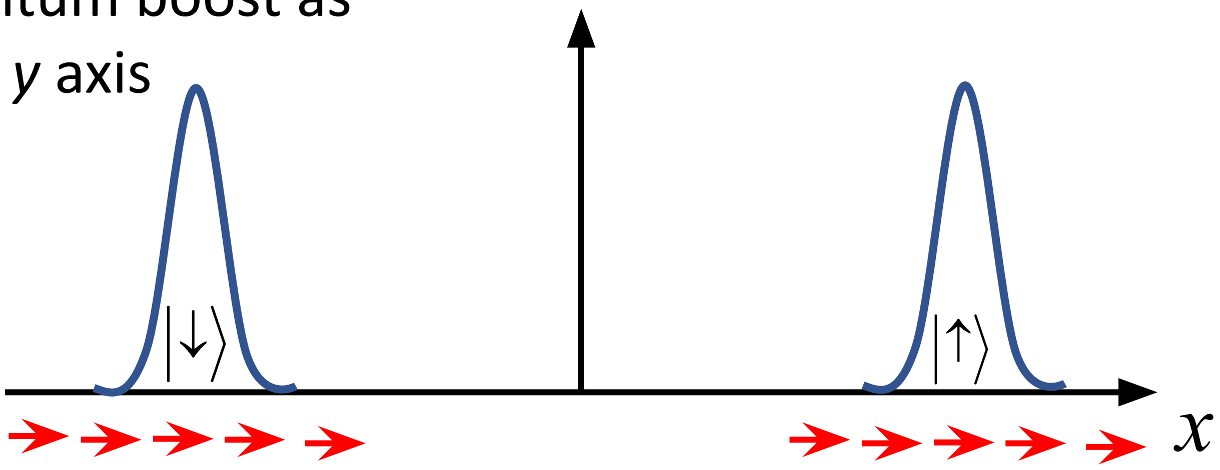


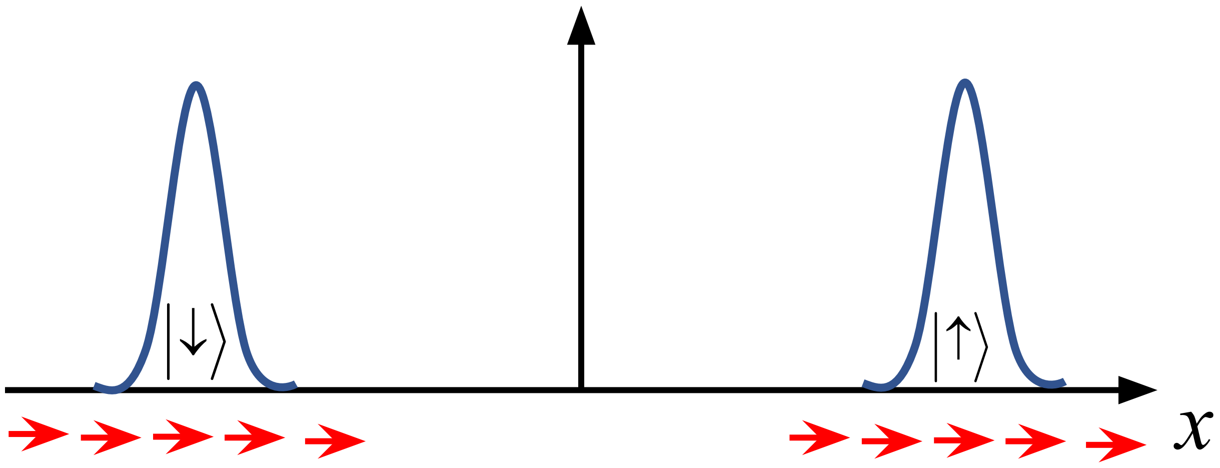
Q: Can we do better?

A: Yes! Use Quantum Signal Processing:

Wimperis (1994) invented the BB1(90) NMR pulse sequence to create spin rotations robust against classical fluctuations in amplifier gain (pulse strength).

We will use it to be robust against quantum fluctuations in position $\hat{x}$ of the oscillator that cause over and under rotations.

$$U = e^{ik\hat{x}\,\sigma^y} \rightarrow e^{if(k\hat{x})\,\sigma^y}$$



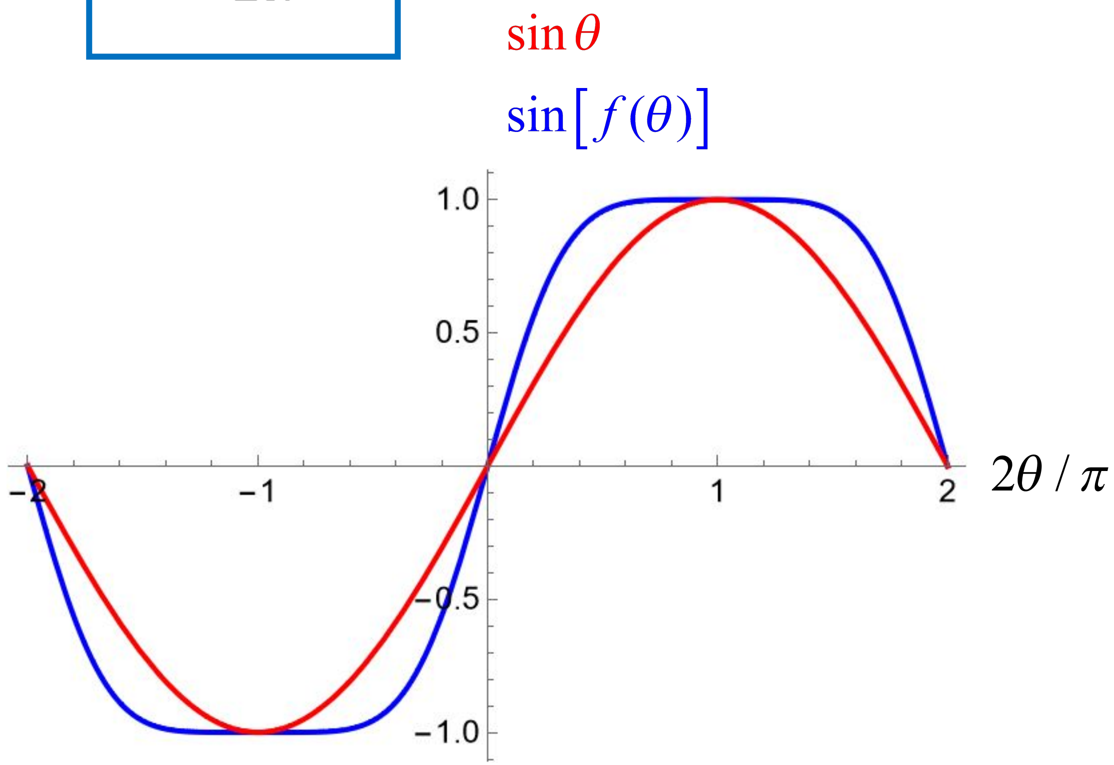
$R_\varphi(\theta) = \exp\left[-i\frac{\theta}{2}\left(\cos\varphi\,\sigma^x + \sin\varphi\,\sigma^y\right)\right]$ rotation by angle θ about an axis in the equatorial plane.

$\mathrm{BB1}(90) = \mathrm{R}_{\varphi_1}(2\theta)\mathrm{R}_{\varphi_2}(4\theta)\mathrm{R}_{\varphi_1}(2\theta)\mathrm{R}_0(\theta)$

$$\theta = \frac{\pi}{2\alpha}\hat{x}$$

$\varphi_1 = 0.54\pi, \quad \varphi_2 = 1.61944\pi$

Produces a robust $\pi/2$ rotation about the x axis of the Bloch sphere.



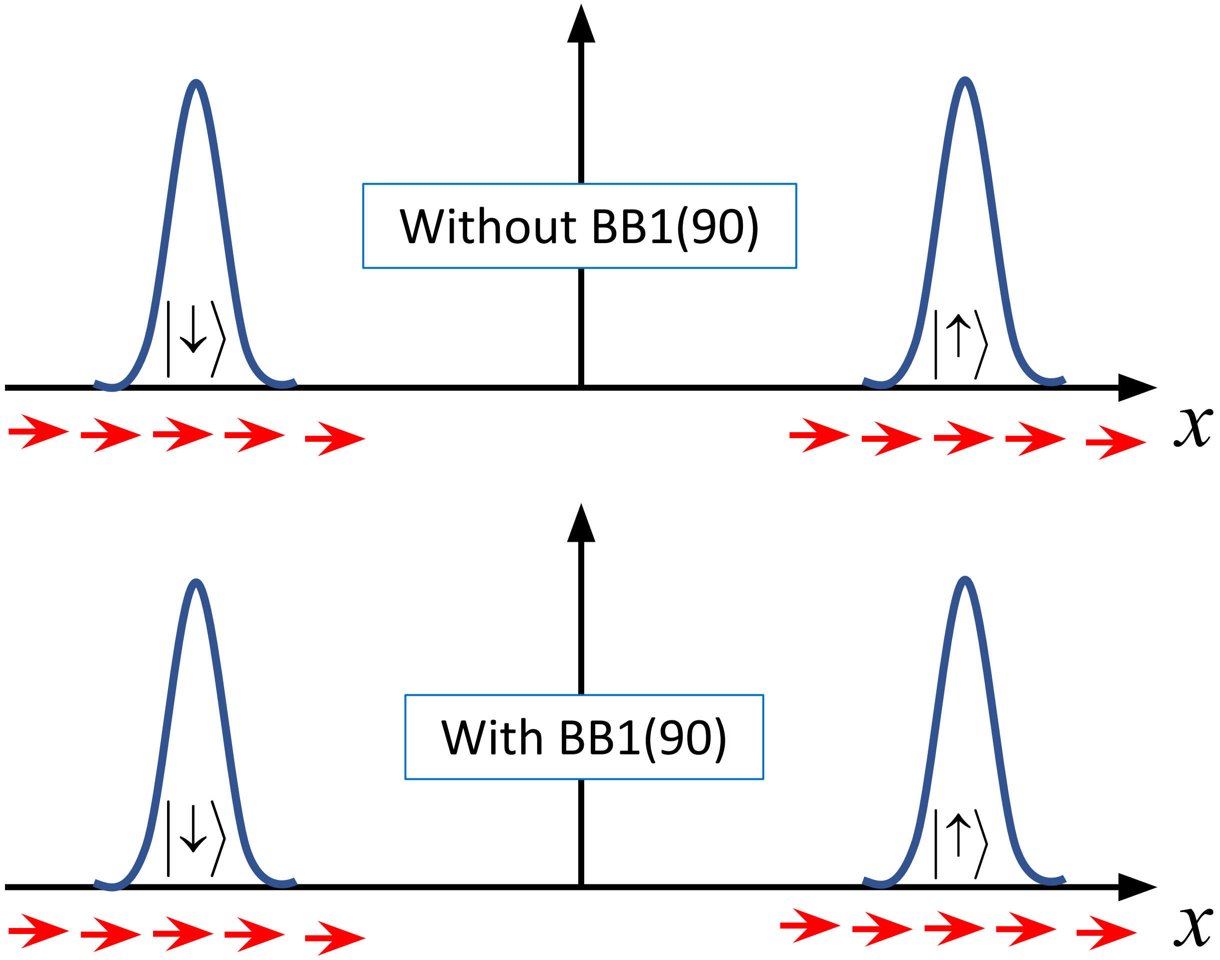
$$U = e^{ik\hat{x}\,\sigma^y} \rightarrow e^{if(k\hat{x})\,\sigma^y}$$

Q: Can we do better?

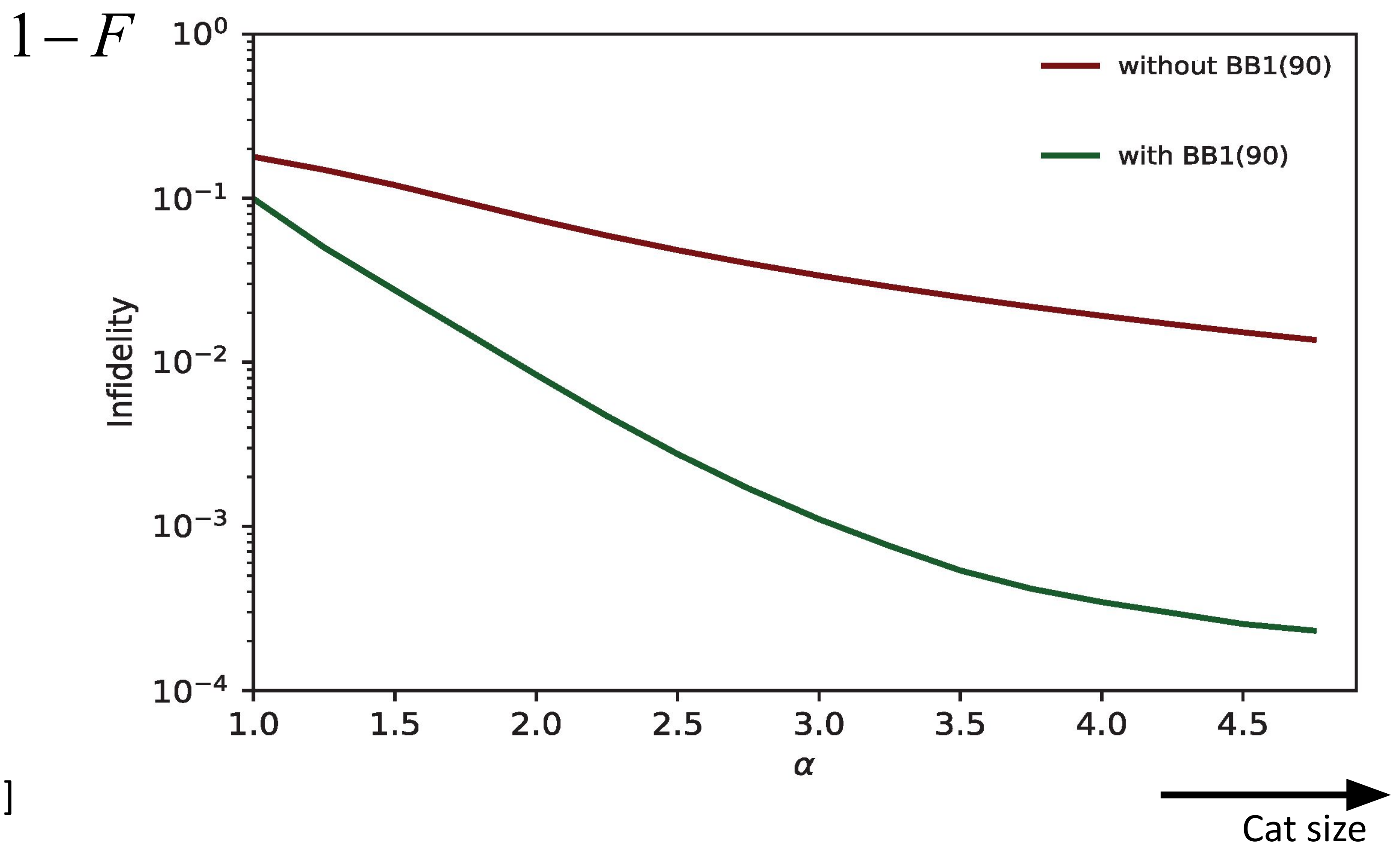
A: Yes! Use Quantum Signal Processing:

Wimperis (1994) invented the BB1(90) pulse sequence to create spin rotations robust against classical fluctuations in amplifier gain (pulse strength).

We will use it to be robust against quantum fluctuations in position $\hat{x}$ of the oscillator that cause over and under rotations.



Infidelity of even cat state preparation with and without quantum signal processing



[Shraddha Singh]

Q: Can we do EVEN BETTER?

A: Yes! Use Non-Abelian Quantum Signal Processing (aka Quantum Quantum Signal Processing)

So far we have used qubit rotation angles that are only a function of the oscillator position. This is a quantum operator, but all the rotation angles commute so classical BB190 protocol still works.

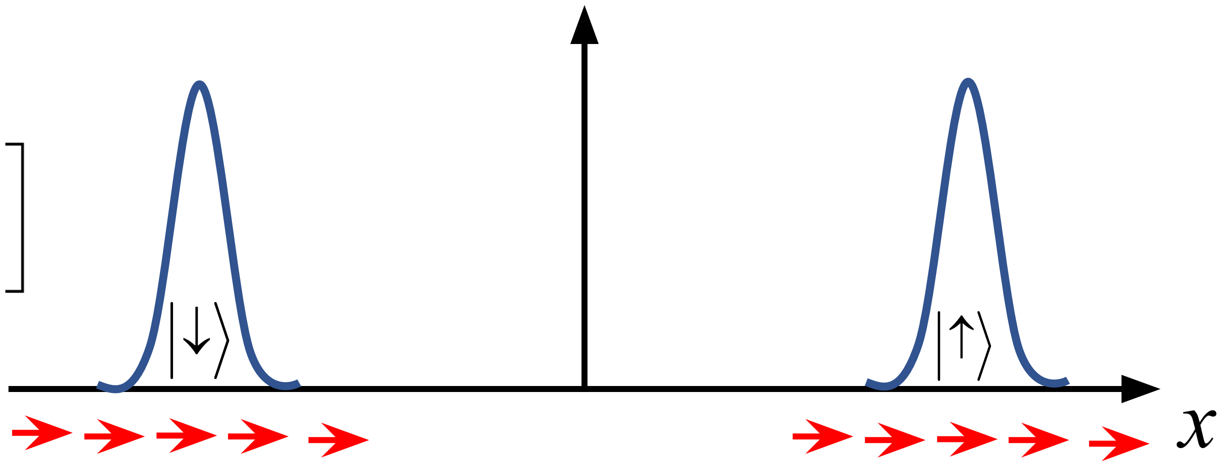
$$U = e^{ik\hat{x}\,\sigma^y} \to e^{if(k\hat{x})\,\sigma^y}$$

Generalize rotations to include angles that depend on the oscillator momentum via conditional position boosts:

$$\tilde{U} = e^{i\Delta\,\hat{p}\sigma^z}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left[e^{i\frac{\pi}{4\alpha}(\hat{x}-\alpha)\sigma^y} \,|+\alpha\rangle_{\mathrm{R}}\,|+\rangle_{\mathrm{Q}} + e^{i\frac{\pi}{4\alpha}(\hat{x}+\alpha)\sigma^y} \,|-\alpha\rangle_{\mathrm{R}}\,|+\rangle_{\mathrm{Q}} \right]$$

Residual over/under rotations around y axis



Entire BB1(90) disentangling sequence can be replaced by a single (very small) conditional displacement!

$$\psi(x) = \langle x \,|\, \pm\alpha\rangle_{\mathrm{Q}} = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{1}{4\sigma^2}(x\mp\alpha)^2)}$$

$$U_3 = e^{i\epsilon\hat{p}\,\sigma^z} \approx 1 + \epsilon\frac{d}{dx}\left[i\sigma^y\sigma^x\right]$$

$$U_3\,\psi_{\pm\alpha}(x)\big|+\big\rangle_Q \approx \left[1 + i\epsilon(x \mp \alpha)\sigma^y\right]\psi_{\pm\alpha}(x)\big|+\big\rangle_Q$$

$$\text{Choose: } \epsilon = -\frac{\pi}{4\alpha}$$

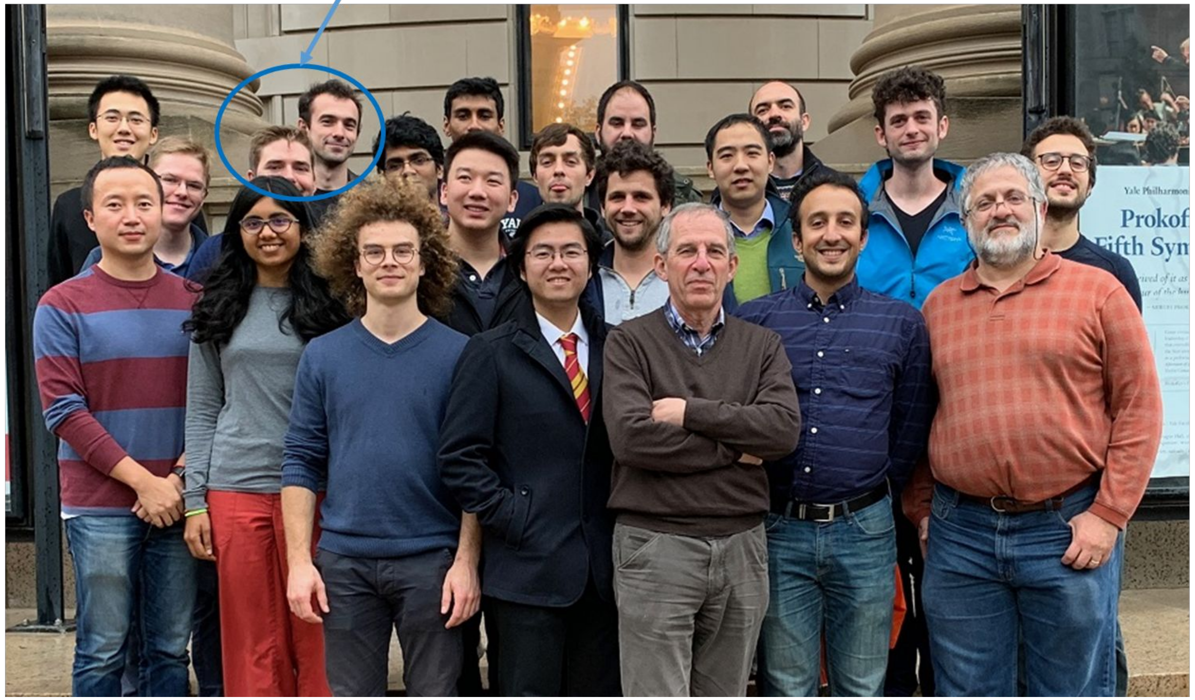
This is a hint towards a potentially powerful generalization of QSP:

Non-Abelian Quantum Signal Processing

Generalization of these Quantum Signal Processing Protocols have been used in Experimental Real-Time Quantum Error Correction Beyond Break Even with the GKP Code

Devoret Lab

Alec Eickbusch & Vlad Sivak



Autonomous QEC protocol:

Baptiste Royer
&
Shraddha Singh

U.S. DEPARTMENT OF ENERGY

QuantumInstitute.yale.edu

LUX ET VERITAS

Primary support:
Army Research Office

OPEN QUESTIONS:

1. Benchmarking:
Gaussian operations are non-compact SU(1,1) not compact SU(n)
Gaussian states do not form a 2-design

2. Formal theory and convergence bounds for non-abelian QSP?
CV operators are unbounded

3. Constructive unitary synthesis with non-abelian QSP?

C^2QA ISA (theory) collaboration

Nathan Wiebe
U. Toronto & PNNL

Tim Stavenger
PNNL

Chris Kang
U. Washington

Eleanor Crane
UCL -> Maryland

Micheline Solely
Yale -> Wisconsin

Kevin Smith
Yale

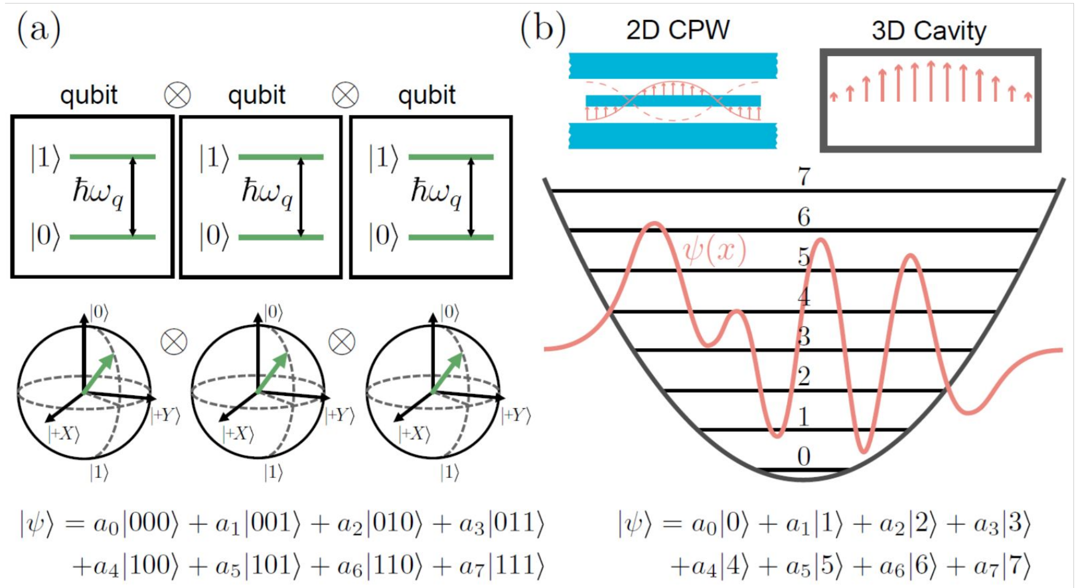
+ Chuang group (MIT)
+ Ali Javadi (IBM)

+ Alec Eickbusch and Devoret Lab

+Shraddha Singh
Baptiste Royer

Discrete variable (transmon qubits)

Continuous variable (microwave or mechanical oscillators)

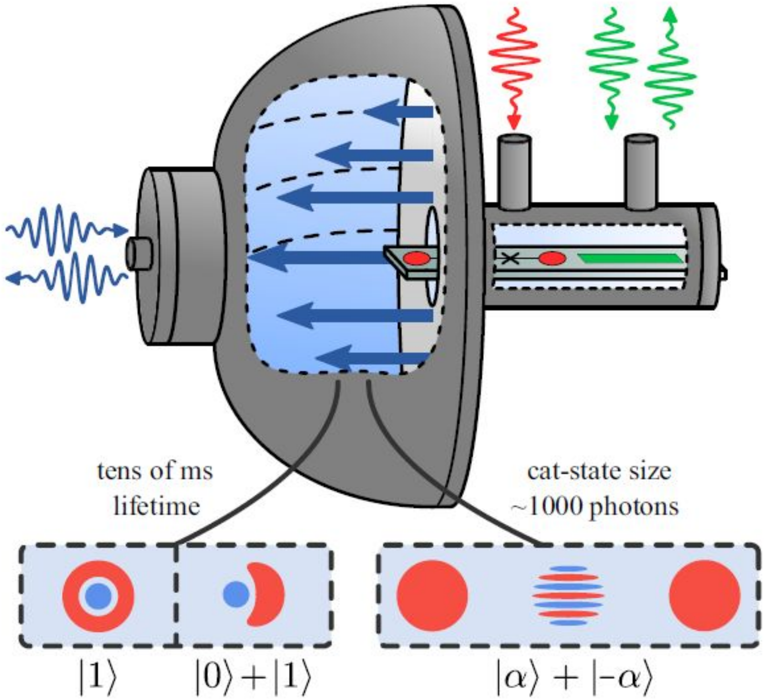


Boson Fock (photon number) states

- Instruction Set Architecture for hybrid qubit/oscillator systems
- Qiskit extension to oscillators
  - Represent $\Lambda = 2^n$ levels of oscillator with a register of $n = \log_2 \Lambda$ qubits
  - Access ISA and Wigner tomography toolkit within Qiskit

Record Schrödinger Cat Size 1024 photons

S. Rosenblum group PRX QUANTUM **4,** 030336 (2023)



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|+\alpha\rangle + |-\alpha\rangle\right]$$

$$|2\alpha|^2 = 1024$$

Strong qubit-oscillator coupling gives amazing control capabilities to 'sculpt' complex quantum states of qubits and oscillators.

Long-lived photon states in high-Q cavity:

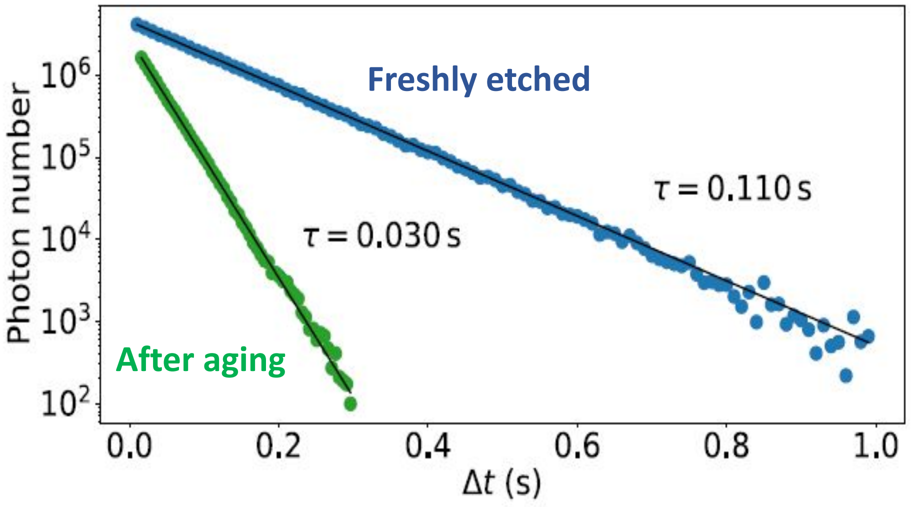
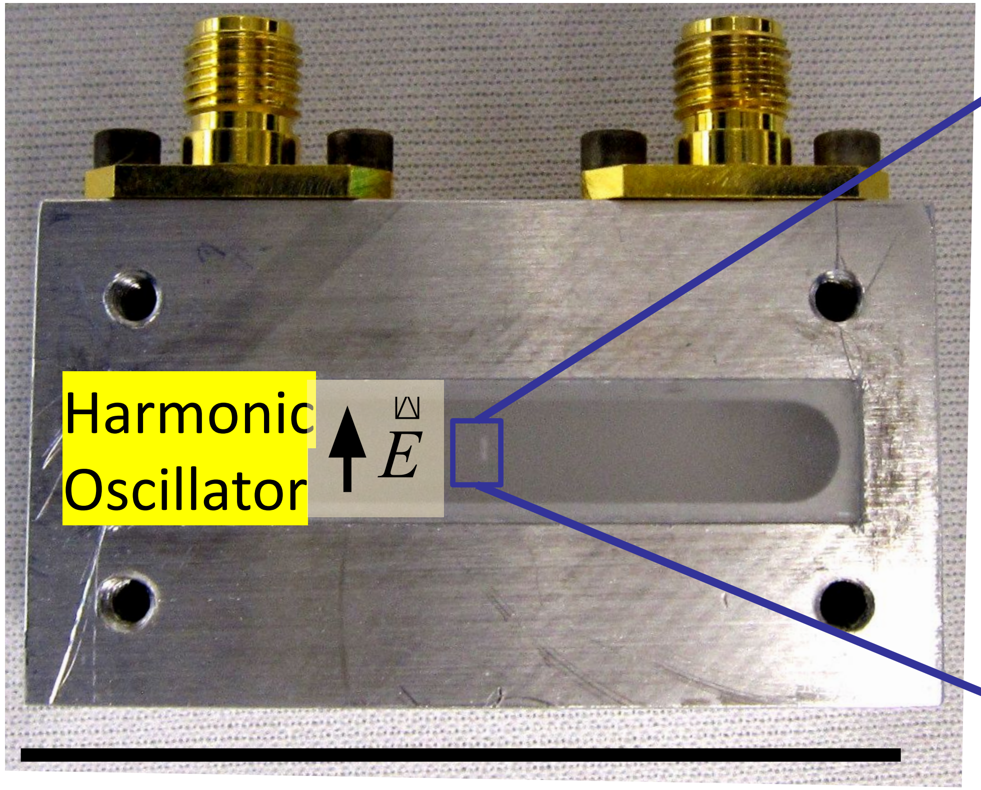
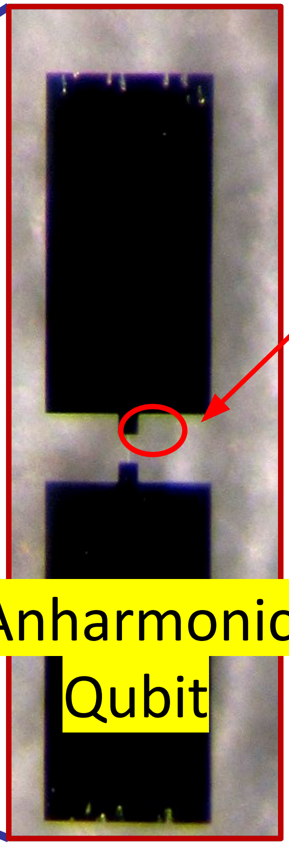


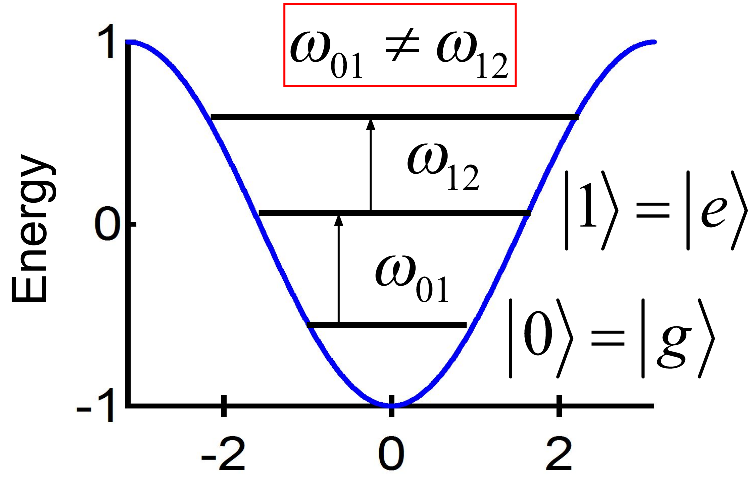
FIG. 2. Classical ring-down measurements of the bare cavity.

Transmon Qubit in 3D Cavity



50 mm





$\omega_{01} \sim 5-10\ \mathbf{GHz}$

$$g = \frac{\vec{d} \bullet \vec{E}_{rms}}{h}$$

$$|\vec{d}| = 2e \times 1\ \text{mm} \approx 10^7 \text{Debye!!}$$

Huge dipole moment: strong coupling

bit flip

$$V_{\text{dipole}} = g\sigma^x (a + a^\dagger)$$

g ~ 100 MHz