Free Dimension

(and the Biane-Voiculescu-Wasserstein distance)

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Classical Case

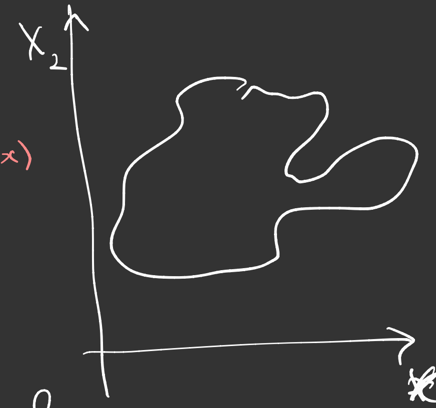
$X_1, \dots, X_d$ $\mathbb{R}$-valued random vars

↑ coordinate functs

$\mu \in P(\mathbb{R}^d)$

μ dim-regular

$\mu(B(x,R)) \sim R^{\alpha(x)}$



H = differential entropy

if $\rho = d\mu / d\lambda$ exists

$H = \int \rho \log \rho \, d\lambda$ infinite if μ singular.

$$\mu_\varepsilon = \mu * \gamma(0, \underbrace{\varepsilon}_{\text{variance}}) \Rightarrow H(\mu_\varepsilon) \text{ finite.}$$

- $H(\mu_\varepsilon) + d \log(\varepsilon^{1/2}) \xrightarrow[\lim \text{ as } \varepsilon \to 0]{}$ discrete entropy of μ if μ discrete.
- $d - \dfrac{H(\mu_\varepsilon)}{\log(\varepsilon^{1/2})}$ ← a fractal dim of μ : $\int \alpha(x) \, d\mu(x)$
[A. Guionnet + D.S. '00)

Another def!

d_W L^2-Wasserstein metric on probab. laws.

$$d_W\left[(X_1, \dots, X_d), (Y_1, \dots, Y_d)\right]^2 = \inf \sum_j \| X_j' - Y_j' \|_{L^2}^2$$

$\mathbb{E}((X_j' - Y_j')^2)$

$(X_1, \dots, X_d) \sim (X_1', \dots, X_d')$
$(Y_1, \dots, Y_d) \sim (Y_1', \dots, Y_d')$ r.vars on some prob. sp.

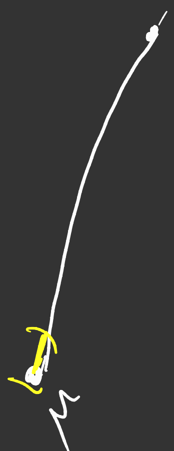
<u>Th</u> (Brenier) if law of $X_1, \dots, X_d$ is "nice enough" (e.g. Leb. a.c.) $\Rightarrow$ optimum is achieved when $Y_j' = f_j(X_1', \dots, X_d')$

$f = (f_1, \dots, f_d)$ is "monotone" $f = \nabla \varphi$, φ convex.

$\mu \in \mathcal{P}(\mathbb{R}^d)$ as before

$X_1, \dots, X_d$ $\boxed{\text{W. Gangbo, A. Palmer, D. Jekel}}$

$B_1, \dots, B_d$ iid $(0,1)$-Gaussian



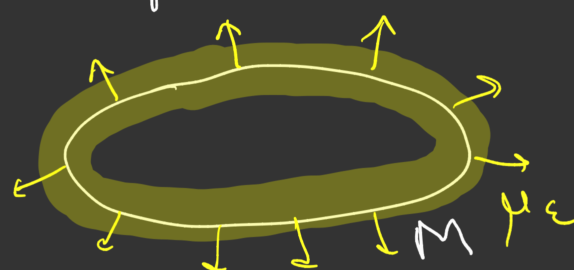
gen: $\mu_\varepsilon = \mu * \gamma(0, \varepsilon)$

$\leq d \cdot \varepsilon$

$$d_W(\mu_\varepsilon, \mu)^2 = d_W\big((X_1 + \sqrt{\varepsilon} B_1, \dots, X_d + \sqrt{\varepsilon} B_d), (X_1, \dots, X_d)\big)^2$$

$$d \sim \frac{d_W(\mu_\varepsilon, \mu)^2}{\varepsilon}$$ [Conj: again a fractal dim].

Eg: $\mu =$ surface meas of some $M \subseteq \mathbb{R}^d$



$\operatorname{dist}_M(\cdot) \leftarrow$ Lipschitz

Obvious plan: project

$\text{cost}^2 \sim \varepsilon$

$$\left(X_1', X_2', \quad Y_1' + \sqrt{\varepsilon} B_1', \quad Y_2' + \sqrt{\varepsilon} B_2' \right)$$

optimal.

$$g = \mathrm{dist}_M(\cdot) \qquad g^2\left(Y_1' + \sqrt{\varepsilon} B_1', \ Y_2' + \sqrt{\varepsilon} B_2' \right) - g\left(X_1', \cancel{X_2'} \right) \sim \varepsilon^2$$

$$\varepsilon^2 = \boxed{\| g(z') - g(z) \|_2^2 \leq \| z - z' \|^2}$$

End of classical part.

Non-Commutative / Free case (Voiculescu).

$X_1, \ldots, X_n \in M \subseteq B(H)$ von Neumann algebra

self-adjoint.

<u>Quant. Info:</u> "density" $Tr(\rho \cdot)$ $\rho \in \mathcal{L}^1_+$

<u>Free Prob:</u> "law" Assume $\tau : M \to \mathbb{C}$ state (typically a trace)

$\tau(xy) = \tau(yx)$ $\forall x, y \in M$

same law

$(X_1, \ldots, X_d) \sim (X_1', \ldots, X_d')$

$\Updownarrow$

$\exists$ map $(W^*(X_1, \ldots, X_d), \tau|_{W^*(X_1, \ldots, X_d)}) \xrightarrow{\alpha} (W^*(X_1', \ldots, X_d'), \tau|_{\ldots})$

α preserves traces & $\alpha(X_j) = X_j'$ $\forall j$. & α $*$-homomph.

Example: Γ - discrete group (eg. $\mathbb{Z}$ or $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}, \ldots$)

$\lambda : \Gamma \curvearrowright \ell^2(\Gamma)$ left translation $\lambda(g) \cdot \delta_h = \delta_{gh}$.

$L(\Gamma) = W^*(\lambda(g) : g \in \Gamma)$ $\tau(x) = \langle x \cdot \delta_1, \delta_1 \rangle$

$\langle e | x | e \rangle$

$\Gamma = \mathbb{Z}$ $L(\Gamma) \overset{\text{Fourier}}{\approx} L^\infty(\mathbb{T})$

τ $\int \cdot \, d\theta$

Free independence: $L(\mathbb{Z} * \mathbb{Z})$

$$\begin{array}{ccc} \cup & & \cup \\ L(\mathbb{Z} * e) & & L(e * \mathbb{Z}) \end{array}$$

Freely independent.

$$\underbrace{X_1, \dots, X_n}_{\downarrow \atop *\text{-alg } A} \text{ freely indep. from } \underbrace{Y_1, \dots, Y_m}_{*\text{-alg } B} \quad \in \quad (M, \tau)$$

$$\tau(\cancel{a_0}\, b_1\, a_1\, b_2 \cdots a_{n-1}\, b_n\, \cancel{a_n}) = 0 \Leftarrow \left[\begin{array}{ll} a_j \in A & \tau(a_j) = 0 \\ b_j \in B & \tau(b_j) = 0 \end{array}\right.$$

" $\tau(ab) = \tau(a)\,\tau(b)$ "

Free CLT (Voiculescu '84):

$X_1, X_2, \dots$ free iid

$\tau(X_j) = 0$

$\tau(X_j^2) = 1$

+ growth cond.

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{\text{law}} \frac{\sqrt{2-t^2}\, dt}{\pi}$$

$-\sqrt{2}$ $\sqrt{2}$

Free entropy theory (Voiculescu) "statistical mech. def via RM"

$\chi(X_1, \dots, X_n)$ — microstates

^ Biane-Capitaine-Guionnet

$\chi^*(X_1, \dots, X_n)$ — non-microstates

free entropy.

free score function

Fisher info.

$\chi(X) = \chi^*(X) = (\text{const}) + \iint \log|s-t|\, d\mu(s)\, d\mu(t)$

Aside: $f(n) = \chi\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right)$ monotone.

$X_1, X_2, \dots$ free iid centered, var $= 1$

$$n \longmapsto \chi\left(\tfrac{1}{\sqrt{n}} \cdot \mu^{\boxplus n}\right)$$

$\mu^{\boxplus t}$ always exists $\forall t \geq 1$

$$f : t \longmapsto \chi\left(\tfrac{1}{\sqrt{t}}\, \mu^{\boxplus t}\right)$$

(D.S., T. Tao).

$\partial_t f \geq 0.$

$A \subseteq W^*(X_1, \ldots, X_d)$ define

- $\chi(X_1 + \sqrt{\varepsilon} S_1, \ldots, X_n + \sqrt{\varepsilon} S_n | A) + d \log \varepsilon^{1/2} \xrightarrow{\text{lim's}} h$ "1-bdd entropy"

$S_1, \ldots, S_n$ free semicircles, free from $X_1, \ldots, X_n$

h W^*-alg. invt (Jung, B. Hayes ...) $h(-)$ $+\infty$? 0?

$h(L(\mathbb{F}_n \times \mathbb{F}_m)) = 0$ $h(L(\mathbb{F}_n)) = +\infty$

- $\delta = d - \dfrac{\chi(X_1 + \sqrt{\varepsilon} S_1, \ldots, X_n + \sqrt{\varepsilon} S_n)}{\log \varepsilon^{1/2}}$ Voiculescu: free dimension.

$\delta^* = d - \dfrac{\chi^* \ldots}{\ldots}$ $\delta \leq \delta^*$

δ^*(group Generators of $\mathbb{C}\Gamma$) $= \beta_1^{(2)}(\Gamma) \boxed{- \beta_0^{(2)} + 1}$

(Mineyev + D.S). $\qquad = \dim_{R(\Gamma)} \{ c : \Gamma \to \ell^2(\Gamma) \quad c(gh) = c(g) + \lambda(g) c(h) \}$.

$= + \dim$ of inner $\quad \xi \in \ell^2(\Gamma)$

$c(g) = \lambda(g)\xi - \xi$

- Γ infinite sofic f. generated f. presented & $\boxed{\beta_1^{(2)}(\Gamma)=0}$

$$\Rightarrow \quad \delta(\mathbb{C}\Gamma) = 1 \quad \& \quad h(L(\Gamma)) = 0.$$

$$\Rightarrow \quad L(\Gamma) \not\cong L(\mathbb{F}_n)$$

$$\beta_1^{(2)}(\Gamma) = 0 \implies h(L(\Gamma)) = 0$$

$$\beta_1^{(2)}(\Gamma) \neq 0 \overset{?}{\implies} h(L(\Gamma)) \neq 0$$

$$d_{BVW}\big((X_1, \dots, X_d), (Y_1, \dots, Y_d)\big)^2 =$$

$$= \inf_{\underbrace{(X_1', \dots, X_d'}_{\sim (X_1, \dots, X_d)}, \underbrace{Y_1', \dots, Y_d'}_{\sim (Y_1, \dots, Y_d)}) \in (M, \tau)} \sum_{i=1}^{d} \tau\big((X_i' - Y_i')^2\big)$$

$$\dim_{BVW}(X_1, \ldots, X_d) = d - \lim \frac{d_{BVW}\big((X_1, \ldots, X_d), (X_1 + \sqrt{\varepsilon} S_1, \ldots, X_d + \sqrt{\varepsilon} S_d)\big)}{\varepsilon}$$

$S_1, \ldots, S_d$ free semicirc.

<u>Results:</u>

- [Conj]: $\dim_{BVW} \geq \delta^*$
- $\dim_{BVW}(\mathbb{C}\Gamma) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$
- May have nicer properties (TBD).
- h? <u>Conj:</u> $\beta_1^{(2)}(\Gamma) = 0$ or $\neq 0$ is remembered by $L(\Gamma)$.

ξ – free score funct.

(classical case: $\nabla \log \rho$ ← density)

$$\xi_{X+\sqrt{\varepsilon} S} = E_{W^*(X+\sqrt{\varepsilon} S)}(S)$$

$$\xi_S = S$$

$$W_2(\tau_{\text{semicular}}, \tau_X) \leq \sqrt{\chi(\tau_X \mid \tau_{\text{semi}})}$$

$$X_1, X_2 \in (M, \tau)$$

$$\tau(X_{i_1} \cdots X_{i_k})$$

$$\inf_{K,\varepsilon} \limsup_{N\to\infty} \frac{1}{N^2} \log \operatorname{Vol}\left(\left\{ (x_1, x_2) \in (M^{sa}_{N\times N})^2 : \forall k \le K,\ \forall i_1, \dots, i_k \ \left| \frac{1}{N} \operatorname{Tr}(x_{i_1} \cdots x_{i_k}) - \tau(X_{i_1} \cdots X_{i_k}) \right| < \varepsilon \right\}\right) + \frac{d}{2} \log N$$

For 1 var:

$$\mu \rightsquigarrow \int t^k \, d\mu(t) = m_k$$

$$x = x^* \quad \text{eigenval } (\lambda_1, \dots, \lambda_n) : \quad \frac{1}{N} \sum \lambda_j^p \approx m_p$$

$$p = 1, \dots, K.$$