On Quantum Symmetry

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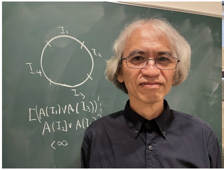
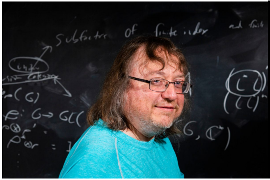
Quantum Symmetry

- symmetries beyond groups, quantum groups — subfactors


Vaughan Jens Böckenhauer Yasu Kawahigashi Mathew Pugh

- K-theory — higher twists




Terry Gannon Andreas Aaserud Ulrich Pennig Corey Jones

Quantum Symmetry

Verlinde ring

- as representation theory of a loop group
- bimodules for subfactors or representation theory of conformal nets
- as twisted equivariant K-theory of equivariant bundles of compact operators

higher twists

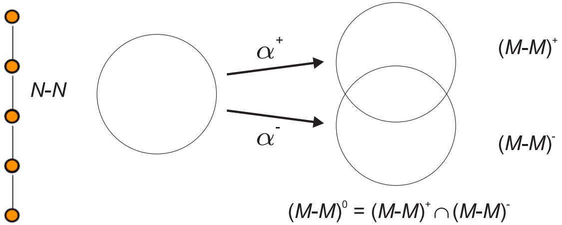
- equivariant bundles of self absorbing C^*-algebras
- their twists and their K-theory

provide

- $\otimes$ categories, fusion modules, CFT conformal field theory
- actions of $\otimes$ categories as modules and bimodules

$N \subset M$ with

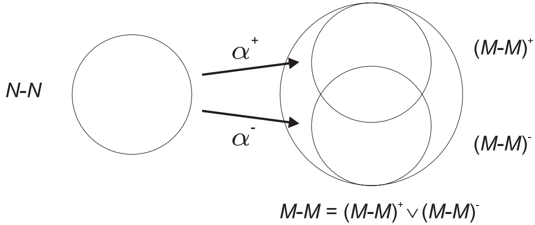
- ${}_N\mathcal{X}_N \subset \mathrm{End}(N)$ braided
- $\theta = \bar{\iota}\iota = {}_N M_N \in \Sigma({}_N\mathcal{X}_N)$ $\quad a.x.b = ax\lambda(b)$ bimods–endos



$$\alpha^{\pm} : \quad \lambda \quad \longrightarrow \quad \alpha^{\pm}_{\lambda}$$

$Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$ is a modular invariant

Bockenhauer-Evans-Kawahigashi, Feng Xu, Ocneanu



- N-M sectors from $\iota\lambda$

$$\iota : N \subset M, \qquad \lambda \in N\text{-}N \text{ system}$$

  N-N sectors act on N-M sectors with multiplicity graph of Cappelli-Itzykson-Zuber CIZ

- M-M sectors from $\iota\lambda\bar{\iota}$ CIZ graph for $ZZ^* = Z_1 + Z_2 + \ldots\ldots$

$Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ is a modular invariant

Bockenhauer-Evans-Kawahigashi, Feng Xu, Ocneanu

- $N \subset M_\pm \subset M$, $N \subset M_\pm$ satisfy chiral locality Böckenhauer-E

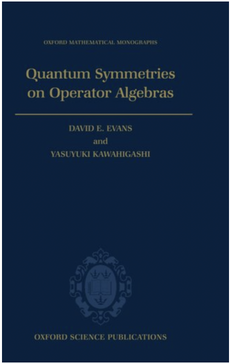
$$Z_{\lambda\mu} = \Sigma_\tau b^+_{\tau\lambda} b^-_{\beta(\tau)\lambda}$$

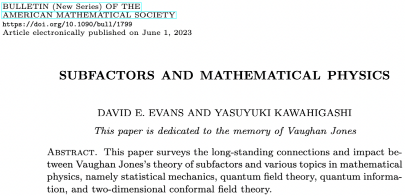
$$N \otimes N^{opp} \subset M_+ \otimes M_-^{opp} \subset B$$

canonical endomorphism of $N \otimes N^{opp} \subset B$ is

$$[\Theta] = \bigoplus_{\lambda,\mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu} \, [\lambda \otimes \mu^{opp}] \, .$$

Quantum Symmetries – 1998 and 2023

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SUBFACTORS AND MATHEMATICAL PHYSICS

DAVID E. EVANS AND YASUYUKI KAWAHIGASHI

This paper is dedicated to the memory of Vaughan Jones

Abstract. This paper surveys the long-standing connections and impact between Vaughan Jones's theory of subfactors and various topics in mathematical physics, namely statistical mechanics, quantum field theory, quantum information, and two-dimensional conformal field theory.

quantum symmetries and K-theory

Verlinde ring for $G = SU(n)$ as

$$\text{Rep } (LG)_{level} = {}^{\text{twist}}K^G(G)$$ Freed-Hopkins-Teleman 00

G a finite group

double of $R \subset R \rtimes G$: $A \subset B = (R \otimes R) \rtimes G \subset (R \otimes R) \rtimes (G \times G)$

categorical equivalence : Kosaki-Yamagami 92
bimodules $\simeq$ equivariant vector bundles
${}_A B_B \cong_G \mathbb{C}_{G \times G}$ vector bundle on G

A-A bimodules $\simeq$ G-G equivariant bundles on $G \times G$

$$\begin{array}{ccccc} & & G \times G & & \\ & \nearrow & & \nwarrow & \\ G & & & & G \end{array}$$

$G \curvearrowright G$ by conjugation

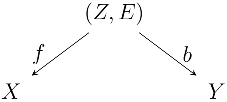
$\mathcal{D}(G)$ is q double:

Verlinde ring $K_G(G) \simeq Rep(\mathcal{D}(G))$, G on G by conjugation

modular invariants, K and KK 

- $\lambda \to \alpha_\lambda^\pm \,, \quad \alpha_\lambda^\pm|_N = \lambda \,,$
- $Z_{\lambda,\mu} = \langle \alpha_\lambda^+ \,, \alpha_\mu^- \rangle$ is mod invt
- $N \subset M_\pm \subset M \,, \quad N \subset M_\pm$ satisfy chiral locality

$$Z_{\lambda\mu} = \Sigma_\tau b^+_{\tau\lambda} b^-_{\beta(\tau)\lambda}$$



$$b_!(f^*(-) \otimes E) \in Hom(K(X), K(Y)) \simeq KK(X, Y)$$

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Modular $\otimes$ Category as modules  or bimodules 

- For a finitely generated braided C*-$\otimes$ category $\mathcal{C}$ constructed a C*-algebra $\mathcal{B}$ such that $K_0(\mathcal{B}) \cong K_0(\mathcal{C})$ and as braided C*-$\otimes$ categories

$$\mathcal{C} \cong \mathrm{Mod}_{\mathcal{B}}{}^{f}$$

  for right Hilbert modules which admit a finite orthonormal basis

  Aaserud–E 19

  e.g. Temperley-Lieb-Jones $\otimes$-category $\mathcal{TLJ}_k \cong \mathrm{Mod}_{\mathcal{B}k}{}^{f}$

- $\otimes$-category $\mathcal{C}$ on C*-algebra A as **bimodules** with $K_0(A) \simeq \mathbb{Z}[dim\,\mathcal{C}]$??
  e.g. Haagerup $\theta^2 = 3\theta + 1$ with $\mathbb{Z}[dim\,\mathcal{C}] = \mathbb{Z} + \mathbb{Z}\theta$
  - Action of Haagerup system $\{1, \alpha, \alpha^2, \rho, \rho\alpha, \rho\alpha^2 : \alpha^3 = 1, \alpha\rho = \rho\alpha^2, \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^2\}$ on irrational rotation algebra A_θ
  - Categorify $A_\theta \otimes A_\theta \longrightarrow A_\theta$ E - Corey Jones in progress

higher twists

$\mathcal{A} \otimes \mathcal{K}$-bundles, $\mathcal{A} \otimes A \simeq A$

e.g. $\mathcal{A} = M_{\mathbb{Q}} = \otimes_{n \in \mathbb{N}} M_n$

$Bun_X(\mathcal{K} \otimes M_{\mathbb{Q}}) \simeq H^1(X, R_+^\times) \oplus (\bigoplus_{k \geq 1} H^{2k+1}(X, R))$

$R = K_0(M_{\mathbb{Q}}) \simeq \mathbb{Q}$

Dadarlat-Pennig 13

higher equivariant twists 

$SU(n)$-equivariant UHF $\otimes\mathcal{K}$-bundles

E-Pennig 19, 23

$\mathrm{UHF} = M_F = \otimes^\infty \mathrm{End}(F(\mathbb{C}^n))$

$F : \mathrm{Vect} \to \mathrm{Vect}\,;\, F(V \oplus W) = F(V) \otimes F(W)$, e.g.

$$\Lambda^{top}(V) = \bigwedge^{\dim V} V = Det(V)$$

$$\Lambda^*(V) = \bigoplus_{k=0}^{\infty} \bigwedge^k V$$

$$\mathcal{G} = \{(g, z_1, z_2) \in G \times Z \times Z \mid z_1\,, z_2 \notin EVg\}\,, \quad Z = \mathbb{T} \setminus \{1\}$$

$$\mathcal{G} \longrightarrow G$$

equivariant $SU(n)$ Fell bundles

$$M_F = \bigotimes_{i=1}^{\infty} \mathrm{End}\left(F(\mathbb{C}^n)\right)$$

$$E(g, z_1, z_2) = \bigoplus_{\substack{z_1 < \lambda < z_2 \\ \lambda \in \mathrm{EV}(g)}} \mathrm{Eig}(g, \lambda)$$

$$\mathcal{E}_{(g, z_1, z_2)} = F(E(g, z_1, z_2)) \otimes M_F$$

$$\mathbb{C}^n = \mathrm{Eig}(g, \lambda) \oplus E(g, \lambda)^{\perp}$$

$$F(\mathbb{C}^n) \cong F(\mathrm{Eig}(g, \lambda)) \otimes F(E(g, \lambda)^{\perp})$$

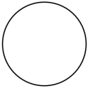
$$M_F \cong End(F(E(g, z_1, z_2))) \otimes M_F$$

Fell bundle $\mathcal{E} \to \mathcal{G}$

$C^*(\mathcal{E}) \otimes \mathcal{K} \cong$ section alg. of locally trivial bundle, fibre $M_F \otimes \mathcal{K}$

Mayer-Vietoris and spectral sequences

$\mathbb{T}$ =



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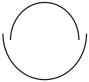
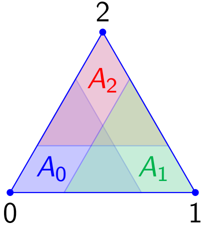
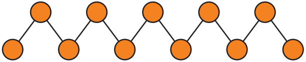
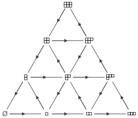








$$R_F(SU(2)) = K_0^{SU(2)}(M_F) \cong \mathbb{Z}[\rho][F(\rho)^{-1}]$$

classical equivariant $SU(2)$ twists

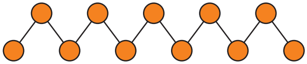
classical case at level $k \in \mathbb{N}$ is $F = \left(\Lambda^{\mathrm{top}}\right)^{\otimes k}$, with $M_F \cong \mathbb{C}$

$$R_F(SU(2)) \cong R(SU(2))$$

$$\rho_{k-1} = (t^k - t^{-k})/(t - t^{-1}) = \det \begin{pmatrix} t^k & t^{-k} \\ 1 & 1 \end{pmatrix} /(t - t^{-1})$$

$$K_1^G(C^*(\mathcal{E})) = R(SU(2))/(\rho_{k-1})$$

Freed-Hopkins-Teleman 00



higher equivariant $SU(2)$ twists

$$K_0^{SU(2)}(C^*(\mathcal{E})) = 0 \ , \quad R_F(SU(2)) = K_0^{SU(2)}(M_F) \cong \mathbb{Z}[\rho][F(\rho)^{-1}]$$

$$K_1^{SU(2)}(C^*(\mathcal{E})) = R_F(SU(2))/(\sigma^F)$$

E-Pennig 19

For $F(V) = \Lambda^*(V)^{\otimes k}$, $F(\rho) = (\rho + 2)^k$, $\sigma^F = \sum_{\ell=1}^{k} \binom{k}{\ell} \rho_{\ell-1}$

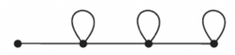
$$k = 3 : \quad 3 + 3\rho_1 + \rho_2 = \rho^2 + 3\rho + 2 = (\rho + 2)(\rho + 1)$$

$$k = 5 : \quad 5 + 10\rho_1 + 10\rho_2 + 5\rho_3 + \rho_4 = (\rho + 2)^2(\rho^2 + \rho - 1)$$

$$k = 7 : \quad 7 + 21\rho_1 + 35\rho_2 + 35\rho_3 + 21\rho_4 + 7\rho_5 + \rho_6$$
$$= (\rho + 2)^3(-1 - 2\rho + \rho^2 + \rho^3)$$

$k = 5$

$k = 7$



$k = 9$

higher equivariant $SU(n)$ twists

$$K^{SU(n)}_{n-1}(C^*(\mathcal{E}))\otimes\mathbb{Q}\cong K^{SU(n)}_*(C^*\mathcal{E}\otimes M_{\mathbb{Q}})$$

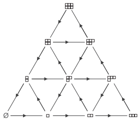
$$\cong (R_F(SU(n))\otimes\mathbb{Q})/<\sigma^F_1,\sigma^F_2\ldots,\sigma^F_{n-1}>$$

$$R_F(SU(n))=K^{SU(n)}_0(M_F)\cong R(SU(n))[F(\rho)^{-1}]$$

$$\sigma^F_i(t_1,\ldots,t_n)=\frac{1}{\Delta}\det\begin{pmatrix} F(t_1)t_1^i & F(t_2)t_2^i & \ldots & F(t_n)t_n^i \\ t_1^{n-2} & t_2^{n-2} & \ldots & t_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \ldots & t_n \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

Vandermonde $\Delta=\Pi_{i<j}(t_i-t_j)$

E-Pennig 23



potentials and Grassmannians

$$K^{SU(n)}_{n-1}(C^*(\mathcal{E})) \otimes \mathbb{Q} \cong K^{SU(n)}_*(C^*\mathcal{E} \otimes M_{\mathbb{Q}})$$
$$\cong (R_F(SU(n)) \otimes \mathbb{Q})/ < \sigma_1^F, \sigma_2^F \ldots, \sigma_{n-1}^F >$$

$$K_0^{SU(n)}(M_F) \otimes \mathbb{Q} \cong \mathbb{Q}[\bar{c}_1, \ldots, \bar{c}_{n-1}, F(\bar{c}_1)^{-1}]$$

$\bar{c}_k(t_1, \ldots, t_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} t_{i_1} t_{i_2} \cdots t_{i_k}$ antisymmetric

Ideal given by potential $< \partial V >$ where

$$\sigma_j^F = (-1)^{n-j} \frac{\partial V}{\partial \bar{c}_{n-(j+1)}}$$

E-Pennig 23

Gepner compared with cohomology ring $H^*(G_k(\mathbb{C}^{n+k}), \mathbb{Z})$ of Grassmann manifolds $G_k(\mathbb{C}^{n+k})$ of k-dimensional subspaces in $\mathbb{C}^{n+k}$

$$G_k(\mathbb{C}^{n+k}) \cong U(n+k)/U(n) \times U(k)$$

$$D = \mathrm{End}\,(V)^{\otimes\infty}\ , \qquad \mathbb{K} = \mathbb{K}(\ell^2(\mathbb{Z}) \otimes H_0)$$

V is a $\mathbb{T}$-representation,

character $p_V = \sum_{i=0}^{d} a_i t^i \in \mathbb{Z}[t] \subset \mathbb{Z}[t, t^{-1}] = R(\mathbb{T})$

$\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})$ is an ∞ loop space with cohomology $E^*_{D,\mathbb{T}}(X)$

E-Pennig 22

$$Bun^{\mathbb{T}}_X(D \otimes \mathbb{K}) \cong E^1_{D,\mathbb{T}}(X) \cong [X, B\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})]$$

$$E_2^{p,q} = H^p(X, \check{E}^q_{D,\mathbb{T}}) \quad \Rightarrow \quad E^{p+q}_{D,\mathbb{T}}(X)\ .$$

Coefficients are:

$$\check{E}^k_{D,\mathbb{T}} \cong \begin{cases} 0 & \text{if } k > 0\ , \\ GL_1(K_0^{\mathbb{T}}(D)_+) & \text{if } k = 0\ , \\ \pi_{-k-1}(U(D^{\mathbb{T}})) & \text{if } k < 0\ . \end{cases}$$

spectral sequence coefficients from homotopy groups

$$K_0^{\mathbb{T}}(D) \cong \mathbb{Z}[t, t^{-1}, p_V(t)^{-1}]$$
$$\pi_{2k}(\mathrm{Aut}_{\mathbb{T}}(D \otimes \mathbb{K})) \cong \pi_{2k-1}(U(D^{\mathbb{T}}))$$
$$\pi_1(U(D^{\mathbb{T}})) \cong K_0(D^{\mathbb{T}}) \cong R_{\mathrm{bdd}}$$

$$\begin{aligned} R_{\mathrm{bdd}} = & \mathrm{K}_0(D^{\mathbb{T}}) = \left\{x \in K_0^{\mathbb{T}}(D) \mid -m[1_D] \leq x \leq m[1_D] \text{ for some } m \in \mathbb{N}\right\} \\ & \subset \mathbb{Z}[t, p_V(t)^{-1}] \quad , \end{aligned}$$
$$R^0_{\mathrm{bdd}} = \{r \in R_{\mathrm{bdd}} \mid r(0) = 0\} \ ,$$
$$R^{\infty}_{\mathrm{bdd}} = \left\{\frac{q}{p_V^k} \in R_{\mathrm{bdd}} \mid q \in \mathbb{Z}[t],\ k \geq 0,\ \deg(q) < kd\right\}$$

- If $a_0 > 1$ and $a_d > 1$, then $\pi_{2k-1}(U(D^{\mathbb{T}})) \cong R_{\mathrm{bdd}}$.
- If $a_0 = 1$ and $a_d > 1$, then $\pi_{2k-1}(U(D^{\mathbb{T}})) \cong R^0_{\mathrm{bdd}}$.
- If $a_0 > 1$ and $a_d = 1$, then $\pi_{2k-1}(U(D^{\mathbb{T}})) \cong R^{\infty}_{\mathrm{bdd}}$.
- If $a_0 = 1$ and $a_d = 1$, then $\pi_{2k-1}(U(D^{\mathbb{T}})) \cong R^{\infty}_{\mathrm{bdd}} \cap R^0_{\mathrm{bdd}}$

equivariant bundles of $D \otimes \mathbb{K}$ on torii

$$
\begin{aligned}
Bun_{\mathbb{T}^n}^{\mathbb{T}}(D \otimes \mathbb{K}) &\cong E^1_{D,\mathbb{T}}(\mathbb{T}^n) \\
&\cong H^1(\mathbb{T}^n, G) \;\oplus\; H^3(\mathbb{T}^n, R_{\mathrm{bdd}}) \;\oplus\; \bigoplus_{k=2}^{\infty} H^{2k+1}(\mathbb{T}^n, R^{0,\infty}_{\mathrm{bdd}}) \\
&\cong \mathbb{Z}^n \oplus \bigoplus_{\substack{q|p_V \\ q \text{ prime}}} \mathbb{Z}^n \;\oplus\; \bigwedge^3 (R_{\mathrm{bdd}})^n \;\oplus\; \bigoplus_{k=2}^{\infty} \bigwedge^{2k+1} (R^{0,\infty}_{\mathrm{bdd}})^n \;,
\end{aligned}
$$

E-Pennig 22

$$
R^{0,\infty}_{\mathrm{bdd}} = \begin{cases} R_{\mathrm{bdd}} & \text{if } a_0 > 1 \text{ and } a_d > 1 \;, \\ R^0_{\mathrm{bdd}} & \text{if } a_0 = 1 \text{ and } a_d > 1 \;, \\ R^\infty_{\mathrm{bdd}} & \text{if } a_0 > 1 \text{ and } a_d = 1 \;, \\ R^\infty_{\mathrm{bdd}} \cap R^0_{\mathrm{bdd}} & \text{if } a_0 = 1 \text{ and } a_d = 1 \;. \end{cases}
$$

equivariant circle bundles of $\mathbb{C} \otimes \mathbb{K}$

$$\begin{aligned}
Bun_X^{\mathbb{T}}(\mathbb{C} \otimes \mathbb{K}) &\cong E^1_{\mathbb{C},\mathbb{T}}(X) \\
&\cong H^1(X,\mathbb{Z}) \oplus H^3(X,\mathbb{Z}) \\
&\cong H^0(B\mathbb{T},\mathbb{Z}) \otimes H^3(X,\mathbb{Z}) \oplus H^2(B\mathbb{T},\mathbb{Z}) \otimes H^1(X,\mathbb{Z}) \\
&\cong H^3(B\mathbb{T} \times X,\mathbb{Z}) \\
&\cong H^3((E\mathbb{T} \times X)/\mathbb{T},\mathbb{Z}) \\
&\cong H^3_{\mathbb{T}}(X,\mathbb{Z})
\end{aligned}$$

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