What's Done Cannot Be Undone:
Non-Invertible Symmetries

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Non-invertible symmetry

- Conventional global symmetries are described by group theory. Wigner's theorem in quantum mechanics states that symmetries are generated by (anti-)unitary operators.
- In recent years, there has been a lot of developments in a new kind of symmetry, the non-invertible symmetry, with applications in quantum field theory, particle physics, condensed matter theory, quantum gravity, and mathematical physics.
- It is implemented by conserved operators that do **NOT** have inverses.
- It leads to new conservation laws, selection rules, constraints on RG and confinement, and more.
- New structural developments in quantum field theory.

Non-invertible symmetry in QFT

- Kramers-Wannier symmetry of the 1+1d critical Ising CFT [Grimm-Schultz ‘93,…, Chang-Lin-**SHS**-Wang-Yin ‘18,…].

$$\mathcal{D}^2 = 1 + \eta, \qquad \eta\mathcal{D} = \mathcal{D}\eta = \mathcal{D}, \qquad \eta^2 = 1$$

- Many examples and application in 1+1d conformal field theory.
- Rapid developments in higher spacetime dimensions since 2021 [Choi-Cordova-Hsin-Lam-**SHS** ‘21, Kaidi-Ohmori-Zheng ‘21, Roumpedakis-Seifnashri-**SHS** ‘22, …].
- 3+1d U(1) gauge theory, QED, QCD, axions, lattice gauge theory, string/M-theory…

See TASI lectures
[**SHS**, 2308.00747] for a review

Non-invertible symmetry in QED

Chiral symmetry in QED

- Consider QED with a massless, unit charge Dirac fermion and $U(1)$ gauge group.

$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}\left(\partial_\mu - iA_\mu\right)\gamma^\mu \Psi$$

- The classical $U(1)_A$ chiral symmetry acts as

$$\Psi \to \exp\left(\frac{i\alpha}{2}\gamma_5\right)\Psi \quad , \qquad \alpha \sim \alpha + 2\pi$$

- The Adler-Bell-Jackiw anomaly implies that the classical $U(1)_A$ chiral symmetry **fails** to be a global symmetry quantum mechanically.

ABJ anomaly

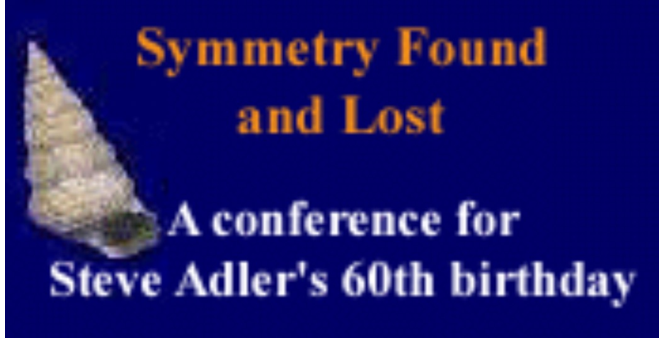
- The ABJ anomaly was discovered in the late 60s to explain the neutral pion decay, $\pi^0 \to \gamma\gamma$.

- It successfully determined the coupling

$$\frac{i}{8\pi^2 f_\pi}\pi^0 F \wedge F = \frac{i}{32\pi^2 f_\pi}\pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

in the pion Lagrangian.

ABJ anomaly?



- Conceptually, there is something *slightly* counterintuitive though.
- Usually, we celebrate when we discover the **existence** of a global symmetry.
- ABJ anomaly states that there is **not** a global symmetry that one would have naively expected.
- So how come we can derive all these quantitative results from the **absence** of a global symmetry?
- Wouldn't it be nice if we can reinterpret these classic results from the existence of a generalized global symmetry (rather than the absence thereof)?

Is "chiral symmetry" a symmetry in massless QED?

- **No**. Period.
- But helicity is conserved in electron-positron scattering...

massless. (The calculation can be done for lower energy, but it is much more difficult and no more instructive.)†

Our starting point for both methods of calculating the polarized cross section is the amplitude

$$i\mathcal{M}\big(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\big) = \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \quad (5.1)$$

We would like to use the spin sum identities to write the squared amplitude in terms of traces as before, even though we now want to consider only one set of polarizations at a time. To do this, we note that for massless fermions, the matrices

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.17)$$

are *projection operators* onto right- and left-handed spinors, respectively. Thus if in (5.1) we make the replacement

$$\bar{v}(p')\gamma^\mu u(p) \longrightarrow \bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p),$$

the amplitude for a right-handed electron is unchanged while that for a left-handed electron becomes zero. Note that since

$$\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p) = v^\dagger(p')\Big(\frac{1+\gamma^5}{2}\Big)\gamma^0\gamma^\mu u(p), \quad (5.18)$$

this same replacement imposes the requirement that $v(p')$ also be a right-handed spinor. Recall from Section 3.5, however, that the right-handed spinor $v(p')$ corresponds to a *left*-handed positron. Thus we see that the annihilation amplitude vanishes when both the electron and the positron are right-handed. In general, the amplitude vanishes (in the massless limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.

Chapter 5 of Peskin&Schroeder

- **Yes**, it is a symmetry in flat spacetime because there is no nontrivial abelian instanton configuration ($\pi_3(U(1)) = 0$).
- But helicity conservation is violated in monopole-electron scattering. Callan-Rubakov effect.
- Fine, it's a symmetry in flat spacetime and if there is no monopole.

Answer: **Yes and No**.
Something is conserved but there isn't an ordinary global symmetry.
Is there a straight answer to this question?

Non-invertible global symmetry

- I will argue that the invertible $U(1)_A$ chiral symmetry of the classical Lagrangian is not entirely broken by the ABJ anomaly.

- Rather, it becomes a non-invertible global symmetry labeled by the rational numbers.

- In the pion Lagrangian, the coupling $\pi^0 F \wedge F$ can be rederived by matching the non-invertible **global** symmetry in the UV QCD.

Noether current

- Consider a conserved Noether current

$$\mu = t, x, y, z$$
$$i = x, y, z$$

$$\partial^\mu j_\mu = -\partial_t j_t + \partial_i j_i = 0$$

- The $U(1)$ unitary operator is

$$U_\alpha = \exp(i\alpha \int d^3x \, j_t)$$

It is conserved,i.e., $\partial_t U_\alpha = 0$, because $\partial_t \int d^3x \, j_t = \int d^3x \partial_i j_i = 0$.

- Written covariantly, the current conservation equation reads $d \star j = 0$, and the conserved operator on a general 3-dimensional space M is:

$$U_\alpha(M) = \exp(i\alpha \oint_M \star j) = \exp(i\alpha \oint_M dn^\mu j_\mu)$$

QED

- The axial current $j_\mu^A = \frac{1}{2}\bar{\Psi}\gamma_5\gamma_\mu\Psi$ obeys the anomalous conservation equation

$$d \star j^A = \frac{1}{8\pi^2} F \wedge F$$

$$\partial^\mu j_\mu^A = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

The field strength is normalized such that the magnetic flux is $\oint F \in 2\pi\mathbb{Z}$.

- Naively, we can define the symmetry operator

$$U_\alpha(M) = \exp(i\alpha \oint_M \star j^A)$$

- However, it is **not** conserved.

$$d \star j^A = \frac{1}{8\pi^2} F \wedge F$$

$$\Rightarrow d\left(\star j^A - \frac{1}{8\pi^2} A \wedge dA\right) = 0$$

QED

- [Adler '69] discussed an operator that is formally conserved, but is **not** gauge invariant for general space M:

$$\text{“ } \hat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{8\pi^2} AdA)] \text{ ”}$$

$$\text{“ } \hat{U}_\alpha(M) = \exp[i\alpha \oint_M dn^\mu (j^A_\mu - \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma)] \text{ ”}$$

Fact: The Chern-Simons action $\exp[i \oint_M (\frac{N}{4\pi} AdA)]$ is gauge invariant iff N is an integer.

Notation: every operator in quotation marks is not gauge-invariant and is awaiting to be fixed.

Rational angles

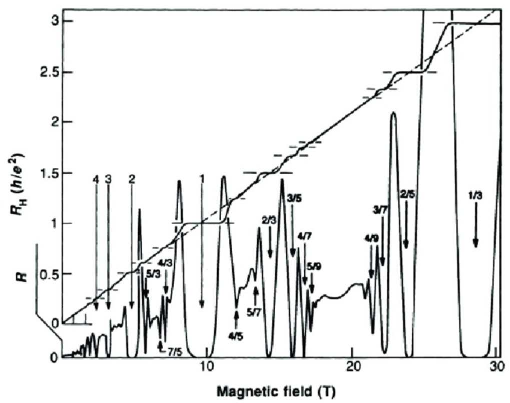
- Let us be less ambitious, and assume the chiral rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\text{“ } \widehat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

- The operator $\widehat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state



$$\text{“} -\frac{i}{4\pi N}\oint_M AdA \text{”}$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix.
- The more precise, gauge invariant Lagrangian for the FQHE is

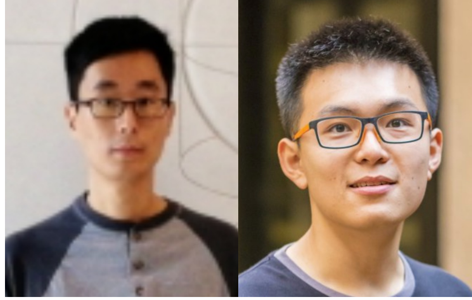
$$\oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

  where a_μ is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.
- The two actions are related by illegally integrating out a to obtain

$$\text{“}a = -\frac{A}{N}\text{”}$$

Back to QED

[Choi-Lam-**SHS** '22, Cordova-Ohmori '22]



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- Motivated by the discussion of FQHE in 2+1d, we define a new operator in 3+1d QED:

$$\text{“ } \widehat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{N} \star j^A - \frac{i}{4\pi N} AdA)] \text{ ”}$$

$$\downarrow$$

$$\mathcal{D}_{1/N}(M) \equiv \int [Da]_M \exp\left[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)\right]$$

a: auxiliary field on M
A: gauge field for photon

- The new operator is gauge-invariant *and* conserved.

*The FQH state **“cures”** the ABJ anomaly.*

- It is easy to generalize to an arbitrary rational chiral rotation $\alpha = 2\pi p/N$, leading to $\mathcal{D}_{\frac{p}{N}}$ labeled by $\frac{p}{N} \in \mathbb{Q}/\mathbb{Z}$.

Non-invertible chiral symmetry

[Choi-Lam-**SHS** '22]

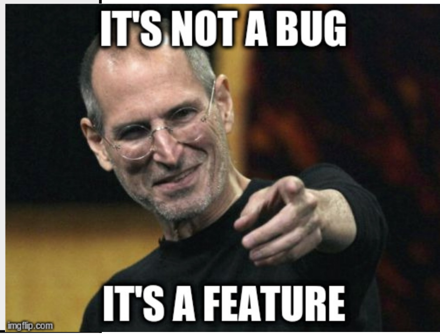
- The price we pay is that it **NOT** unitary:

$$\mathcal{D}_{1/N} \times \mathcal{D}_{1/N}^{\dagger} \equiv \int [Da]_M \int [D\bar{a}]_M \exp[\oint_M (\frac{iN}{4\pi} ada - \frac{iN}{4\pi} \bar{a} d\bar{a} + \frac{i}{2\pi}(a - \bar{a})dA)] \neq 1$$

- Furthermore, the operator has a zero eigenvalue in the presence of magnetic flux. It is non-invertible.

Non-invertible chiral symmetry

Operator	Gauge-invariant?	Conserved?	Invertible?
$U_\alpha(M) = \exp(i\alpha \oint_M \star j^A)$	✓	✗	N/A
“ $\hat{U}_\alpha(M) = \exp[i\alpha \oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$ ”	✗	✓	✓
$\mathcal{D}_{\frac{1}{N}}(M) = \int [Da]_M$ $\exp[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$	✓	✓	✗



Non-invertible chiral symmetry in QED

- In Euclidean signature, $\mathcal{D}_{p/N}$ acts invertibly on the fermions:

$$\mathcal{D}_{p/N}\colon\ \ \Psi \mapsto \exp\left(\frac{2\pi i p}{2N}\gamma_5\right)\Psi$$

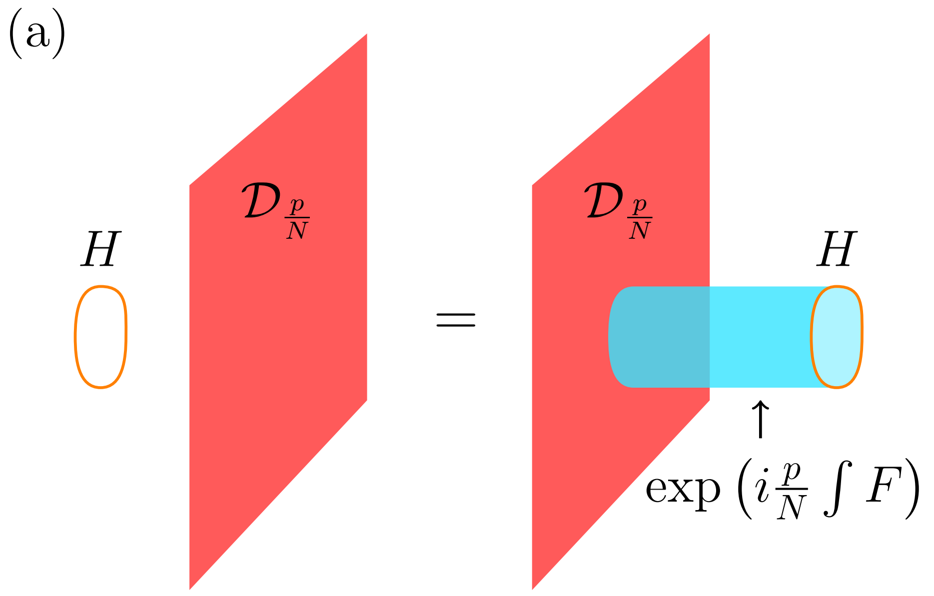
leading to selection rules such as the helicity conservation.

- It acts on the infinitely heavy monopole worldline, ‘t Hooft lines H, by the Witten effect:

$$\mathcal{D}_{p/N}\colon\ \ H \mapsto H\exp\left(\frac{ip}{N}\int F\right)$$

monopole dyon

- This explains why helicity is not conserved in the Callan-Rubakov process. It’s an unconventional symmetry that acts both on local operators and line defects.



‘t Hooft Naturalness

- Naturalness [‘t Hooft 1980]: Impose a global symmetry G. The Lagrangian should include all G-invariant terms with coefficients of order one with no fine-tuning.
- QED Lagrangian: $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu \Psi$
- The electron mass term $m\bar{\Psi}\Psi$ violates the non-invertible global symmetry.
- Therefore, electron is naturally massless in QED because of the non-invertible global symmetry.

III2. NATURALNESS IN QUANTUM ELECTRODYNAMICS

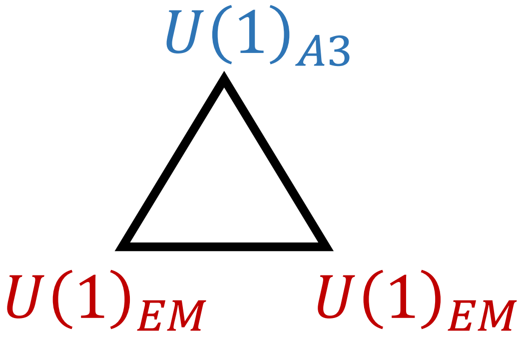
Quantum Electrodynamics as a renormalizable model of electrons (and muons if desired) and photons is an example of a "natural" field theory. The parameters α, m_e (and m_μ) may be small independently. In particular m_e (and m_μ) are very small at large μ. The relevant symmetry here is chiral symmetry, for the electron and the muon separately. We need not be concerned about the Adler-Bell-Jackiw anomaly here because the photon field being Abelian cannot acquire non-trivial topological winding numbers[4)].

‘t Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking* (1980)

Why do pions decay?

Non-invertible symmetry in QCD

[Choi-Lam-**SHS** '22]



- Below the electroweak scale, the massless QCD Lagrangian for the up and down quarks has a classical chiral symmetry (corresponding to π^0)

$$U(1)_{A3}: \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\alpha\gamma_5\sigma_3) \begin{pmatrix} u \\ d \end{pmatrix}$$

- ABJ anomaly with the electromagnetic $U(1)_{EM}$ gauge symmetry.
- By the exact same construction, we conclude that there is a non-invertible global symmetry $\mathcal{D}_{p/N}$ in QCD from $U(1)_{A3}$.
- How does the IR pion Lagrangian capture this non-invertible global symmetry?

Pion

[Choi-Lam-**SHS** '22]

$$\mathcal{L}_{IR} = \frac{1}{2}(\partial_\mu \pi^0)^2 + ig\, \pi^0 F \wedge F$$

$$\mathcal{D}_{1/N}(M) = \int [Da]_M \exp[\oint_{x=0} (\frac{2\pi i}{N} \star j^{A3} + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- Inserting $\mathcal{D}_{1/N}$ at $x = 0$ as a defect, the equations of motion are
  - π^0 EOM: $\pi^0|_{x=0^+} - \pi^0|_{x=0^-} = -\frac{2\pi}{N} f_\pi$
  - a EOM: $Nda + F = 0$
  - A EOM: $2ig(\pi^0|_{x=0^+} - \pi^0|_{x=0^-})F = \frac{i}{2\pi} da$
- Combining the above, it fixes

$$g = \frac{1}{8\pi^2 f_\pi}$$

Pion decay

[Choi-Lam-**SHS** '22]

- Conventionally, the pion decay $\pi^0 \to \gamma\gamma$ is explained by the ABJ anomaly.

- We have provided an alternative explanation for the pion decay as a direct consequence from matching the non-invertible global symmetry in the UV QCD.

- The non-invertible global symmetry gives an invariant characterization of the ABJ anomaly in terms of the **existence** of a generalized global symmetry, rather than the **absence** thereof.

What is it good for?

Non-invertible symmetries of axions

[Choi-Lam-**SHS** ’22, Cordova-Ohmori ‘22, Choi-Forslund-Lam-**SHS** ‘23]

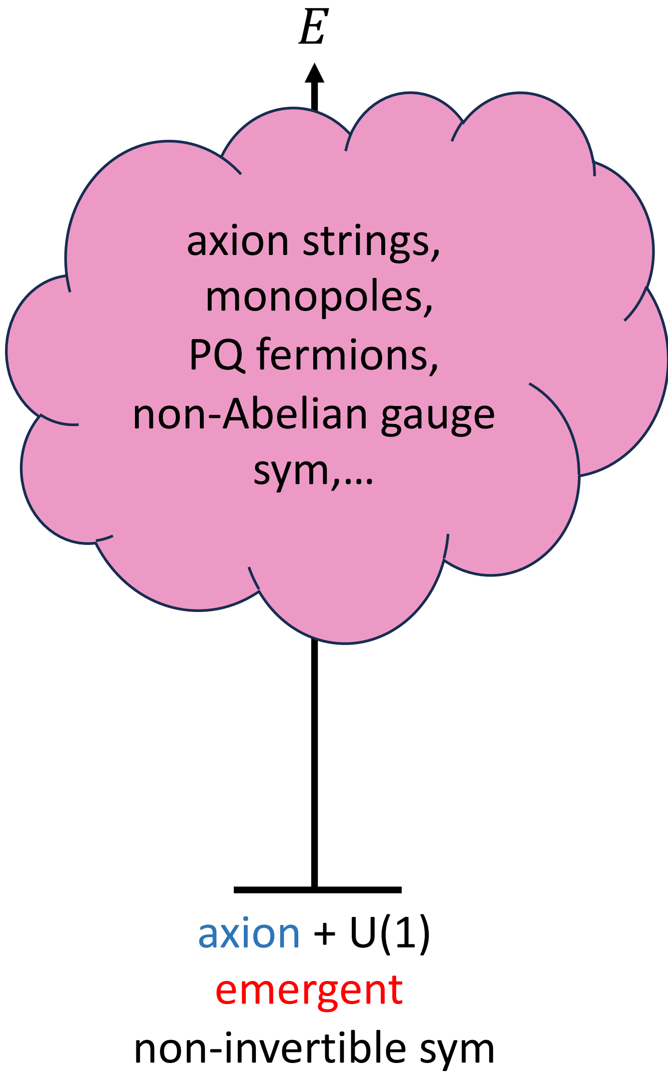
- Axion has been a major target for new physics beyond the Standard Model. Many phenomenological models contain a sector of a dynamical axion field $\theta(x)$ coupled to a U(1) gauge field:

$$\frac{f^2}{2}(\partial_\mu\theta)^2+\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}-\frac{i}{32\pi^2}\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

- The axion model has many non-invertible symmetries. In particular, the Peccei-Quinn symmetry is a (spontaneously broken) non-invertible global symmetry:

$$\mathcal{D}_{p/N}\colon \theta(x)\to\theta(x)+2\pi p/N$$

- These symmetries are typically emergent.



Constraints on axion physics

[Choi-Lam-**SHS** '22, Choi-Forslund-Lam-**SHS** '23]

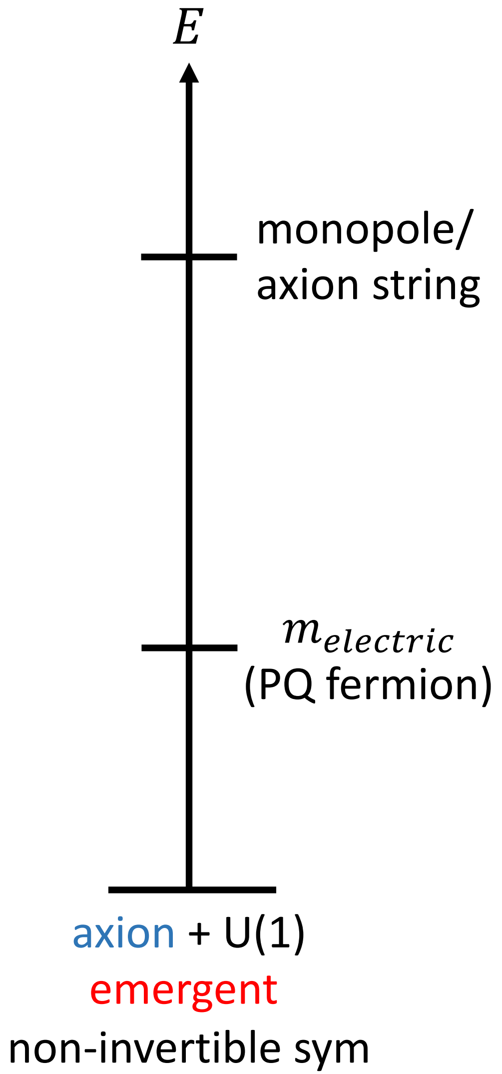
- The emergent non-invertible symmetries **mix** with other (higher-form) invertible symmetries, leading to universal constraints on the possible UV completions of the axion model [Choi-Lam-**SHS** '22]:

$$m_{electric} \lesssim m_{monopole}$$

$m_{electric}$ is the mass of the lightest electrically charged particle (e.g., Peccei-Quinn fermions).

- It also constrains the tension T of the axion string [Brennan-Cordova '20, Choi-Lam-**SHS** '22]:

$$m_{electric} \lesssim \sqrt{T}$$



Constraints on axion-photon decay

[Choi-Forslund-Lam-**SHS** ’23, Reece ‘23]

- Generalized symmetry can help constrain the effective axion-photon coupling $g_{a\gamma\gamma}$.

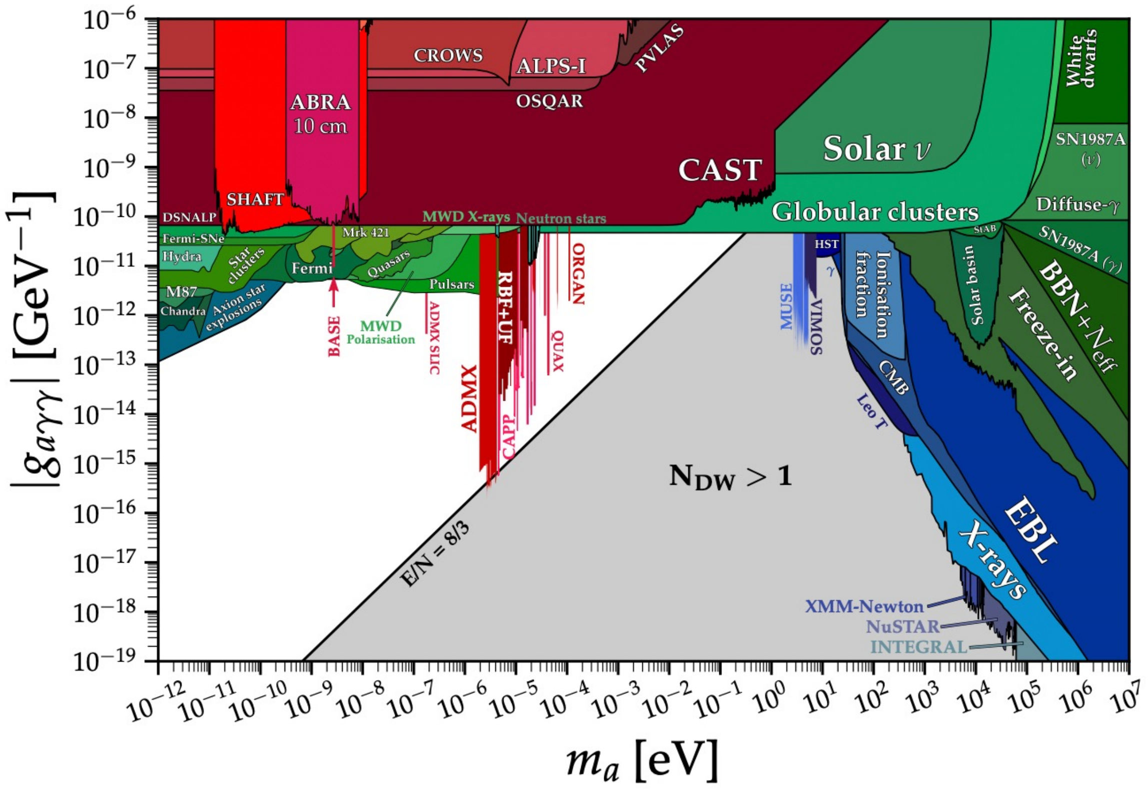
$$g_{a\gamma\gamma} = \frac{\alpha N}{\pi f}\left[\frac{E}{N} - 1.92\right]$$

- We show that any QCD axion model that violates

$$\frac{|g_{a\gamma\gamma}|}{m_a} \geq 0.15(1) GeV^{-2}$$

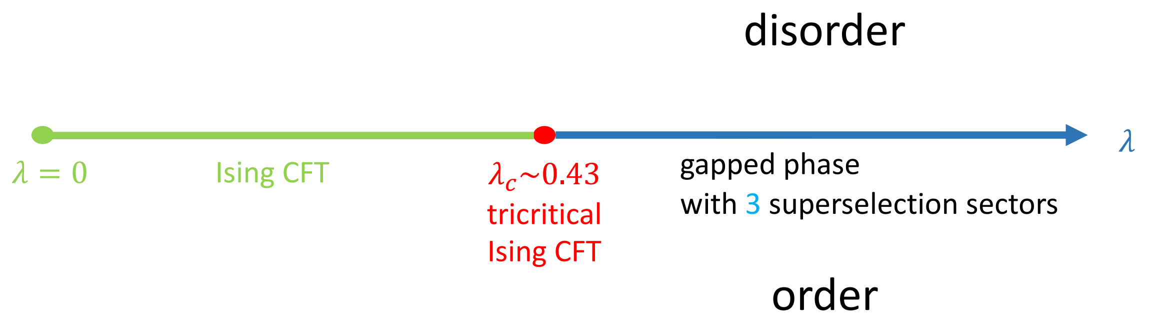
must necessarily face the axion domain wall problem in a post-inflationary scenario.

- The inequality is saturated by the SU(5) GUT model (with $\frac{E}{N} = \frac{8}{3}$).



a is the axion field on this slide, not the previous auxiliary gauge field. Sorry!

Qubits



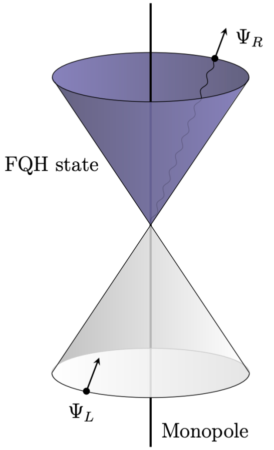
- The simplest non-invertible symmetry in 1+1d quantum lattice models of qubits, including the critical Ising model, is the Kramers-Wannier duality operator [Seiberg-SHS '23]:

$$D = \sqrt{2}\, e^{-2\pi i \frac{N}{8}} \left(\prod_{j=1}^{N-1} \frac{1 + i\, X_j}{\sqrt{2}} \frac{1 + i\, Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 + i\, X_N}{\sqrt{2}} \frac{1 + \prod_j X_j}{2}$$

- $D X_j = Z_j Z_{j+1} D, \quad D Z_j Z_{j+1} = X_{j+1} D$
- In the thermodynamic limit, we show that any Hamiltonian commuting with D must be [Seiberg-Seifnashri-SHS '24]:

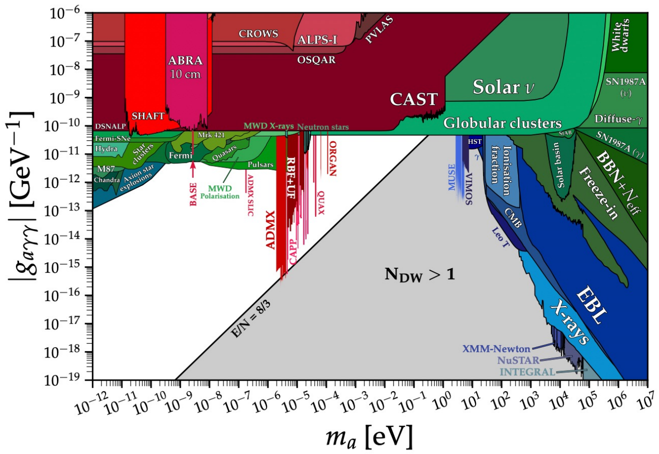
1. Gapless, or
2. Gapped with the number of superselection sectors a multiple of 3.

Applications



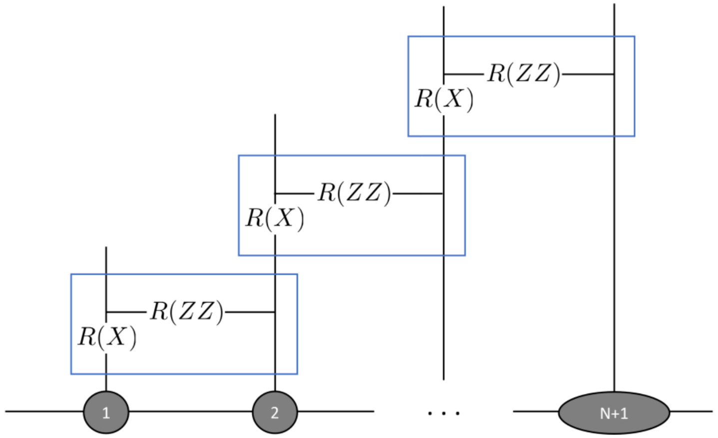
Quantum Field Theory:

- ‘t Hooft anomaly matching [Chang-Lin-**SHS**-Wang-Yin ’18, Choi-C. Cordova-Lam-Hsin-**SHS** ’21,…]
- Confinement in low dimensional QCD [Komargodski et al ‘20]
- Monopole-fermion scattering [Komargodski et al ’23, vanBeest et al ‘23]
- Crossing symmetry [Copetti-L. Cordova-Komatsu ‘24]
- Axion-photon decay [Choi-Forslund-Lam-SHS ‘23]
- …



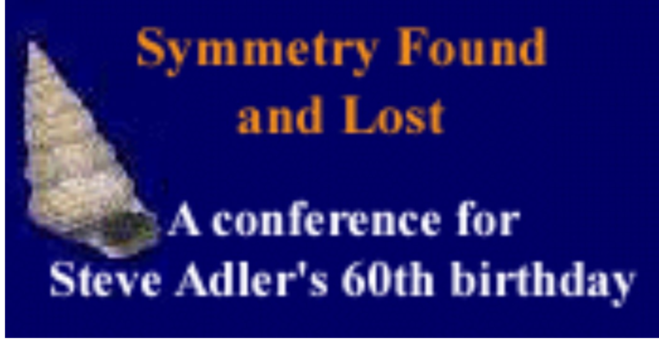
Quantum Gravity:

- Consolidating conjectures in quantum gravity [Rudelius-**SHS** ‘20, Heidenreich et al. ’21,…]
- (Approximate) selection rules in string theory [Kaidi-Tachikawa-Zhang ‘24, Heckman et al ‘24]
- Page charge in string theory [Apruzzi-Bah-Bonetti-SchaferNameki ‘22, Choi-Lam-**SHS** ‘22]
- …



Condensed matter theory:

- Lieb-Schultz-Matthias-type constraints and the sequential circuit [Seiberg-Seifnashri-**SHS** ‘24]
- New symmetry-protected topological (SPT) phases [Fechisin-Tantivasadakarn-Albert ’23, Seifnashri-**SHS** ‘24]
- …



Symmetry Lost and Found

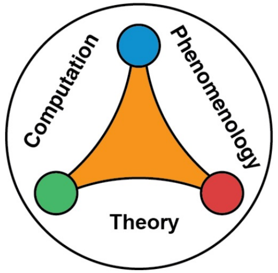
- Symmetry can be non-invertible. *What's done cannot be undone.*
- In massless QED and QCD, the chiral $U(1)_A$ symmetry is broken by the ABJ anomaly into a non-invertible symmetry:

$$\mathcal{D}_{\frac{1}{N}}(M) \equiv \int [Da]_M \exp[\oint_M (\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

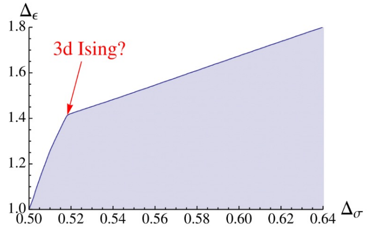
- The non-invertible symmetry is a composition of the naïve chiral rotation with a fractional quantum Hall state.
- To put it in the maximally offensive way, the neutral pion decays $\pi^0 \to \gamma\gamma$ because of the non-invertible global symmetry.
- Application: universal constraints in axion physics including bounds on the axion-photon coupling.

Activities

- TASI 2023
  Aspects of Symmetry
- Aspen workshop 2023
  Traversing the Particle Physics Peaks - Phenomenology to Formal
- SwissMAP school 2023
  Categorical Symmetries in QFT
- IHES school 2024
  Symmetries and Anomalies
- PiTP school at IAS 2024
  Ultra-Quantum Matter
- KITP workshop 2025
  Generalized Symmetries in QFT



Simons Collaboration on Ultra-Quantum Matter



Generalized Symmetry

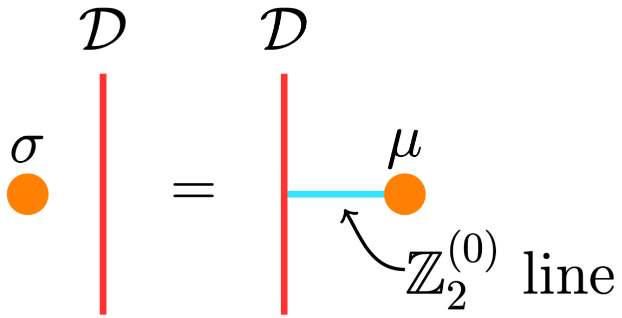
Quantum Field Theory
anomaly matching, selection rule, confinement,…

Mathematics
higher category theory, cobordism hypothesis, Witt class…

CMT&QI
anyons, SPT, MPO, sequential quantum circuit, LSM,…

Phenomenology
axions, naturalness, CP, neutrino mass, domain wall problem…

Quantum Gravity String Theory
no global symmetry conjecture, completeness hypothesis, …

	1+1d Ising lattice model $H = -\sum_j (X_j + Z_j Z_{j+1})$	**3+1d QED** $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu \Psi$
Non-invertible symmetry	Kramers-Wannier symmetry [Seiberg-**SHS** '23, Seiberg-Seifnashri-SHS '24] $D = \sqrt{2}\, e^{\frac{-2\pi i N}{8}} \left(\prod_{j=1}^{N-1} \frac{1 + i X_j}{\sqrt{2}} \frac{1 + i Z_j Z_{j+1}}{\sqrt{2}} \right) \frac{1 + i X_N}{\sqrt{2}} \frac{1 + \prod_j X_j}{2}$	Non-invertible chiral symmetry [Choi-Lam-**SHS** '22, Cordova-Ohmori '22] $\mathcal{D}_{\frac{1}{N}}(M) = \int [Da]_M \exp\left[\oint_M \left(\frac{2\pi i}{N} \star j^A + \frac{iN}{4\pi} a da + \frac{i}{2\pi} a dA \right) \right]$
Anomaly origin	Kitaev chain anomaly involving lattice translation	Dirac fermion ABJ anomaly
Action on observables	order <-> disorder 	monopole <-> dyon