Preparing solvable anyons with constant-depth adaptive quantum circuits

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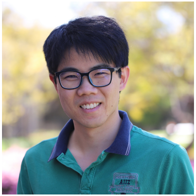

Nathanan Tantivasadakarn
Caltech

Plan

Motivations

Example: Toric code

Example: the Ising TO

Nilpotent anyon theory

Solvable anyon theory

Example: $\mathcal{D}(S_3)$ anyons

Tambara-Yamagami (TY) Category

General procedure for defectification and gauging

Summary and outlook

Motivation (1): Classification of TOs

Different long-range entanglement in TOs can be classified using the equivalence class of Finite-Depth Local Unitaries [CGW10].

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e.g. Product state, ground states of the quantum double of $\mathbb{Z}_2$ (toric code), D_4 and S_3 can be seen in the same class, whereas the Fibonacci TO is of another type[VTV22, TVV23].

Motivation (2): Quantum Simulation and Quantum Codes

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One can also view this effort as a quantum simulation TOs or TQFTs. And such a system can be used to encode logical information in the field of Quantum Error Correction.
Advantage: Constant depth circuit w.r.t the anyon separation/distance bounds the number of qudit errors during the operation.

Plan

Preliminary example: Gauging the $\mathbb{Z}_2$ symmetry to get the toric code

We will sometimes use $\mathbf{1}$ and ψ instead of 0 and 1 for the computional basis, as the category of Ising can be obtained from making $\mathbb{Z}_2$ extensions

$$\underbrace{\mathbf{1}}_{\mathcal{V}ec} \overset{\mathbb{Z}_2}{\to} \underbrace{1 \oplus \psi}_{\mathcal{V}ec_{\mathbb{Z}_2} \cong \mathcal{TC}} \overset{\mathbb{Z}_2}{\to} \underbrace{(1 \oplus \psi) \oplus \sigma}_{\text{Ising}} \tag{2.1}$$

This can serve as a warm-up to get us familiar with the notation later.

Preliminary example: Ground State of the toric code on a honeycomb lattice

Let the Hilbert space be a tensor product of qubits on edges of the lattice.

$$\mathcal{H} = \bigotimes_{e} \mathcal{H}_e$$

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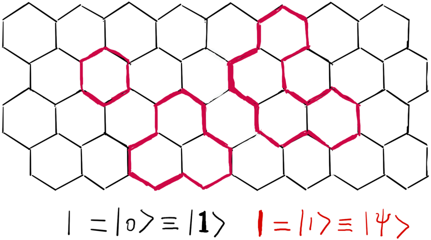
The easiest ground state is a equal superposition of all loop configurations $|\Omega\rangle = \sum_\Gamma |\Gamma\rangle$.

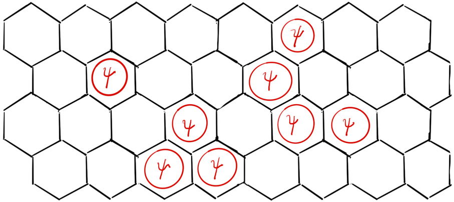


Preliminary example: Ground State of the toric code on a honeycomb lattice

Another way to write down the ground state wavefunction is by gauging or Kramers-Wannier (KW) duality [VTV22, TVV23],

$$KW_{EP}^{\mathbb{Z}_2} = \langle + |_P \prod_p (\prod_{e \in \partial p} CX_{p \to e}) | \mathbf{1} \rangle_E , \tag{2.2}$$

$$| \Omega \rangle = KW_{EP}^{\mathbb{Z}_2} | + \rangle_P , \quad | + \rangle = (\underbrace{| \mathbf{1} \rangle}_{= |0\rangle} + \underbrace{| \psi \rangle}_{|1\rangle}) / 2. \tag{2.3}$$



Each nontrivial plaquette $|\psi\rangle_p$ is shown, which results in a fusion of a ψ loop to the hexagon edges, operated by Control-$X^{\otimes 6}$.

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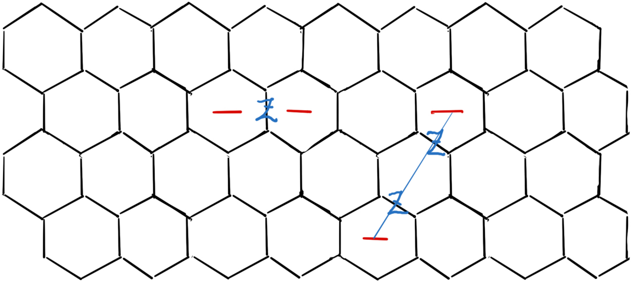
How do we correct the unwanted $\langle - |_p$ for some plaquette p by using adpative local unitary U?
One can simply use strings of Z-operator corresponding to the $\mathbb{Z}_2$ grading to connect pairs of $\langle - |_p \langle - |_{p'}$.



Plan

Ground state preparation of the double Ising TO

Recall that the Ising category can be obtained from the trivial TO by two steps of extensions or two levels of gradings.

$$\underbrace{\mathbf{1}}_{\mathcal{V}ec} \xrightarrow{\mathbb{Z}_2} \underbrace{1 \oplus \psi}_{\mathcal{V}ec_{\mathbb{Z}_2} \cong \mathcal{TC}} \xrightarrow{\mathbb{Z}_2} \underbrace{(1 \oplus \psi) \oplus \sigma}_{\text{Ising}} \tag{3.1}$$

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where the second grading means:

$$(1 \oplus \psi) \mapsto 0 \in \mathbb{Z}_2 \tag{3.2}$$

$$\sigma \quad \mapsto 1 \in \mathbb{Z}_2. \tag{3.3}$$

Graded sectors form a rep of the grading group

$$B_p^0 := \frac{1}{2}(B_p^{\mathbf{1}} + B_p^{\psi}), \quad B_p^1 := \frac{\sqrt{2}}{2} B_p^{\sigma} \tag{3.4}$$

They admit the algebra of $\mathbb{Z}_2$ sign-representation :

$$(B_p^1)^2 = B_p^0, \tag{3.5}$$

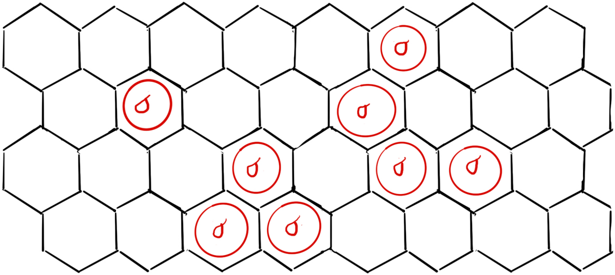
resembling $(-1) \times (-1) = +1$.

$$CB_p \left| i \right\rangle_p \left| \psi \right\rangle = B_p^i \left| \psi \right\rangle, \quad i = 0, 1. \tag{3.6}$$

Ground state preparation of the double Ising TO

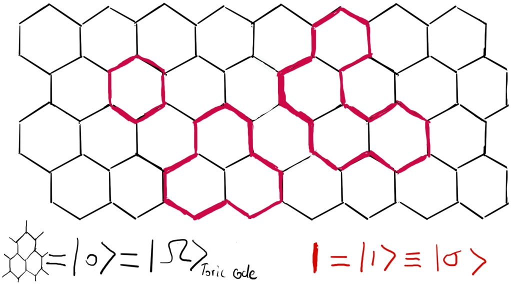
$$KW^{\mathbb{Z}_2}_{\text{Ising}} := \langle + |_P \prod_p CB_p \, |\Omega\rangle_{\mathbb{Z}_2, E} \tag{3.7}$$

$$|\Omega\rangle_{\text{Ising}} = KW^{\mathbb{Z}_2}_{\text{Ising}} \, |+\rangle_P \, , \tag{3.8}$$



Closer look of the ground state wavefunction

Again one can think of as an equal superposition σ-loops on top of the toric code (black background).



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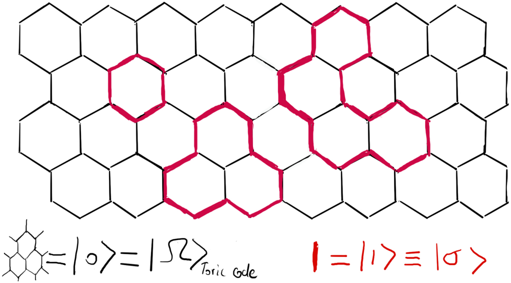
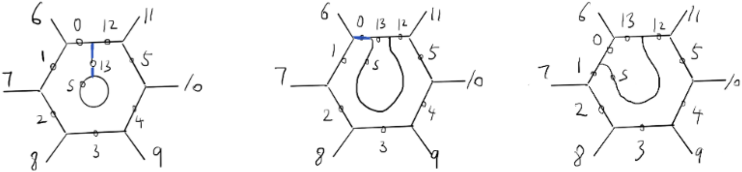


Or one may think of as a equal superposition of various red loops, which partitions the lattice, and in each partitioned area sits a (local) toric code wavefunction.

See also for a similar construction [SZBV22].



Introduce three ancilla $|\mathbf{1}\rangle_{12}, |\mathbf{1}\rangle_{13}, |\sigma\rangle_s$.

Quantum Circuit to apply Controlled-B_p

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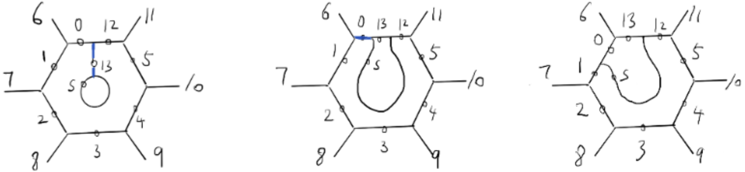

Introduce three ancilla $|\mathbf{1}\rangle_{12}, |\mathbf{1}\rangle_{13}, |\sigma\rangle_s$.
Apply unitary $U_{0,12}$ s.t. $U_{0,12} : |i\rangle_0 |\mathbf{1}\rangle_{12} \mapsto |i\rangle_0 |i\rangle_{12}$.

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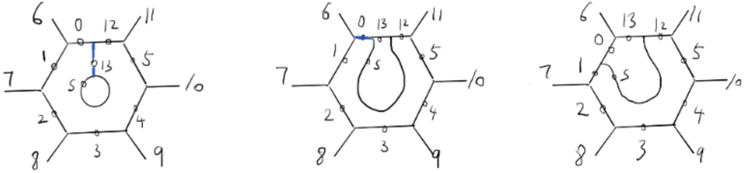

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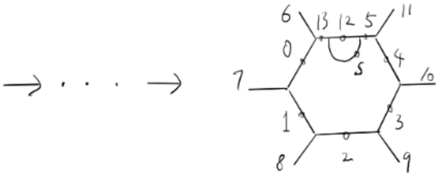
Apply unitary $U_{0,12}$ s.t. $U_{0,12} : |i\rangle_0 |\mathbf{1}\rangle_{12} \mapsto |i\rangle_0 |i\rangle_{12}$.
Apply the controlled-F operator to go to the 2nd figure. Then apply another six Controlled-F.

Quantum Circuit to apply Controlled-B_p



Eventually we arrive at a full circuit realization of CB_p with the control acting on $|+\rangle_p$ at the center of plaquette p.

Plan

Nilpotent anyon theory

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Let $(G^{(0)}, G^{(1)}, \cdots, G^{(k)})$ be a serious of groups. Suppose the anyon theory admits a series of **abelian grading**:

$$\mathcal{V}ec \equiv \mathcal{B}^{(0)} \subset \mathcal{B}^{(1)} \subset \cdots \subset \mathcal{B}^{(k)} \equiv \mathcal{Z}(\mathcal{C}), \tag{4.1}$$

$$\text{s.t. } \mathcal{B}^{(j)} = \bigoplus_{\mathbf{g} \in G^{(j)}} B_{\mathbf{g}}^{(j)}, \quad \mathcal{B}^{(j-1)} = \mathcal{B}_{\mathbf{1}}^{(j)}. \tag{4.2}$$

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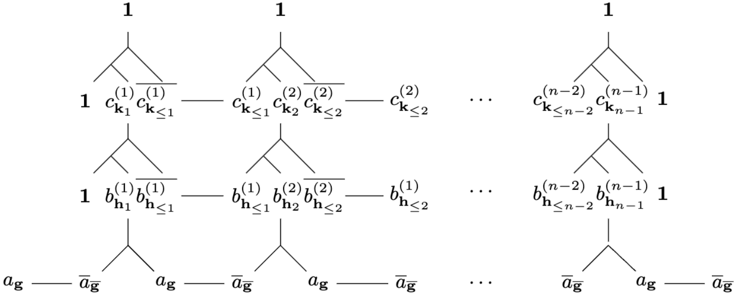
If for any anyon $a_{\mathbf{h}} \in \mathcal{B}^{(j)}$ graded by $\mathbf{h} \in G^{(j)}$, there exists an anyon $b_{\overline{h}}$ with the inverse grading $\overline{\mathbf{h}} \in G^{(j)}$, s.t. all of their fusion product are in $\mathcal{B}_{\mathbf{1}}^{(j)} \equiv \mathcal{B}^{(j-1)}$ graded by $G^{(j-1)}$, then we say it's a **nilpotent anyon theory**.

Anyon can always be prepared in finite depth in a nilpotent theory

A theory of grading group series that has length 3 $(G^{(1)}, G^{(2)}, G^{(3)})$.



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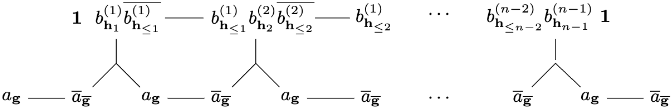

Let us pay attention to only one fusion step, and focus on the fusion of the 1st $\overline{a}_{\overline{\mathbf{g}}} \in \mathcal{B}_{\overline{g}}^{(3)}$ and the 2nd $a_{\mathbf{g}} \in \mathcal{B}_{\mathbf{g}}^{(3)}$.

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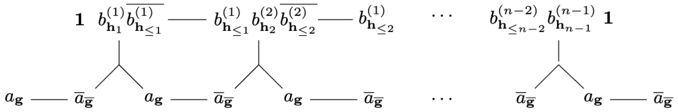

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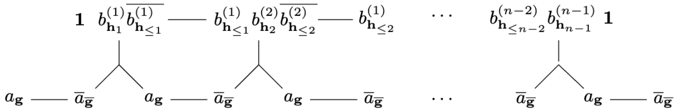

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We then create a pair of anyons $\overline{b^{(1)}_{\mathbf{h}_{\leq 1}}} - - - b^{(1)}_{\mathbf{h}_{\leq 1}}$. So that the fusion of $b^{(1)}_{\mathbf{h}_1} \otimes \overline{b^{(1)}_{\mathbf{h}_{\leq 1}}} \in \mathcal{B}^{(2)}_{\mathbf{1}}$.

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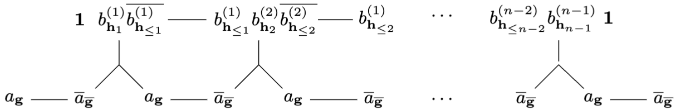

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Similalry, create $\overline{b^{(1)}_{\mathbf{h}_{\le 2}}} - - - b^{(1)}_{\mathbf{h}_{\le 2}}$ so that $b^{(1)}_{\mathbf{h}_{\le 1}} b^{(2)}_{\mathbf{h}_2} \overline{b^{(2)}_{\mathbf{h}_{\le 2}}}$ has the trivial grading $\mathbf{1}$.

Keep repeating to the right, and continue to descend to the lowest grading group $G^{(0)} = \{e\}$, we are done.

Examples

Toric code TO and the Ising TO are examples of nilpotent anyon theories. One can prepare any anyon ribbons in finite depth.

Plan

Solvable anyons

Knowing how to prepare nilpotent ones is not the end of the story. There are harder ones. Consider for example anyon C in $\mathcal{D}(S_3) = (C = \{e\}, R = Std)$. The fusion rule is

$$C \otimes C = A + B + C.$$

Namely anyon C is **cyclic**.

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For a mathematician, you can view an excitation as a simple object in $\mathcal{R}ep(\mathcal{D}(G))$ in this case.

Solvable anyons

The **solvable** anyon theory is defined by the following properties: Unlike the nilpotent case, this time the center category is not graded by a group series; however, the input category still accepts a grading series.

$$\mathcal{C}^{(0)} \subset \cdots \subset \mathcal{C}^{(k)}, \tag{5.1}$$

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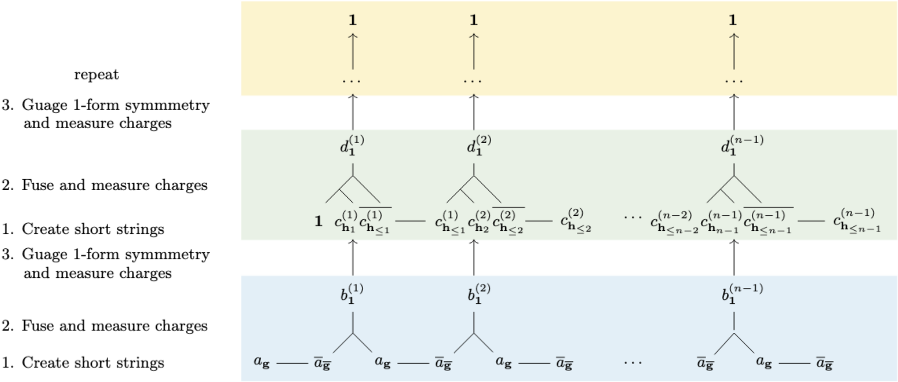
One example is that $\mathcal{Z}(S_3)$ can be obtained by gauging the $\mathbb{Z}_2$-charge conjugation symmetry of $\mathcal{Z}(\mathbb{Z}_3)$.

Solvable anyons

How do we prepare anyons/ribbons in finite depth then?
The answer lies in ungauging and re-gauging.

Solvable anyons



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$\mathcal{D}(S_3)$ anyons: A little bit of technical details to convince you :D

Recall the ribbon s of anyon $e \in \mathcal{D}(\mathbb{Z}_3)$ is a product of Pauli Z in the $\mathbb{Z}_3$ toric code:

$$W_s^{e,\mathbb{Z}_3} = \prod_{\ell \in s} \mathcal{Z}_\ell \tag{6.3}$$

So let $|\psi_1\rangle := |\psi_0\rangle$ be the state of such a ribbon acting on the ground state of $\mathbb{Z}_3$ TO, then the claim is that the state $|\phi\rangle$ after gauging is ribbon C acting on the ground state of $\mathcal{D}(S_3)$.

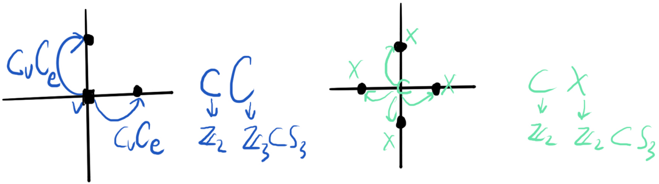
$\mathcal{D}(S_3)$ anyons: A little bit of technical details to convince you :D

$$|\phi\rangle = \widehat{G}\,|\psi_1\rangle := KW_{EV}^{\mathbb{Z}_2} U_{EV}\,|\psi_1\rangle\,|+\rangle_V^{\mathbb{Z}_2}\,, \tag{6.4}$$

$$U_{EV} = \prod_v \prod_{v\to e} C_v^{\mathbb{Z}_2} \mathcal{C}_e^{\mathbb{Z}_3} \quad \text{to couple } \mathbb{Z}_2 \text{ and } \mathbb{Z}_3 \text{ systems} \tag{6.5}$$

$$KW_{EV} = \langle +|_V \prod_v \prod_{e\ni v} (CX_{v\to e})\,|0\rangle_E \tag{6.6}$$

$$\text{gauging } \mathbb{Z}_2 \text{ symm. same as preparing toric code} \tag{6.7}$$



For more details of this KW map, see [VTV22, TVV23].

Plan

The definition and features of TY category

The Tambara-Yamagami $TY(\mathbb{Z}_N)$ of the abelian group $\mathbb{Z}_N$ has N invertible simple objects $a \in \mathbb{Z}_N$ with quantum dimension $d_a = 1$ and one non-invertible object σ with quantum dimension $d_\sigma = \sqrt{N}$.

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The fusion rule is

$$a \otimes b = (a+b) \mod N, \tag{7.1}$$

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For example the Ising category is a TY when we take $N = 2$, by labeling $\{\mathbf{1}, \psi\} \mapsto \{0, 1\} \in \mathbb{Z}_2$.

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We will mostly focus on $\mathbb{Z}_3$ and the G-crossed braiding data have already been given in [LLB21, BBCW19].

TY TO ground state

By definition of the Levin-Wen model (String-Net model), the ground state of $\mathcal{Z}(TY(\mathbb{Z}_3))$ is given by fusing possible loops B^s for simple object $s \in TY(\mathbb{Z}_3)$ to each plaquette, weighted by the quantum dimension d_s

$$|\Omega\rangle_{TY(\mathbb{Z}_3)} = \prod_p \sum_{s \in TY} d_s B_p^s \, |0\rangle_E \tag{7.4}$$

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As one can guess, this can be given by the protocol we defined before

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For each plaquette p, half of the time we don't touch the local $\mathbb{Z}_3$ TO ground state, and half of the time we fuse the σ-loop to p.

TY TO ground state wavefunction

Can we make the ground state wavefunction even more clear?

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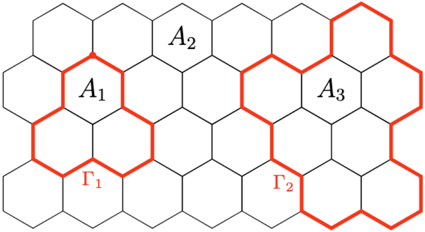

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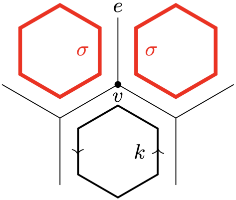

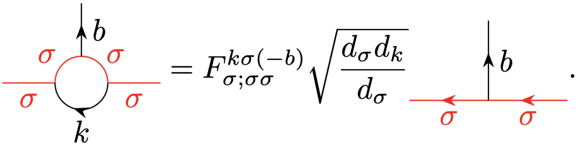

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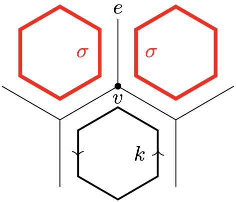

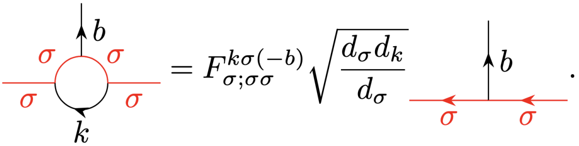

One of simplest nontrivial Levin-Wen example that has a clear explicit ground state wavefunction (instead of written in terms of B_p operators and F-symbols).

Emergent anyons of TY from defect SET + Gauging

To see the deep relation between TY and $\mathbb{Z}_3$ Toric code, we take a second look of the anyons in $\mathcal{D}(\mathbb{Z}_3)$. We can divide them into two sectors via the (in)variance under e-m duality

$$\mathbb{Z}_3^{(1)} = \{1, em, e^*m^*\}, \quad \mathbb{Z}_3^{(-1)} = \{1, e^*m, em^*\}. \tag{7.10}$$

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The centralizer group are $\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2$. Hence their respective anyons in the 2nd layer of $\mathcal{D}(TY(\mathbb{Z}_3))$:

$$\mathbf{1} = (O_1, +), \quad z = (O_1, -), \quad \phi := (O_{em}, +), \quad \sigma_\pm := (O_\sigma, \pm). \tag{7.12}$$

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$$\phi \times \phi = 1 + z + \phi, \tag{7.14}$$

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Together one can identify them with objects in $\overline{SU(2)_4}$ theory:

$$\{1, \sigma_+, \phi, \sigma_-, z\} \equiv \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}. \tag{7.17}$$

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We only pay attention to ϕ for it is cyclic and hence the hardest to prepare.

Emergent anyons of TY from defect SET + Gauging

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Recall there's another half of the system $\mathbb{Z}^{(1)}$ that's invariant under em-duality (or the action of σ-defect before gauging). Therefore we can conclude that

$$\mathcal{Z}(TY(\mathbb{Z}_3)) = \mathbb{Z}_3^{(1)} \boxtimes \overline{SU(2)_4}.$$

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The total number of anyon excitaitons are $3 \times 5 = 15$.

String operator in $\mathcal{Z}(TY(\mathbb{Z}_3))$

The anyon strings in toric code or double semion model are well-known and can be written explicitly. But what does a string operator in $TY(\mathbb{Z}_3)$ look like?

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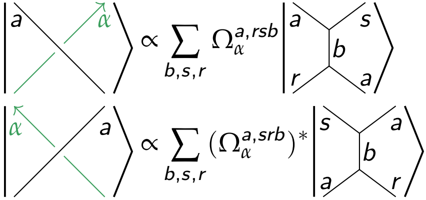
The general formula has been mentioned in Levin's paper, by using the half braiding data Ω_α for anyon α. We now give an explicit expression to help the audience build an intuitive picture of what a string look like.

String operator in $\mathcal{Z}(TY(\mathbb{Z}_3))$

Type 1 (7.19)

Type 2 (7.20)



String operator in $\mathcal{Z}(TY(\mathbb{Z}_3))$

A ϕ ribbon operator acting on the ground state:

$$W_\gamma^r |\psi\rangle = \omega^{2(N_1 - N_2)} \left(\bigotimes_i |\Gamma_i\rangle \right) \left(\bigotimes_k W_{\gamma_k, \mathbb{Z}_3}^{(-1)^k r} |\Omega\rangle_{A_k} \right) \tag{7.21}$$

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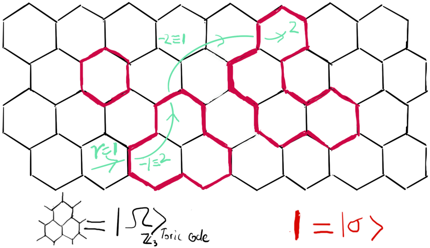

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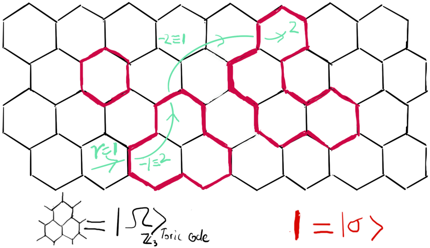

Recall Γ_i are the σ-loops; and $\left|\Omega\right\rangle_{A_k}$ are the $\mathbb{Z}_3$ TC areas.

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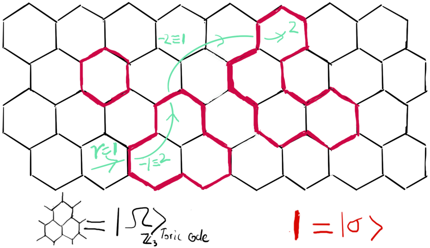

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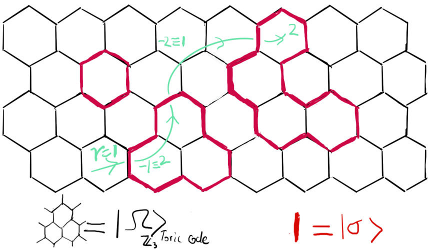

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$\omega = e^{2\pi i/3}$, $N_1 = 2$ and $N_2 = 1$. "Counting the toll we need to pay".

Can we obtain the ϕ-string through gauging?

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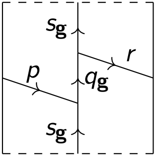
But before that, we need more math machinery to reach a general procedure.

Plan

Define a defect tube element:

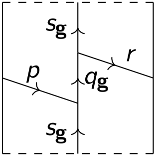
$$\mathcal{T}^{s_{\mathbf{g}}}_{p q_{\mathbf{g}} r} :=$$



$$p, r \in \mathcal{C}_{\mathbf{1}},\ s_{\mathbf{g}}, q_{\mathbf{g}} \in \mathcal{C}_{\mathbf{g}}. \tag{8.1}$$

Defect tube algebra and domain wall actions

Define a defect tube element:

$\mathcal{T}^{s_\mathbf{g}}_{p q_\mathbf{g} r} :=$  $p, r \in \mathcal{C}_\mathbf{1},\ s_\mathbf{g}, q_\mathbf{g} \in \mathcal{C}_\mathbf{g}.$ (8.1)

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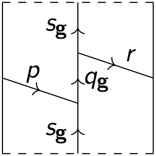
$$\mathcal{T}^{s_{\mathbf{g}}}_{pq_{\mathbf{g}}r} :=$$ 

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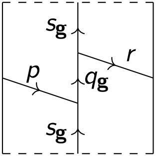

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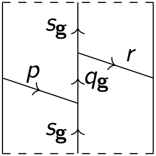

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Note that we have trivially-graded p and r. So we only care about the subalgebra of the action of these graded domain walls $s_{\mathbf{g}}$.

Defect tube algebra and domain wall actions

Using the recipe in [WBV17], we define the following operator acting on the trivial defects ($p, r \in \mathcal{C}_{\mathbf{1}}$)

$$\mathcal{B}^{\mathbf{g}} := \sum_{pqrs} c^{s}_{pqr} \mathcal{T}^{s_{\mathbf{g}}}_{pq_{\mathbf{g}}r} \tag{8.3}$$

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In particular, these operator take into account the case where there's a nontrivial charge/label $p \in \mathcal{C}_{\mathbf{1}}$ deposited to the plaquette. In the case of $TY(\mathbb{Z}_3)$, $p = 1$ or 2 as shown before.

Defect tube algebra and domain wall actions

Define this generalized controlled-plaquette operator:

$$(CB)_p \left|\mathbf{g}\right\rangle \left|\psi\right\rangle := B_p^{\mathbf{g}} \left|\mathbf{g}\right\rangle \left|\psi\right\rangle . \tag{8.5}$$

$$KW_{EP}^{G} := \langle + |_P \left(\prod_p (CB)_p \left| + \right\rangle_P \right) \tag{8.6}$$

Then the claim is that the protocol prepares anyon correctly.
In the case of $TY(\mathbb{Z}_3)$:

$$W_\gamma^\phi \left|\Omega\right\rangle_{TY(\mathbb{Z}_3)} = KW_{EP}^{G}(W^{em^*} + W^{e^*m}) \left|\Omega\right\rangle_{\mathbb{Z}_3} \tag{8.7}$$

Plan

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Summary

- Extended the idea of preparing groundstates via gauging to preparing all solvable anyon in finite depth.
- Worked out examples including $\mathcal{Z}(\mathit{Ising})$, $\mathcal{D}(S_3)$, and $\mathcal{Z}(TY(\mathbb{Z}_3))$.
- We conjecture this construction captures all 2D topological phases that can be created exactly via Adaptive Finite Depth Local Unitaries.

Outlook

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- Maybe just prove that a single nonsolvable phase i.e. of Fibonacci theory or of a nonsolvable group, lies outside the reach of AFDLU?

Outlook

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- Can these protocols be made fault tolerant for scalable practical QC implementations?

References I

[BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang, Symmetry fractionalization, defects, and gauging of topological phases, Phys. Rev. B **100** (2019), 115147.

[CGW10] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, Phys. Rev. B **82** (2010), 155138.

[DNO11] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik, On the structure of the witt group of braided fusion categories, 2011.

[LLB21] Chien-Hung Lin, Michael Levin, and Fiona J. Burnell, Generalized string-net models: A thorough exposition, Phys. Rev. B **103** (2021), 195155.

[SZBV22] Alexis Schotte, Guanyu Zhu, Lander Burgelman, and Frank Verstraete, Quantum error correction thresholds for the universal fibonacci turaev-viro code, Phys. Rev. X **12** (2022), 021012.

References II

[TVV23] Nathanan Tantivasadakarn, Ashvin Vishwanath, and Ruben Verresen, Hierarchy of topological order from finite-depth unitaries, measurement, and feedforward, PRX Quantum **4** (2023), 020339.

[VTV22] Ruben Verresen, Nathanan Tantivasadakarn, and Ashvin Vishwanath, Efficiently preparing schrödinger's cat, fractons and non-abelian topological order in quantum devices, 2022.

[WBV17] Dominic J. Williamson, Nick Bultinck, and Frank Verstraete, Symmetry-enriched topological order in tensor networks: Defects, gauging and anyon condensation, 2017.