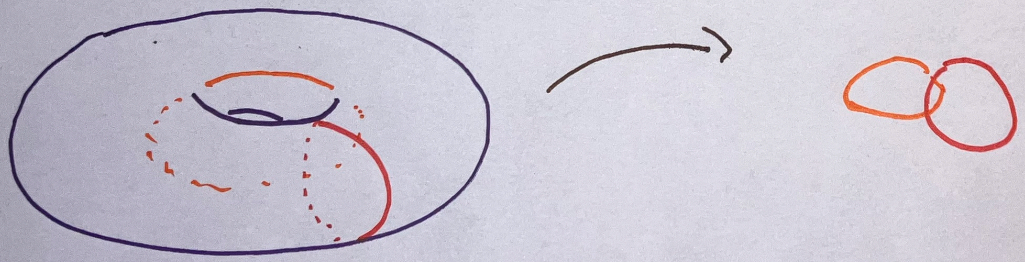
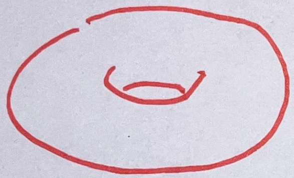
Quantum algorithms for computing homology

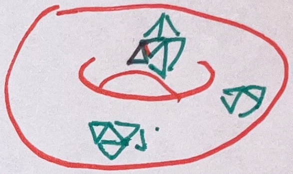


Seth Lloyd
MIT

Homology → fundamental method for calculating topological invariants

Euler → Riemann → Betti → Poincare → Noether, Vietoris Mayer





$$\to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1}$$

origins: Count # connected components, holes, + voids

now: wide set of applications in pure + applied mathematics + data science

Chain complex

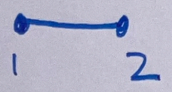
boundary map

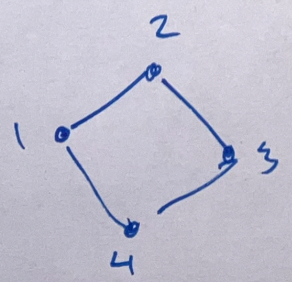
$$\cdots \longrightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \cdots$$

Abelian groups

$\partial_k \partial_{k+1} = 0$ 'The boundary of a boundary is zero'

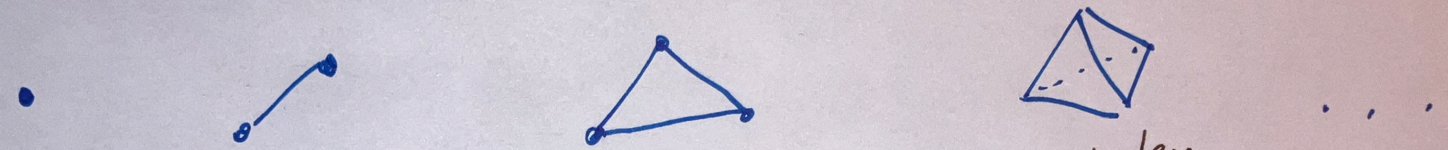
e.g.

∂_1  $= \dot{1} - \dot{2}$

∂_1  $= \dot{1} - \dot{2} + \dot{2} - \dot{3} + \dot{3} - \dot{4} + \dot{4} - \dot{1}$

$= 0$

Simplicial homology

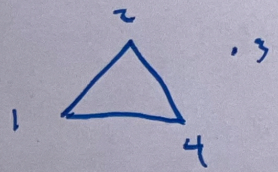


0 simplex   1 simplex   2 simplex   3 simplex   . . .

$\binom{n}{k+1}$ possible k simplices

2^n possible simplices

Quantum representation $\longrightarrow$ map simplices into vectors in $\mathbb{C}^{2^n}$ $\longrightarrow$ n qubits



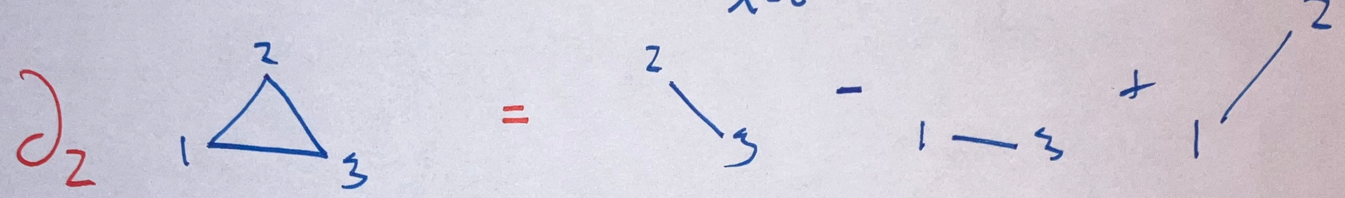
$\longrightarrow$ $|124\rangle$

(keep track of order!)

Boundary map

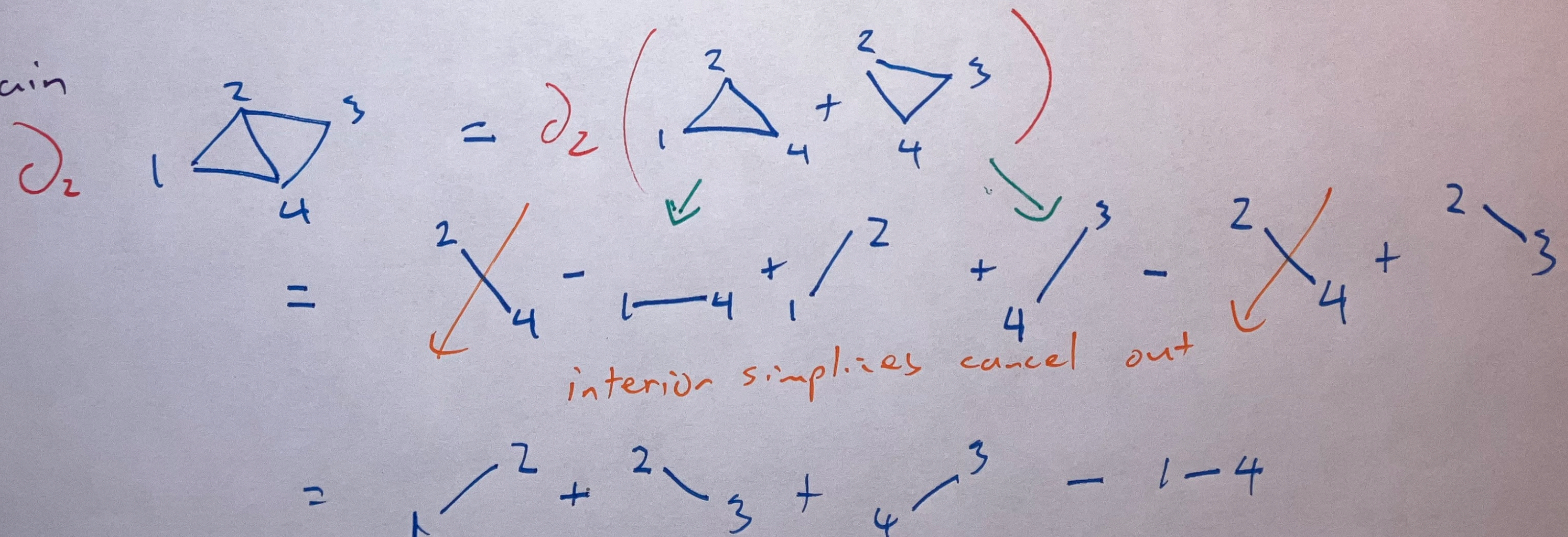
$$\partial_k |i_0 i_1 \dots i_k\rangle = |i_1 \dots i_k\rangle - |i_0 i_2 \dots i_k\rangle + \dots$$

$$= \sum_{l=0}^{k} (-1)^l |i_0 \dots \underbrace{\hat{i}_l}_{\text{omit}} \dots i_k\rangle$$

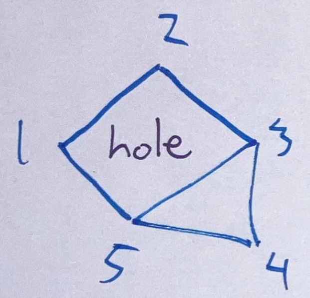


Minus signs keep track of orientation

Chain



Boundary map is a sparse linear operator on a 2^n dimensional space: $\partial_{k-1}\partial_k = 0$.



Goal: find and count holes/voids

- hole is bounded by chain

$$b(h) = [1,2] + [2,3] + [5,3] + [1,5]$$

$$\Rightarrow \partial_1 b(h) = 0$$

- boundary of hole is not the boundary of any 2-chain in the complex

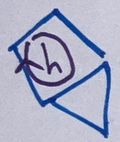
$$b(h) \neq \partial_2 c_2$$

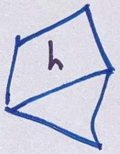
That is

$$(\partial_1 + \partial_2^{+})\, b(h) = 0$$

boundariless

not a boundary





Note:

$$\underbrace{(\partial_1^{+} + \partial_2)}_{D_1^{+}}\underbrace{(\partial_1 + \partial_2^{+})}_{D_1} = \underbrace{\partial_1^{+}\partial_1 + \partial_2\partial_2^{+}}_{L_1}$$

"Dirac operator"

Laplacian

$\Rightarrow$ k-dimensional boundaries
of k+1 dimensional holes

lie in the Kernel of $\underset{D_k}{(\partial_k + \partial_{k+1}^{+})}$, $\underset{L_k}{\partial_k^{+}\partial_k + \partial_{k+1}\partial_{k+1}^{+}}$

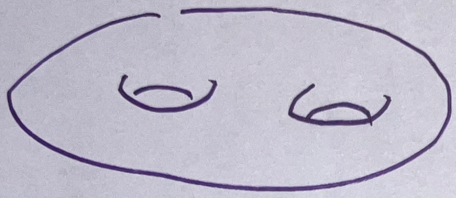
$\Rightarrow$ representatives of the homology

and the dimension of $\mathrm{Ker}(D_k) = \mathrm{Ker}(L_k)$

= the number of k+1 dimensional holes/voids

β_k = k'th Betti number

Recap:



Find + count holes
in k dimensions

$\Rightarrow$ find Kernel of

$D_k = (\partial_k + \partial_{k+1}^{+})$, $L_k = \partial_k^{+}\partial_k + \partial_{k+1}\partial_{k+1}^{+}$

and calculate its dimension

$2^n \times 2^n$ sparse matrices

¡Algebraic Topology!

Quantum computers

can perform linear algebra on $2^n \times 2^n$ sparse matrices

exponentially faster than classical computers !!

In particular, standard techniques of quantum simulation allow one

to apply the unitary transformations

$$e^{-iDt}, \quad e^{-iLt} \qquad D = \bigoplus_k D_k, \quad L = \bigoplus_k L_k$$

and to project an arbitrary state

onto the eigenvectors of D, L

and estimate the corresponding eigenvalues

von Neumann, quantum phase estimation

Von Neumann 1930

$H = H^{\dagger}$, $H|j\rangle = E_j|j\rangle$

Hermitian

eigenvector eigenvalue

$$e^{-iH\otimes\hat{p}} \sum_j \psi_j |j\rangle \otimes |x=0\rangle$$

$\hat{p} = \frac{i}{\hbar}\frac{\partial}{\partial x}$

momentum operator

$$= \sum_j \psi_j |j\rangle |x = E_j\rangle$$

measure to obtain eigenvalue

this register contains the corresponding eigenvector

"pointer variable model"

The Quantum Homology algorithm

(SL, P. Zanardi, S. Garnerone
Nat Comm 7, 1~~[illegible]~~ (2016))

- map k simplices $s_k \to |s_k\rangle$ quantum states
- use von Neumann / quantum phase estimation

  to project $\frac{1}{\sqrt{\#_k}} \sum_{s_k} |s_k\rangle$

  onto Kernel of D_k, L_k

- successful projection yields representatives of the homology (chains ~~[illegible]~~ bounding $k+1$ dimensional holes)
- probability of success yields dimension of Kernel / Betti numbers

Computational complexity for the quantum homology algorithm scales as

\# k simplices

\# vertices

$$O\left(\frac{|S_k|^2}{\beta_k^2}\frac{n^3}{\delta^2}\right)$$

Betti \#

multiplicative accuracy

versus

$O(2^{2n})$ for classical algorithm

Features:

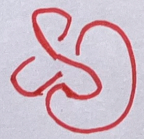
- Works best for large Betti numbers

  (use thermalization/quantum annealing to find Kernel?)

- Don't expect Q Algorithm to work in worst case

  exact Betti numbers => #P-hard, multiplicative error => NP-hard

Widely applicable:

- quantum data analysis (edge lengths -> clique complex)
- list of simplices (e.g., Facebook groups)
- knot theory  Khovanov homology => detect unknot
- Algebraic geometry / Hodge theory

Improvements/variants:

- A Schmidhuber, SL PRX Quantum 4, 040349, 2023
  complexity theory analysis
- S. Ubaru et al. arXiv: 2108.02811
  use fermions to keep track of signs
- R. Hayakawa arXiv: 2111.00433
  refreshing simplifications
- S. McArdle et al. arXiv: 2209.12887
  fewer qubits
- D.W. Berry et al. arXiv: 2209.13581
  comprehensive analysis

+ many more!

- M. Crichigno + T. Kohler arXiv: 2209.11793
  supersymmetric field theory (Witten)