Insights on Analysis Meeting Topology in Quantum Field Theory

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Program: Functional Integral Construction of TQFT

We propose a new program to address a long-standing open problem: how to construct a meaningful unitary topological quantum field theory (TQFT) in arbitrary dimension.

We construct a unitary $n+1$ TQFT from a functional integral Z on an n-dimensional lattice model, for arbitrary n. The configuration space is mathematically defined as labelled, regular, stratified, piecewise linear, n-manifolds.

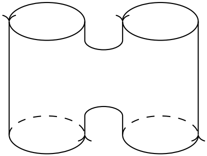
It provides new analytical insights in additional to the previous topological/categorical approaches to TQFT.

Topological Quantum Field Theory

Witten initiated Topological Quantum Field Theory (TQFT) and constructed a 2+1 TQFT using Chern-Simons theory and obtained an invariant of links in 3-manifolds as a path integral [**Wit89**], generalizing the Jones polynomial originated from subfactor theory [**Jon83**, **Jon85**, **Jon87**], and other link invariants from the representation theory of Drinfeld-Jimbo quantum groups [**Jim85**, **Dri86**, **HOMFLY85**, **PT88**, **Kau90**].
Feynman's path integral is a powerful method in physics, but the measure of the path space is only mathematically defined for a few case, such as the remarkable work of Glimme-Jaffe in constructive QFT [**GliJaf87**].
Reflection Positivity of the path integral is a crucial condition to implement the Wick rotation between Euclidean field theory and relativistic field theory, as proved by Osterwalder-Schrader [**OstSch73**].

Atiyah TQFT

Atiyah 88: *Quantum* Physics and *Topology* phenomenon emerging from the continuum limit.
Atiyah's $n+1$ TQFT is defined as a symmetrical monoidal functor from **Cob** to **Vec**.
Object: n-manifolds without boundary $\rightarrow$ vector spaces
Morphisms: $n+1$ cobordisms $\rightarrow$ linear transformations
The TQFT is called unitary, if the partition function is reflection positive.
In this case, the (finite dimensional) vector spaces are Hilbert spaces.



$\rightarrow \quad \mathrm{Hom}(V \otimes V, V \otimes V)$

lower dimensional TQFT

The topological invariant of Witten's 2+1 TQFT can be rigorously defined using the link invariants from quantum groups and the Lickorish-Wallace surgery theory, known as the Witten-Reshetikhin-Turaev TQFT [**ResTur91**].
The Turaev-Viro-Barrett-Westbury 2+1 TQFT from a spherical fusion category [**TurVir92**, **BarWes96**] is a state sum construction over a triangulation, which is a combinatorial analogue of path integral.
The Turaev-Viro state sum construction has been generalized to construct 3+1 TQFT using a braided fusion category by Crane-Yetter in [**CraYet93**, **CKY97**, **Cui19**] or a fusion 2-category by Duoglas-Reutter in [**DogReu18**].
The 2+1 TQFT is exceptionally successful due to the fruitful examples of quantum symmetries coming from the representation theory of quantum groups, subfactors, hopf algebras, vertex operator algebras, conformal field theory, etc.

Cobordism Hypothesis

Baez and Dolan proposed the cobordism hypothesis in 1995 to study manifolds in all co-dimensions in an n-dim TQFT as a symmetric monoidal n-categories with a fully dualizable object.
Lurie introduced a fruitful theory of (∞, n) category in [**Lur09**] to study non-invertible higher symmetries and to answer the cobordism hypothesis of Baez and Dolan [**BaeDol95**], generalizing the two-dimensional result of Hopkins-Luire.

n+1 TQFT and spherical n-category

It is widely believed that the state sum construction of 2+1 TQFT from spherical 2-categories will work in any dimension e.g. [**Wal21**], and the n-dimensional topological order should be characterized by a unitary n-category and an $n+1$ unitary TQFT [**KWZ15**].
However, there is no agreement on the mathematical definition of a (unitary) spherical n-category; see a recent discussion of Ferrer et al in [**FHJ24**].
A fundamental question of Wen: How to characterize topological orders by ground states, instead of categorical symmetries?

Challenging Questions

It is highly expected, but remains challenging, to generalize the framework in lower dimensions to higher dimensions, which should provide examples of higher quantum/non-invertible symmetries.
Questions: ?? $\rightarrow$ Higher algebras $\rightarrow$ Higher representation theory $\rightarrow$ unitary/spherical n-categories $\rightarrow$ unitary n+1 TQFT

Function Integral Construction of TQFT

Theorem (Liu 2024, arXiv:2409.17103v1)

Suppose Z is a linear functional on labelled stratified manifolds with support S^n over the field $\mathbb{C}$, satisfying the three conditions

1. *(RP) reflection positivity;*
2. *(HI) homeomorphic invariance;*
3. *(CF) complete finiteness. (finite entanglement rank)*

Then we obtain an $n+1$ unitary alterfold TQFT.
(For a general field $\mathbb{K}$, we replace RP by strong semisimplicity.)

Answer: Linear Functional on Lattice Models $\rightarrow$ D^n algebras $\rightarrow$ Higher representation theory $\rightarrow$ unitary/spherical n-categories $\rightarrow$ unitary n+1 Alterfold TQFT

The alterfold TQFT has alternating A/B-colors of $n+1$ cells. The n-dim lattice model lives on the domain wall between A and B.

An answer to Wen's question

Our functional integral construction suggests a mathematical definition of unitary/spherical n-categories.

In our approach, we do not need a unitary n-category as the input data to construct an $n+1$ unitary TQFT. Instead, a unitary n-category is an emergent quantum symmetry coming from the physical quantized space, which captures the quantum symmetry of the lattice model in an n-disc.

The three conditions (RP) (HI) (CF) of the linear functional Z provides a characterization of topological orders from its ground states which is an answer to the fundamental question of Wen.

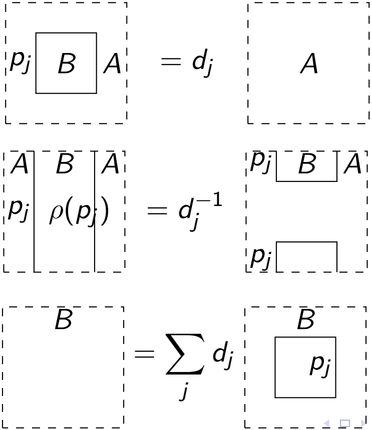
Toy Model: 1+1 case

Principle: The partition function Z tells everything.
Linear Functional on Lattice Models $\rightarrow$ D^n algebras $\rightarrow$ Higher representation theory $\rightarrow$ unitary/spherical n-categories $\rightarrow$ unitary n+1 Alterfold TQFT
1D Lattice Model: Labelled Stratified 1-Manifolds
Spins: vectors in the label space L.
Free D^1 algebra: Fock space $\mathcal{F}(L)$
Linear Functional (RP, HI): Tracial state on $\mathcal{F}(L)$.
D^1 algebra: Free Algebra quotient by the Kernel of Z.
Representation Category: Idempotent Completion
We end up with an associate semisimple algebra $\bigoplus_{j\in J} M_{n_j}(\mathbb{C})$, whose minimal idempotent p_j has positive trace d_j.
The Complete Finiteness condition corresponds to the finiteness of J and n_j.

Toy Model: 1+1 alterfold construction

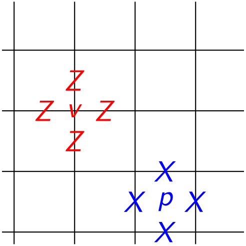
Applying the skein relations (higher resolution of identity) to Vertices, Edges, Faces of a triangulation of an oriented surface S, we obtain the invariant

$$Z(S_B)/Z(S_A) = \sum_j d_j^{\#V-\#E+\#F} = \sum_j d_j^{\chi(S)}.$$



Toric Code

Kitaev proposed topological error corrections and topological quantum computations using his toric code in 1997. (Ann. Phys. 2003)
See also Freedman, Kitaev, Larsen, Wang (Bull. AMS 2003)



Each edge in the $D \times D$ lattice has a physical qubit.
Stabilizers: $A_v = \prod_{v \in \partial e} Z_e$, $B_p = \prod_{e \in p} X_e$.
Hamiltonian $H = -\sum_v A_v - \sum_v B_p$
The logical 2-qubits are encoded by the stabilizer states

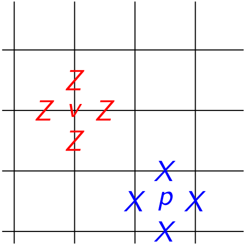
$$(\mathbb{C}^2)^2 \cong \{|\varphi\rangle \in (\mathbb{C}^2)^{\#E} \mid A_v |\varphi\rangle = |\varphi\rangle, B_p |\varphi\rangle = |\varphi\rangle, \forall\ v, p\}.$$

From Lattice Model to TQFT

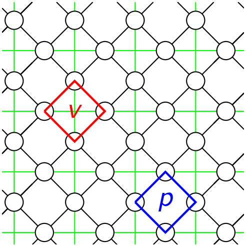
The toric code can be generalized to lattices of a general shape on a surface. The ground state is a sum of all configurations whose spin-1 edges form loops. It defines a linear functional Z on the configuration space.

2D lattice models $\rightarrow$ linear functional Z (ground state) $\rightarrow$ 2+1 TQFT

Periodic Square Lattice



Refined Lattice

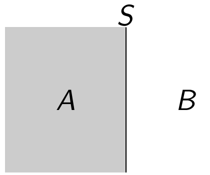


$\mathbb{Z}_2$ symmetry $\longleftrightarrow$ Ising Symmetry

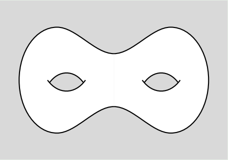
Alterfold TQFT

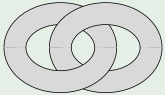
An n-alterfold consist of

- A closed oriented n-manifold M;
- A separating hyper surface $S \subset M$;
- S separate M into A-B colored regions. (S may not be connected.)



Example





B-colored handlebody in A A-colored Hopf Link in B

An alterfold TQFT is a functor from the cobordism category of alterfolds to the category of vector spaces. It reduces to Atiyah's TQFT when the manifold is fully B-color.

2+1 Alterfold TQFT

For the 2+1 dimensional case, if we take the linear functional Z to be the evaluation map of the string-nets in a spherical multi-fusion category $\mathscr{C}$, then, in recent joint work with Shuang Ming, Yilong Wang and Jinsong Wu [**LMWW23a**, **LMWW23b**], we proved that the alterfold 2+1 TQFT contains both

- Turaev-Viro-Barrett-Westbury TQFT of $\mathscr{C}$
  (blow up the skeleton of the triangulation to B-color handles)
- Witten-Reshetikhin-Turaev TQFT of its Drinfeld center of $\mathscr{C}$
  (blow up the framed links to A-color handles)

Moreover, the two sub TQFT are identical on Atiyah's TQFT of cobordisms.

$$\begin{array}{ccc} \text{WRT TQFT} & \subset & \text{Alterfold TQFT} \\ \cup & & \cup \\ \text{Atiyah TQFT} & \subset & \text{TVBW TQFT} \end{array}$$

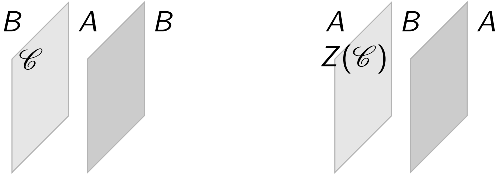
1. Walker 91,03 Turaev 94, Robert 95 for MTC;
2. Kawahigashi-Sato-Wakui 05 for unitary SFC;
3. Turaev-Virelizier 17, Balsam-Kirillov 10 for SFC.

Meanings of A/B colors

Take $Z(M_A) \equiv 1$, namely the partition function of any A-color manifold M is constant 1. Then the partition function is irrelevant to the A-color part, which encodes to the surgery theory in the 2+1 TQFT.
The B-color encodes the 1-dim higher center.
We call A the trivial color and B the bulk color.
If we run the alterfold construction twice, then the $n+2$ manifold invariants is trivia, due to the triviality of A-color. This phenomenon has been considered as the center of the center is trivial.

Embedding Theory

The alterfold TQFT contains various embedding theorems.
For 1+1 dimensions: Gelfund-Naimark-Segal construction
For 2+1 dimensions:



Case 1: Planar algebras $\subseteq$ Graph planar algebras [**Kaw00, JonPen10, MorWal12** etc]
Case 2: α-induction. [**Lon95, Xu98, BEK98-99**]
Case 3: Ground states of Levin-Wen model [**LevWen05**]

Program: Functional Integral Construction of TQFT

We construct a unitary $n+1$ alterfold TQFT from a linear functional Z on labelled, regular, stratified, piecewise linear, n-manifolds, with three properties RP, HI, CF.
We propose a new program on the functional integral construction of $n+1$ altefold TQFT from such a linear functional Z and propose new methods to construct Z.

1. The linear functional Z can be considered as the Feynman path-integral on a lattice model.
2. The linear functional Z can be considered the ground state of a Hamiltonian in topological orders.
3. The linear functional Z can be considered as the evaluation map in a higher category.

Ising 3+1 unitary TQFT

We construct a concrete linear functional Z with the three properties (RP), (HI), (CF).

Theorem (Liu24)

For connected closed surfaces S_i in S^3 with Euler number E_i, $1 \leq i = n$, we define the linear functional as

$$Z(\bigcup_{i=1}^{n} S_i \subset S^3) := \prod_{i=1}^{n} 2^{1-E_i/4}.$$

Then Z is isotopy invariant, reflection positive and completely finite. Therefore, it produces an 3+1 unitary (alterfold) TQFT.

Remark: We assume that the surfaces may intersect. Otherwise we lose the CF condition.

Indeed, the 3D Ising model can be construct as a lattice model on the boundary of this 3+1 TQFT.

Δ^1: 1-morphisms

There is one 0-morphism, corresponding to the background D^3.
There are three 1-morphisms, $\{1, \tau, g\}$, with Ising type fusion rule:
$\tau^2 = 1 \oplus g$, $g^2 = 1$, $g\tau = \tau$.
Here τ is the 2-dimensional face. And g is the 1-cell transversal section of a double-surface attached with a red line:

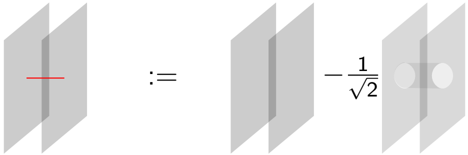


Figure: The red line in the double-surfaces

Δ^2: 2-morphisms

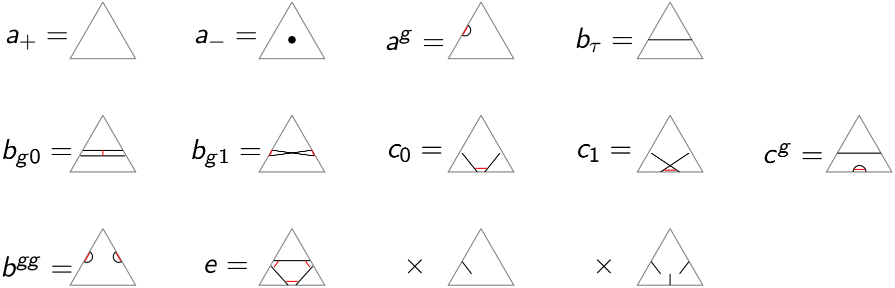


Figure: 2-morphisms

Both b^{gg} and e are zero, as they are in the kernel of Z. The last two are not admissible.

Δ^3: 3-morphisms

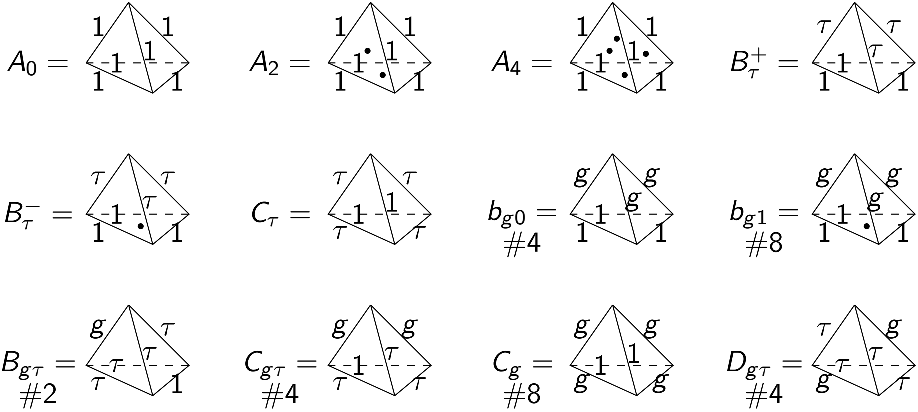
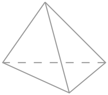
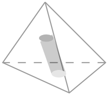
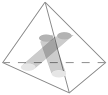


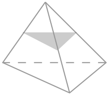
Figure: 3-morphisms

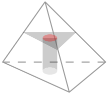
Δ^4: 4-morphisms

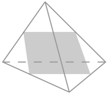
$A_0 =$ 

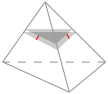
$A_2 =$ 

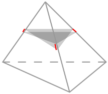
$A_4 =$ 

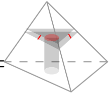
$B_\tau^+ =$ 

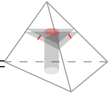
$B_\tau^- =$ 

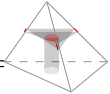
$C_\tau =$ 

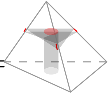
$B_{g0}^+ =$ 

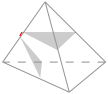
$B_{g1}^+ =$ 

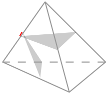
$B_{g0}^{-(a)} =$ 

$B_{g0}^{-(b)} =$ 

$B_{g1}^{-(a)} =$ 

$B_{g1}^{-(b)} =$ 

$B_{g0\tau} =$ 

$B_{g1\tau} =$ 

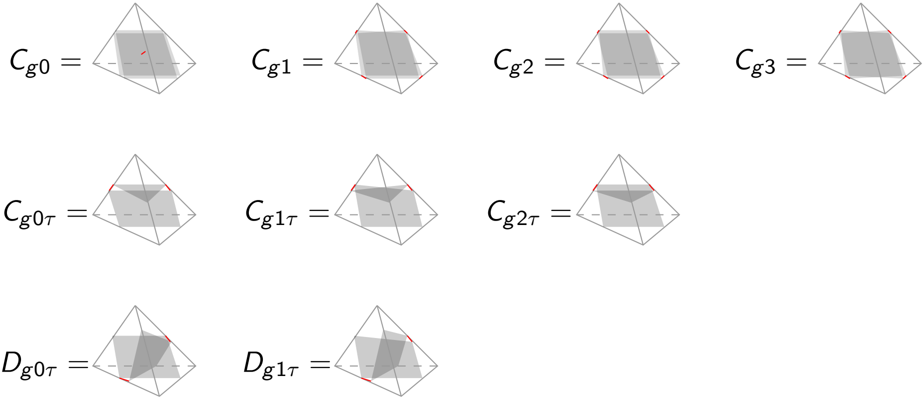


Figure: 4-morphims

Δ^5: 20j-symbols

id	12	13	14	15	23	24	25	34	35	45	123	124	125	134	135	145	234	235	245	345	1234	1235	1245	1345	2345
0^a	1	1	1	1	1	1	1	1	1	1	a_+	a_+	a_+	a_+	a_+	a_+	a_+	a_+	a_+	a_+	A_0	A_0	A_0	A_0	A_0
0^b	1	1	1	1	1	1	1	1	1	1	a_-	a_+	a_+	a_-	a_-	a_+	a_+	a_+	a_+	a_+	A_0	A_0	A_0	A_0	A_0
0^c	1	1	1	1	1	1	1	1	1	1	a_-	a_-	a_+	a_+	a_-	a_-	a_+	a_+	a_+	a_+	A_2	A_2	A_0	A_2	A_0
0^d	1	1	1	1	1	1	1	1	1	1	a_-	a_-	a_+	a_-	a_-	a_+	a_-	a_+	a_-	a_+	A_2	A_2	A_2	A_2	A_0
0^e	1	1	1	1	1	1	1	1	1	1	a_-	a_-	a_-	a_-	a_-	a_+	a_-	a_-	a_+	a_+	A_4	A_4	A_2	A_2	A_2
0^f	1	1	1	1	1	1	1	1	1	1	a_-	a_-	a_-	a_-	a_-	a_-	a_-	a_-	a_-	a_-	A_4	A_4	A_4	A_4	A_4
1_τ^a	τ	τ	τ	τ	1	1	1	1	1	1	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_+	a_+	a_+	a_+	B_τ^+	B_τ^+	B_τ^+	B_τ^+	A_0
1_τ^b	τ	τ	τ	τ	1	1	1	1	1	1	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_-	a_-	a_+	a_+	B_τ^-	B_τ^-	B_τ^+	B_τ^+	A_2
1_τ^c	τ	τ	τ	τ	1	1	1	1	1	1	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_-	a_-	a_-	a_-	B_τ^-	B_τ^-	B_τ^-	B_τ^-	A_4
2_τ^+	τ	1	τ	1	τ	1	τ	τ	1	τ	b_τ	b_τ	b_τ	b_τ	a_+	b_τ	b_τ	b_τ	b_τ	b_τ	C_τ	B_τ^+	C_τ	B_τ^+	C_τ
2_τ^-	τ	1	τ	1	τ	1	τ	τ	1	τ	b_τ	b_τ	b_τ	b_τ	a_-	b_τ	b_τ	b_τ	b_τ	b_τ	C_τ	B_τ^-	C_τ	B_τ^+	C_τ
1_{g0}^a	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	a_+	a_+	a_+	a_+	B_{g0}^+	B_{g0}^+	B_{g0}^+	B_{g0}^+	A_0
1_{g1}^a	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g1}	b_{g0}	b_{g1}	b_{g1}	a_+	a_+	a_+	a_+	B_{g0}^+	B_{g1}^+	B_{g1}^+	B_{g1}^+	A_0
1_{g0}^b	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	a_-	a_-	a_+	a_+	B_{g0}^-	B_{g0}^-	B_{g0}^+	B_{g0}^+	A_2
1_{g1}^b	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g1}	b_{g0}	b_{g1}	b_{g1}	a_-	a_-	a_+	a_+	B_{g0}^-	B_{g1}^-	B_{g1}^+	B_{g1}^+	A_2
1_{g0}^c	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	a_-	a_-	a_-	a_-	B_{g0}^-	B_{g0}^-	B_{g0}^-	B_{g0}^-	A_4
1_{g1}^c	g	g	g	g	1	1	1	1	1	1	b_{g0}	b_{g0}	b_{g1}	b_{g0}	b_{g1}	b_{g1}	a_-	a_-	a_+	a_+	B_{g0}^-	B_{g1}^-	B_{g1}^-	B_{g1}^-	A_4
$1_{g0\tau}^+$	g	τ	τ	τ	τ	τ	τ	1	1	1	c_0	c_0	c_0	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_+	$B_{g0\tau}$	$B_{g0\tau}$	$B_{g0\tau}$	B_τ^+	B_τ^+
$1_{g0\tau}^+$	g	τ	τ	τ	τ	τ	τ	1	1	1	c_1	c_1	c_1	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_+	$B_{g1\tau}$	$B_{g1\tau}$	$B_{g1\tau}$	B_τ^+	B_τ^+
$1_{g0\tau}^-$	g	τ	τ	τ	τ	τ	τ	1	1	1	c_0	c_0	c_0	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_-	$B_{g0\tau}$	$B_{g0\tau}$	$B_{g0\tau}$	B_τ^-	B_τ^+
$1_{g0\tau}^-$	g	τ	τ	τ	τ	τ	τ	1	1	1	c_0	c_0	c_0	b_τ	b_τ	b_τ	b_τ	b_τ	b_τ	a_-	$B_{g1\tau}$	$B_{g1\tau}$	$B_{g1\tau}$	B_τ^-	B_τ^+
$2_{g\tau}^+$	g	τ	g	g	τ	1	1	τ	τ	1	c	b_g	b_g	c	c	b_g	b_τ	b_τ	a_+	b_τ	$C_{g\tau}$	$C_{g\tau}$	B_g^+	$C_{g\tau}$	B_τ^+
$2_{g\tau}^-$	g	τ	g	g	τ	1	1	τ	τ	1	c	b_g	b_g	c	c	b_g	b_τ	b_τ	a_-	b_τ	$C_{g\tau}$	$C_{g\tau}$	B_g^-	$C_{g\tau}$	B_τ^-
$2_{g\tau}^0$	g	τ	g	τ	τ	1	τ	τ	1	τ	c	b_g	c	c	b_τ	b_τ	b_τ	c	b_τ	b_τ	$C_{g\tau}$	$B_{g\tau}$	$C_{g\tau}$	$B_{g\tau}$	C_τ
2_{g0}^+	g	1	g	g	g	1	1	g	g	1	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	b_{g0}	a_+	b_{g0}	C_{g0}	C_{g0}	B_{g0}^+	C_{g0}	B_{g0}^+
2_{g1}^+	g	1	g	g	g	1	1	g	g	1	b_{g1}	b_{g0}	b_{g0}	b_{g1}	b_{g1}	b_{g0}	b_{g0}	b_{g0}	a_+	b_{g0}	C_{g1}	C_{g1}	B_{g0}^+	C_{g1}	B_{g0}^+
2_{g0}^+	g	1	g	g	g	1	1	g	g	1	b_{g0}	b_{g0}	b_{g1}	b_{g0}	b_{g0}	b_{g1}	b_{g0}	b_{g1}	a_+	b_{g1}	C_{g1}	C_{g1}	B_{g1}^+	C_{g1}	B_{g1}^+
2_g^-	g	1	g	g	g	1	1	g	g	1	b_g	b_g	b_g	b_g	b_g	b_g	b_g	b_g	a_-	b_g	C_g	C_g	B_g^-	C_g	B_g^-
$3_{g\tau}$	g	1	g	τ	g	1	τ	g	τ	τ	b_g	b_g	c	b_g	b_τ	c	b_g	c	b_τ	c	C_g	$C_{g\tau}$	$C_{g\tau}$	$C_{g\tau}$	$C_{g\tau}$
$4_{g\tau}$	1	τ	g	g	τ	g	g	τ	τ	1	b_τ	b_τ	b_g	b_τ	b_τ	c	b_g	c	b_τ	c	C_g	$C_{g\tau}$	$C_{g\tau}$	$C_{g\tau}$	$C_{g\tau}$

id	1234	1235	1245	1345	2345	σ^2	$\widetilde{F}$	F
0^a	A_0	A_0	A_0	A_0	A_0	$\emptyset$	1	1
0^b	A_2	A_2	A_0	A_2	A_0	T_{-1}	1	1
0^c	A_2	A_2	A_2	A_2	A_0	T_{-1}	1	1
0^d	A_4	A_2	A_2	A_2	A_2	$2T_{-1}$	1	1
0^e	A_4	A_4	A_2	A_2	A_2	$2T_{-1}$	1	1
0^f	A_4	A_4	A_4	A_4	A_4	$3T_{-1}$	1	1
1_τ^a	B_τ^+	B_τ^+	B_τ^+	B_τ^+	A_0	S_τ	$\sqrt{2}$	$\sqrt{2}/2$
1_τ^b	B_τ^-	B_τ^-	B_τ^+	B_τ^+	A_2	$T_{\tau-}$	1	$\sqrt{2}/2$
1_τ^c	B_τ^-	B_τ^-	B_τ^-	B_τ^-	A_4	$T_{\tau-}^2$	$\sqrt{2}/2$	$\sqrt{2}/2$
2_τ^+	C_τ	B_τ^+	C_τ	B_τ^+	C_τ	S_τ	$\sqrt{2}$	$\sqrt{2}/2^{5/4}$
2_τ^-	C_τ	B_τ^-	C_τ	B_τ^-	C_τ	$T_{\tau-}$	1	$1/2^{3/4}$
1_{g0}^a	B_{g0}^+	B_{g0}^+	B_{g0}^+	B_{g0}^+	A_0	S_g	1	1
1_{g0}^b	B_{g0}^-	B_{g0}^-	B_{g0}^+	B_{g0}^+	A_2	$T_{g-}=S_\tau T_{\tau-}-\frac{1}{\sqrt{2}}T_{\tau-}$	$\sqrt{2}-\sqrt{2}/2$	1
1_{g0}^c	B_{g0}^-	B_{g0}^-	B_{g0}^-	B_{g0}^-	A_4	$T_{g-}^2=S_\tau T_{\tau-}^2-\frac{1}{\sqrt{2}}T_{\tau-}^2$	$1/2$	1
$1_{g0\tau}^+$	$B_{g0\tau}$	$B_{g0\tau}$	$B_{g0\tau}$	B_τ^+	B_τ^+	S_g	$2-1$	$1/\sqrt{2}$
$1_{g0\tau}^-$	$B_{g0\tau}$	$B_{g0\tau}$	$B_{g0\tau}$	B_τ^-	B_τ^-	$\frac{1}{\sqrt{2}}S_g$	$1/\sqrt{2}$	$1/\sqrt{2}$
$2_{g0\tau}^+$	$C_{g0\tau}$	$C_{g0\tau}$	B_{g0}^+	$C_{g0\tau}$	B_τ^+	S_g	$2-1$	$1/2^{1/4}$
$2_{g0\tau}^-$	$C_{g0\tau}$	$C_{g0\tau}$	B_{g0}^-	$C_{g0\tau}$	B_τ^-	$\frac{1}{\sqrt{2}}S_g$	$1/\sqrt{2}$	$1/2^{1/4}$
$2_{g0\tau}^0$	$C_{g0\tau}$	$B_{g0\tau}$	$C_{g0\tau}$	$B_{g0\tau}$	C_τ	S_g	$2-1$	$1/2^{1/4}$
2_{g0}^+	C_{g0}	C_{g0}	B_{g0}^+	C_{g0}	B_{g0}^+	S_g	$2-1$	1
2_{g0}^-	C_{g0}	C_{g0}	B_{g0}^-	C_{g0}	B_{g0}^-	$T_{g-}=S_{\tau-}T_{\tau-}-\frac{1}{\sqrt{2}}T_{\tau-}$	$\sqrt{2}/2$	1
$3_{g0\tau}$	C_{g0}	$C_{g0\tau}$	$C_{g0\tau}$	$C_{g0\tau}$	$C_{g0\tau}$	S_g	$2-1$	1
$4_{g0\tau}$	C_{g0}	$C_{g0\tau}$	$C_{g0\tau}$	$C_{g0\tau}$	$C_{g0\tau}$	S_g	$2-1$	1

TABLE 2. Table of bulks, surfaces and values of unnormalized F-symbol

Δ^6: Pancher Move

Theorem (Liu)

The 20j symbols satisfies the (3-3) (4-2) (5-1) Pancher Move, a one-dimensional higher analogue of the (3-2) pentagon equation.

We have verified all Pachner moves of 20j-symbols on a computer. There are 2044 (1-5) equations, 30464 (4-2) equations and 50709 (3-3) equations.
Remark: Scalar invariant of 2-knots in smooth 4-manifolds can be computed explicitly using the 20j-symbols.

Conjecture

Conjecture

For connected closed hyper surfaces S_i in S^n with Euler number E_i, $1 \leq i = n$, we define the linear functional as

$$Z(\bigcup_{i=1}^{n} S_i \subset S^n) := \prod_{i=1}^{n} 2^{1-E_i/4}.$$

Then Z is isotopy invariant, reflection positive and completely finite. Therefore, it produces an n+1 unitary alterfold TQFT.

Concluding Remarks

We propose a new program to construct a meaningful unitary topological quantum field theory (TQFT) in arbitrary dimension. It is a lattice model approach from a functional integral point of view.

We construct a unitary $n+1$ alterfold TQFT from a linear functional Z from a linear functional Z on labelled, regular, stratified, piecewise linear, n-manifolds, with three properties RP, HI, CF.

1. (RP) reflection positivity;
2. (HI) homeomorphic invariance;
3. (CF) complete finiteness.

It provides an answer to Wen's question on the characterization of topological orders from ground states.

We construct an Ising type unitary 3-category, which is the first example of non-invertible unitary 3-category with explicit 20j symbols. Moreover, we obtain a 3+1 unitary alterfold TQFT, which provides invariants of 2-knots in smooth 4-manifolds.

Further Connections

Lattice Hamiltonian $\rightarrow$ Linear Functional on Lattice Models $\rightarrow$ D^n algebras $\rightarrow$ Higher representation theory $\rightarrow$ unitary/spherical n-categories $\rightarrow$ unitary n+1 Alterfold TQFT

Construct the linear functional Z from a lattice Hamiltonian H and a boundary state Ω,

$$Z = e^{\beta H}\Omega.$$

Remark: $H = 0$ in Atiyah's TQFT and $\beta = \infty$ in topological orders. The conditions are not required in our approach.

Remark: The theory is topological, because the local Hamiltonian is independent of the matrix of the manifold. It will be interesting to adjust this condition and further explore the geometric structure and quantum gravity.

Functional Integral Construction of TQFT
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Thank you for your attention!