# Time-reversal symmetry for quantum measurements

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based on joint work with James Fullwood (Hainan University)

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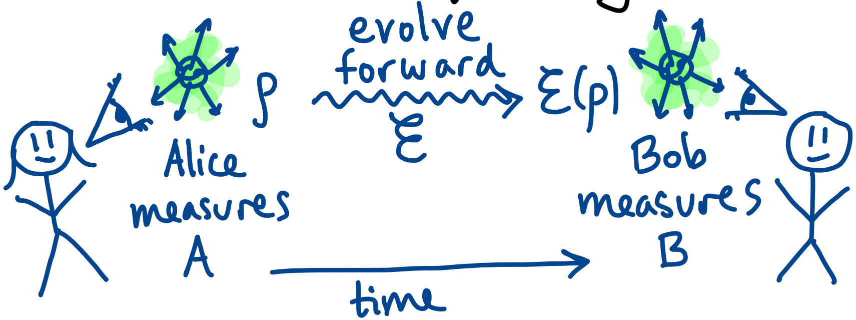
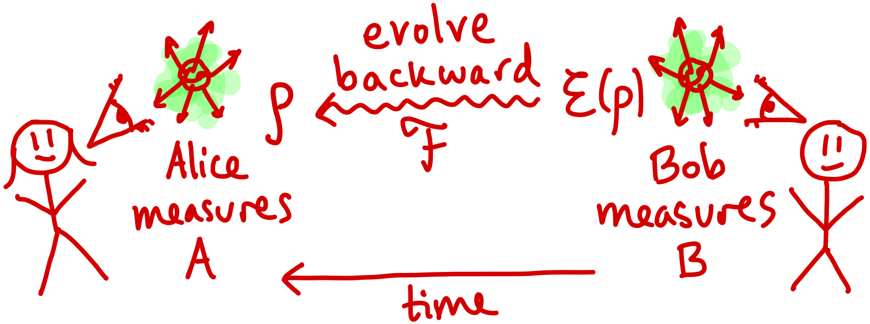
# Main idea

## Quantum symmetry?




? reverse procedure ↓ symmetric expectation values ?




When does $\mathcal{F}$ exist?

## Classical symmetry

$X, Y$ finite sets of configurations

$\mathbb{P}(y|x)$ transition probability

$\mathbb{P}(x)$ probability of state $x \in X$

$\mathbb{P}(y) = \sum_x \mathbb{P}(y|x)\mathbb{P}(x)$ probability of $y \in Y$

$f, g$ observables on $X, Y$, respectively

$\Rightarrow \langle f, g \rangle = \sum_{x,y} \mathbb{P}(y|x)\mathbb{P}(x) f(x) g(y)$

expected value of <u>first</u> measuring $f(x)$ and <u>then</u> $g(y)$

Bayes' rule

$= \sum_{x,y} \mathbb{P}(x|y)\mathbb{P}(y) g(y) f(x)$

$= \langle g, f \rangle \leftarrow$ <u>first</u> $g(y)$ <u>then</u> $f(x)$

$\mathbb{P}(x|y)$ always exists!

# Static expectation values in quantum

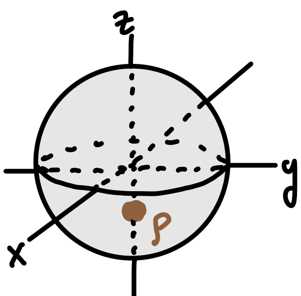
**Notation** $\mathcal{A}, \mathcal{B}$ denote matrix algebras

**Example** $\mathcal{A} = \mathbb{M}_2$ is $2 \times 2$ complex matrices (algebra of a qubit)

**Defn** Let $\rho \in \mathcal{A}$ be a density matrix. Let $A \in \mathcal{A}$ be an observable. The **expectation value** of $A$ with respect to $\rho$ is the real number

$$\langle A \rangle_\rho := \mathrm{Tr}[\rho A] = \sum_i \lambda_i \mathrm{Tr}[\rho P_i] = \sum_i \lambda_i \mathbb{P}(i), \text{ with } A = \sum_i \lambda_i P_i \text{ spectral decomp.}$$

**Example** $\rho = \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}$ describes a state in which a qubit has 1/3 (2/3) chance of being spin up (down) along $z$. Hence,




$$\langle \sigma_z \rangle_\rho = \mathrm{Tr}\left[\begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = -1/3$$

is the expected value of the spin in the $z$ direction. Meanwhile,

$$\langle \sigma_x \rangle_\rho = \mathrm{Tr}\left[\begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = 0$$

is the expected value of the spin in the $x$ direction.

# Dynamic expectation values in quantum

**Defn** Let $\rho \in \mathcal{A}$ be a density matrix and let $\mathcal{E}: \mathcal{A} \to \mathcal{B}$ be a quantum channel, i.e., a completely positive trace-preserving (CPTP) map.
Let $A \in \mathcal{A}$, $B \in \mathcal{B}$ be observables with spectral decompositions
$A = \sum_i \lambda_i P_i$ and $B = \sum_j \mu_j Q_j$ ($\lambda_i, \mu_j \in \mathbb{R}$ & $P_i \in \mathcal{A}$, $Q_j \in \mathcal{B}$ projectors).
The <u>two-time expectation value</u> of $A$ and $B$ with respect to the pair $(\rho, \mathcal{E})$ is the real number $\langle A, B \rangle_{(\rho,\mathcal{E})} := \sum_{i,j} \lambda_i \mu_j \operatorname{Tr}\left[\mathcal{E}(P_i \rho P_i)\, Q_j\right]$.

**Remark** The state after measuring $A$ and getting outcome $\lambda_i$ is $\rho_i = \frac{P_i \rho P_i}{\operatorname{Tr}[\rho P_i]}$ due to the state-update rule, while the probability is $\mathbb{P}(i) = \operatorname{Tr}[\rho P_i]$.
Thus, $\langle A \rangle_\rho = \sum_i \lambda_i \mathbb{P}(i)$ and $\langle B \rangle_{\mathcal{E}(\rho_i)} = \operatorname{Tr}[\mathcal{E}(\rho_i) B]$. Hence,
$\langle A, B \rangle_{(\rho,\mathcal{E})} = \sum_i \lambda_i \mathbb{P}(i) \langle B \rangle_{\mathcal{E}(\rho_i)} = \sum_{i,j} \lambda_i \mu_j \mathbb{P}(j|i)\, \mathbb{P}(i)$, where
$\mathbb{P}(j|i) = \operatorname{Tr}[\mathcal{E}(\rho_i) Q_j]$ is the conditional probability of $j$ given $i$.

# Example with amplitude-damping channel I

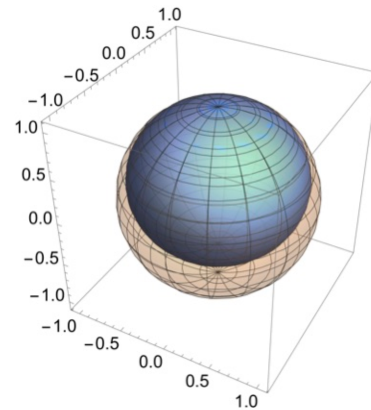
A qubit density matrix $\rho$ is of the form $\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$ with $\|(x,y,z)\|^2 \leq 1$ and $\therefore$ identifies with a point in Bloch ball. Let $\mathcal{E}$ be the evolution

$\mathcal{E} = \mathrm{Ad}_{E_0} + \mathrm{Ad}_{E_1}$ $(\mathrm{Ad}_V(A) = VAV^\dagger)$ w/

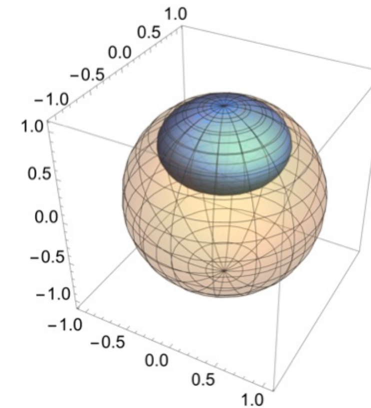
$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$, $\gamma \in (0,1)$.

A generic state $\rho$ gets sent to

$$\mathcal{E}(\rho) = \frac{1}{2}\begin{pmatrix} 1+\gamma+z(1-\gamma) & (x-iy)\sqrt{1-\gamma} \\ (x+iy)\sqrt{1-\gamma} & 1-\gamma-z(1-\gamma) \end{pmatrix}.$$




$\gamma = 0.2$




$\gamma = 0.6$

The image of the Bloch ball is shown in blue for two values of $\gamma \in [0,1]$.

Before calculating two-time expectation values for this example, let us go back to the general setup.

# Calculating two-time expectation values

Calculating $\langle A, B\rangle_{(\rho,\mathcal{E})} = \sum_i \lambda_i \operatorname{Tr}[\mathcal{E}(P_i \rho P_i) B]$ involves a lot of work (eg. spectral decomp of $A$ & evaluating $\mathcal{E}(P_i \rho P_i)$ $\forall i$).

A more convenient approach can be achieved by choosing a special set of (tomographically complete) observables.

**Defn** An observable $A \in \mathcal{A}$ is ***light touch*** iff its spectrum is either $\{-\lambda, \lambda\}$ for some $\lambda > 0$ or $\{\lambda\}$ for some $\lambda \in \mathbb{R}$.

**Example** $\mathcal{A} = \mathbb{M}_2$ Pauli matrices $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$ are all light touch.

Also, if $\vec{n} \in S^2 \subseteq \mathbb{R}^3$, then $\vec{n}\cdot\vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$ is, too.

**Theorem** Let $(\rho, \mathcal{E})$ be a process. Then there exists a unique matrix $X \in \mathcal{A} \otimes \mathcal{B}$ such that $\langle A, B\rangle_{(\rho,\mathcal{E})} = \operatorname{Tr}[X\, A \otimes B]$ for all light touch observables $A \in \mathcal{A}$ and all observables $B \in \mathcal{B}$.

# The operator representing two-time expectation values

**Theorem** Let $(\rho, \mathcal{E})$ be a process. Then there exists a unique matrix $X \in \mathcal{A} \otimes \mathcal{B}$ such that $\langle A, B \rangle_{(\rho,\mathcal{E})} = \mathrm{Tr}[X \, A \otimes B]$ for all light touch observables $A \in \mathcal{A}$ and all observables $B \in \mathcal{B}$.

**Remark** The operator $X$ in this theorem is a "state over time" and is calculated by $X = \frac{1}{2}\{\rho \otimes \mathbb{1}_B, \mathcal{J}[\mathcal{E}]\}$, where $\{f,g\} = fg + gf$ and

$$\mathcal{J}[\mathcal{E}] = (\mathrm{id}_A \otimes \mathcal{E})(\mathrm{SWAP}) \equiv \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}(|j\rangle\langle i|) = (\underset{\text{transpose}}{T} \otimes \mathrm{id}_B)(\overbrace{\mathcal{C}[\mathcal{E}]}^{\text{Choi}(\mathcal{E})}).$$

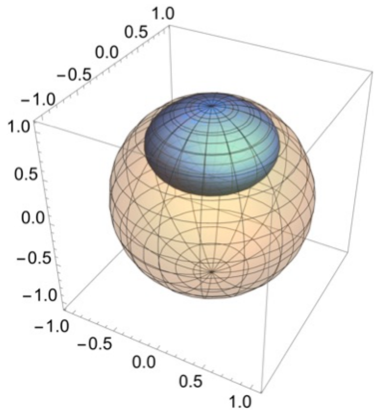
**Defn** A state over time for $(\rho, \mathcal{E})$ is an $X \in \mathcal{A} \otimes \mathcal{B}$ s.t. $\mathrm{Tr}_B(X) = \rho$ & $\mathrm{Tr}_A(X) = \mathcal{E}(\rho)$.

**Example** For qubits

$$\mathcal{J}[\mathcal{E}] = \begin{pmatrix} \mathcal{E}(|0\rangle\langle 0|) & \mathcal{E}(|1\rangle\langle 0|) \\ \mathcal{E}(|0\rangle\langle 1|) & \mathcal{E}(|1\rangle\langle 1|) \end{pmatrix} = \begin{pmatrix} \mathcal{E}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \mathcal{E}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \mathcal{E}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \mathcal{E}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

I now want to show you how to compute the expectation values!

# Example with amplitude-damping channel II




$\mathcal{E}$ w/ $\gamma = 0.6$

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}, \quad \mathcal{E} = \mathrm{Ad}_{E_0} + \mathrm{Ad}_{E_1}, \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$\rho \otimes \mathbb{1}_2 = \frac{1}{2}\begin{pmatrix} 1+z & 0 & 0 & 0 \\ 0 & 1+z & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix}, \quad \mathcal{J}[\mathcal{E}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & \sqrt{1-\gamma} & \gamma & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix}$$

$$X = \frac{1}{2}\{\rho \otimes \mathbb{1}_2, \mathcal{J}[\mathcal{E}]\} = \frac{1}{2}\begin{pmatrix} 1+z & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & \sqrt{1-\gamma} & \gamma(1-z) & 0 \\ 0 & 0 & 0 & (1-\gamma)(1-z) \end{pmatrix}$$

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|---|---|---|---|---|
| $\sigma_0$ | $1$ | $0$ | $0$ | $z+\gamma(1-z)$ |
| $\sigma_1$ | $0$ | $\sqrt{1-\gamma}$ | $0$ | $0$ |
| $\sigma_2$ | $0$ | $0$ | $\sqrt{1-\gamma}$ | $0$ |
| $\sigma_3$ | $z$ | $0$ | $0$ | $1-\gamma(1-z)$ |

Two-time expectation values

$\langle \sigma_\alpha, \sigma_\beta \rangle_{(\rho,\mathcal{E})} = \mathrm{Tr}[X\, \sigma_\alpha \otimes \sigma_\beta]$ w/ Pauli $\sigma_\alpha$ in top row measured first and then (after $\mathcal{E}$) Pauli $\sigma_\beta$ in left column measured second.

# Operational and Bayesian inverses

**Defn** An *operational inverse* of $(\rho, \mathcal{E})$ is a channel $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$ s.t. $\langle A, B \rangle_{(\rho, \mathcal{E})} = \langle B, A \rangle_{(\mathcal{E}(\rho), \mathcal{F})}$ for all light touch observables $A \in \mathcal{A}$, $B \in \mathcal{B}$.

This is a form of time-reversal symmetry! (quantum Bayes' rule)

**Theorem** Given $(\rho, \mathcal{E})$, an operational inverse exists if and only if there exists a channel $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$ s.t. $\{\mathbb{1}_{\mathcal{B}} \otimes \rho, \mathcal{J}[\mathcal{E}^*]\} = \{\mathcal{E}(\rho) \otimes \mathbb{1}_{\mathcal{A}}, \mathcal{J}[\mathcal{F}]\}$, where $\mathcal{E}^* = \sum_n \mathrm{Ad}_{E_n^\dagger}$ is the Hilbert-Schmidt adjoint of $\mathcal{E}$.

**Procedure** To solve for $\mathcal{J}[\mathcal{F}]$, write $\rho = \sum_i p_i |i\rangle\langle i|$, $\mathcal{E}(\rho) = \sum_k q_k |k\rangle\langle k|$, and $\mathcal{J}[\mathcal{E}^*] = \sum_{k,l} |k\rangle\langle l| \otimes \mathcal{E}^*(|l\rangle\langle k|)$ using this eigenbasis of $\mathcal{E}(\rho)$.

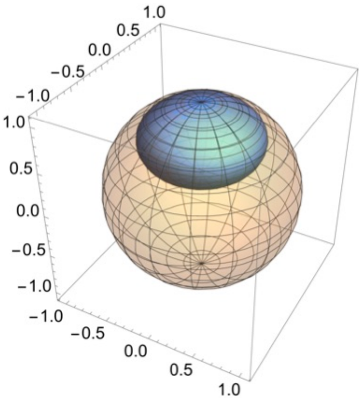
Comparing the two sides and using linear independence of $\{|k\rangle\langle l|\}$ gives $\mathcal{F}(|k\rangle\langle l|) = \dfrac{\{\rho, \mathcal{E}^*(|k\rangle\langle l|)\}}{q_k + q_l}$ which determines $\mathcal{F}$ by linearity.

# Example with amplitude-damping channel III

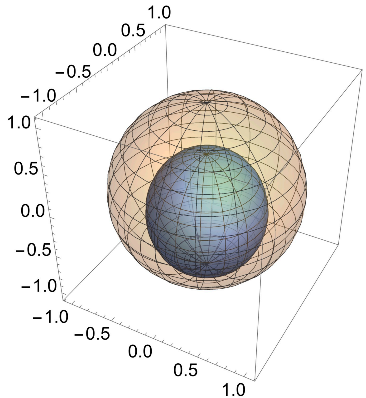
**Theorem** Given $\rho = \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}$, an operational inverse of $(\rho, \mathcal{E})$ exists if and only if $z \geq \frac{\gamma}{\gamma - 2}$, in which case $\mathcal{F} = \mathrm{Ad}_{F_0} + \mathrm{Ad}_{F_1} + \mathrm{Ad}_{F_2}$, where

$$F_0 = \begin{pmatrix} \sqrt{\frac{1+z}{1+z'}} & 0 \\ 0 & \sqrt{\frac{(1-\gamma)(1+z')}{1+z}} \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{\gamma(1-z)}{1+z'}} & 0 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{\gamma(z+z')}{1+z}} \end{pmatrix}, \text{ and } z' = z + \gamma(1-z).$$



original $\mathcal{E}$ w/ $\gamma = 0.6$



Bayesian inverse $\mathcal{F}$ w/ $\gamma = 0.6, z = 0.2$

This is a bit-flipped amplitude-damping channel with dephasing.

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|---|---|---|---|---|
| $\sigma_0$ | 1 | 0 | 0 | $z'$ |
| $\sigma_1$ | 0 | $\sqrt{1-\gamma}$ | 0 | 0 |
| $\sigma_2$ | 0 | 0 | $\sqrt{1-\gamma}$ | 0 |
| $\sigma_3$ | $z$ | 0 | 0 | $1-\gamma(1-z)$ |

← Two-time expect. values $\langle \sigma_\alpha, \sigma_\beta \rangle$ w/ Pauli $\sigma_\alpha$ in top row measured first and then (after $\mathcal{E}$) Pauli $\sigma_\beta$ in left column measured second.

Same but time-reversed $\langle \sigma_\beta, \sigma_\alpha \rangle$ using Bayesian inv. $\mathcal{F}$ →

| | $\sigma_0$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ |
|---|---|---|---|---|
| $\sigma_0$ | 1 | 0 | 0 | $z$ |
| $\sigma_1$ | 0 | $\sqrt{1-\gamma}$ | 0 | 0 |
| $\sigma_2$ | 0 | 0 | $\sqrt{1-\gamma}$ | 0 |
| $\sigma_3$ | $z'$ | 0 | 0 | $1-\gamma(1-z)$ |

# Summary & Key points

- Just like expectation values of observables at a fixed time are encoded in a density matrix, two-time expectation values of sequential measurements of light-touch observables are encoded in a "state over time".
- Light touch observables are just as robust as projections in that their expectation values determine density matrices and states over time.
- Our theory of quantum Bayesian inverses provides a new and mostly unexplored aspect of quantum theory with operational consequences for time-reversal symmetric multi-time expectation values.
- Immense plethora of open questions and low-hanging fruit!

  Eg. ① Verify the predictions with our experimental proposal. ② Extend results for other types of observables. Great problems for grad students!

# Thank you!

All joint work with James Fullwood

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Perimeter Institute (Canada)

"On quantum states over time"

2206.03607

Bypasses a no-go theorem of Horsman et al, proving existence of a state over time that satisfies many conditions

"Operator representation of spatiotemporal quantum correlations"

2405.17555

Provides an operational meaning to states over time and extends pseudo-densities

"From time-reversal symmetry to quantum Bayes' rules"

2212.08088

A modern reference on states over time and how they give rise to a quantum extension of Bayes' rule

"Time-symmetric correlations for open quantum systems"

2407.11123

What this talk was mostly based on (prediction using quantum Bayes' rule)