Local Quantum Error Correcting Codes

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Solution: Fix a set of allowable messages, called code words, ahead of time. The code word closest to the received message should be the original message if not too many bit flips occurred.

Classical linear codes

Let C_0, C_1 be $\mathbb{F}_2$ vector spaces with a distinguished set of basis representatives.
Let H_1 be a linear map between them.

$$C_1 \xrightarrow{H_1} C_0$$

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A bit of information = a coordinate / basis vector in C_1.

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Distance of $\mathcal{C}$ is,

$$dist(\mathcal{C}) = min\{|w| : w \in ker(H_1)\}$$

where $|w|$ is Hamming weight of w.

This code can correct $< dist(\mathcal{C})/2$ bit flip errors.

Representation of quantum error correcting codes

A quantum CSS code $\mathcal{C}$ is represented by a 3-term chain complex of $\mathbb{F}_2$ vector spaces, with a distinguished set of basis representatives.

$$C_2 \xrightarrow{H_2^T} C_1 \xrightarrow{H_1} C_0$$

$$H_1 \circ H_2^T = 0$$

A qubit = basis vector of C_1.

A logical qubit = basis vector in $ker(H_1)/Im(H_2^T)$
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Connection to quantum physics

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Each row of H_2 corresponds to X-type Pauli operator and each row of H_1 corresponds to Pauli Z-type operator in S. Commutativity is given by the chain complex condition. The Stabilizer code is

$$\mathcal{C} = span(\sum_{y \in Im(H_2^T)}' |z + y\rangle : z \in Ker(H_1))$$

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Measuring X-type operator detects phase flip errors. Applying Hadamard gate and measuring Z-type operator detects bit flip errors.

Good quantum codes

Panteleev-Kalachev, Leverrier-Zemor '22

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Difficult construction that combines expander graphs with symmetries and some special random classical codes. Also uses some analogies from topology.

Additional physical restrictions

$$C_2 \xrightarrow{H_2^T} C_1 \xrightarrow{H_1} C_0$$

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In a quantum computer, qubits are positioned on lattice points of a 3-dimensional lattice.

A CSS code $\mathcal{C}$ is called **local** if there is an injective map,

$$f : \{\textbf{qubits}\} \to \mathbb{Z}^3$$

and the image of the qubits in any check lies in some ball of radius $O(1)$.

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The distance is sharp up to a constant by Bravyi-Terhal '08.

Much easier to understand being local from a geometric perspective.

Geometry of manifolds

Let M be a triangulated manifold.

The space of i-chains: $C_i(M) = \mathbb{F}_2^{\{\text{i-simplices of } M\}}$

Boundary maps: $\partial_i : C_i(M) \to C_{i-1}(M)$ are given by the triangulation.

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Example (Kitaev's toric code):
Let T be a standard $2d$-dimensional torus with V simplices.
We get a CSS code from it:

$$C_{d+1}(T) \xrightarrow{H_2^T = \partial_{d+1}} C_d(T) \xrightarrow{H_1 = \partial_d} C_{d-1}(T)$$

Geometry of manifolds

The **k-homology group** of M is

$$H_k(M;\mathbb{F}_2) = ker(\partial_k)/Im(\partial_{k+1})$$

The **k-systole** of M is

$$sys_k(M) = min\{|w| : w \in ker(\partial_k)/Im(\partial_{k+1})\}$$

k-homology is the space of k-dimensional holes in M.
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Hard (but solved) problem in systolic geometry:
Is there a $2d$-manifold of size V with $sys_d(M) >> V^{1/2}$?

Geometry of manifolds

A map $F : M \to \mathbb{R}^3$ is **coarse** if the image of each simplex has diameter $O(1)$ and the pre-image of each unit ball in $\mathbb{R}^3$ intersects $O(1)$ simplices in M.

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Idea: A complicated manifold admits a coarse map to $\mathbb{R}^3$ if it looks 2-dimensional.

Correspondence between codes and manifolds

CSS code $\mathcal{C}$	$2d$-dimensional triangulated manifold M
qubit	d-simplex in M
logical qubit	basis element in d dimensional homology $H_d(M; \mathbb{F}_2)$
$dist(\mathcal{C})$	$\mathrm{sys}_d(M)$
$\mathcal{C}$ is local	there is a coarse map $F : M \to \mathbb{R}^3$

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$2d$-dimensional manifold $M \to$ CSS code C:

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CSS code $\mathcal{C} \to$ Manifold: Result of Freedman-Hastings '20.

Not a natural construction. Requires $d \geq 6$.
Also requires a technical condition on the CSS code, that is satisfied by most important codes we know.

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Apply FH construction to get a $2d$-dimensional triangulated manifold M with V simplices and $sys_d(M) \approx V$.

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Step 2: Analysis of the FH construction shows that there is a 2-dimensional simplicial complex X which approximates M.

Given by a coarse map

$$G : M \to X$$

The pre-image of each simplex in X intersects $O(1)$ simplices in M and the image of each simplex intersects $O(1)$ simplices in X.

d-cycles which represent the d systole in M roughly get mapped to 1-cycles in X.

Sketch of proof of main theorem

Step 3: Generalizing a result of Kolmogorov and Barzdin '93, we can construct a map

$$I : X \to B^3(V)$$

so that the pre-image of each unit ball intersects $O(1)$ simplices in X. However, I streches simplices of X to diameter $\approx V$ and area $\approx V^2$.

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Step 4: Subdivide X to $\tilde{X}$ and M to $\tilde{M}$ so that

$$F = I \circ G : \tilde{M} \to \tilde{X} \to B^3(V)$$

is coarse. Most simplices in M get subdivided into $\approx V^2$ simplices.

As a result $\tilde{M}$ has $\approx V^3$ simplices and $sys_d(\tilde{M}) \geq V^2$

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Step 5: Convert $\tilde{M}$ to a new code with V^3 qubits and distance V^2.

The Kolmogorov-Barzdin theorem

Kolmogorov-Barzdin '93

Let Γ be a graph with V vertices and degree $O(1)$.
Let Λ be a unit grid in $\mathbb{R}^3$. Then there is an injective map

$$I : \Gamma \to B^3(O(1)V^{1/2}) \cap \Lambda$$

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3) Let $I(e)$ be composed of 2 segments going from $I(\partial e) \subset H_0$ to $H_{L(e)}$, and a path inside $H_{L(e)}$ connecting their endpoints.

Future direction

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Open question: Is there $3d$-manifold M with V simplices so that

1. $sys_{2d}(M) >> V^{2/3}$
2. There are three $2d$-homology classes which have a non-trivial mutual intersection in M

Thank you !