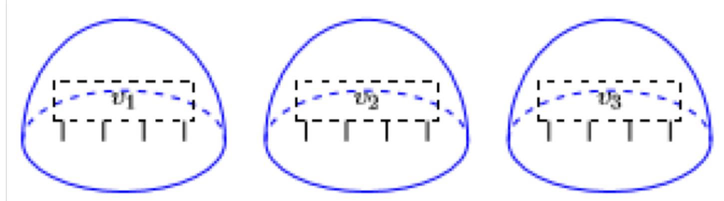
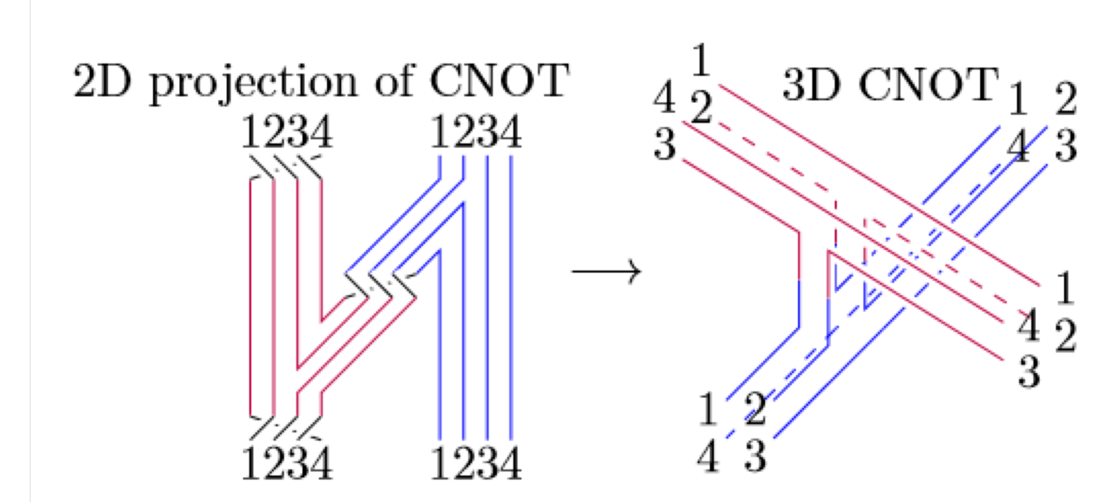
Mathematical Picture Language Project



Harvard University Department of Physics

https://mathpicture.fas.harvard.edu

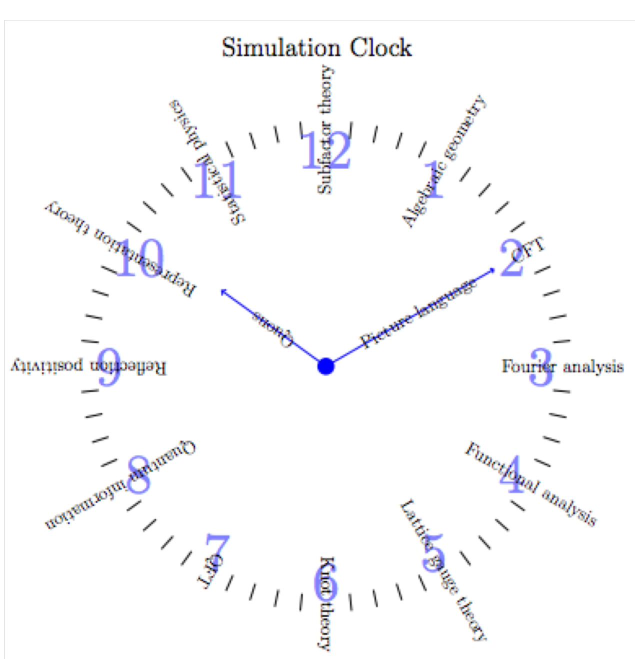


Abstract

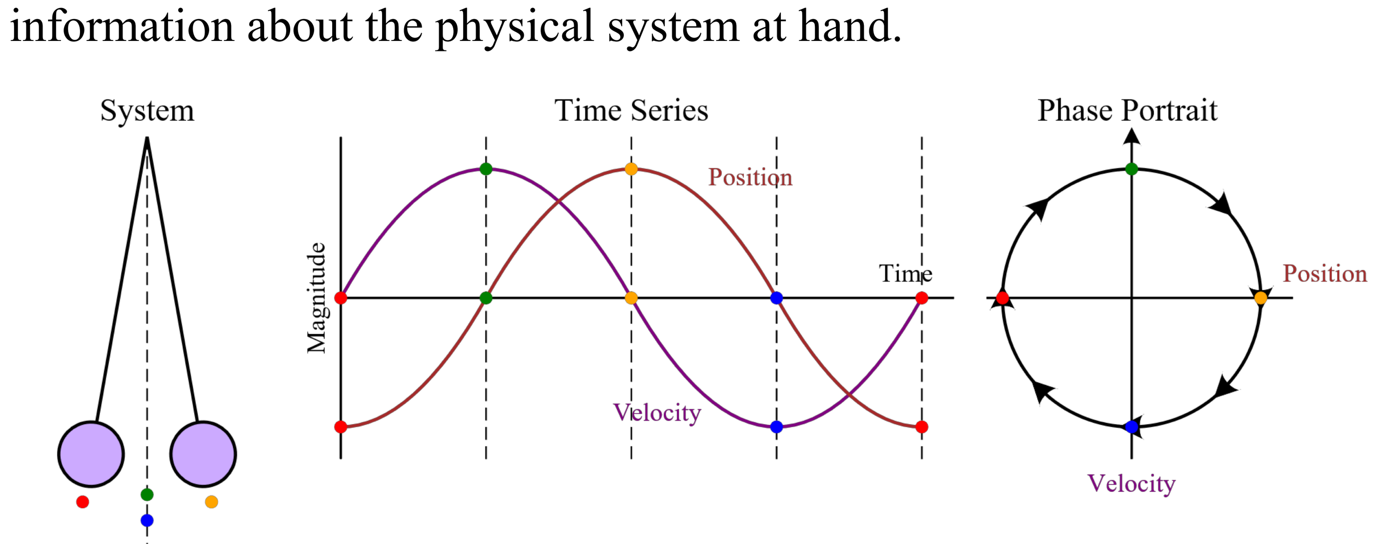
We reevaluate ways that one can use pictures, not only to gain mathematical insights, but also to prove mathematical theorems. As an example, we describe ways that the Quon language, invented to study quantum information, sheds light on several other areas of mathematics. It results in proofs and new algebraic identities of interest in several fields. We explain how this picture language affords mathematical insights. Motivated by this success, we outline a picture program for further research, with the goal of unifying ideas from different subjects in mathematics and physics.

Abstraction and simulation

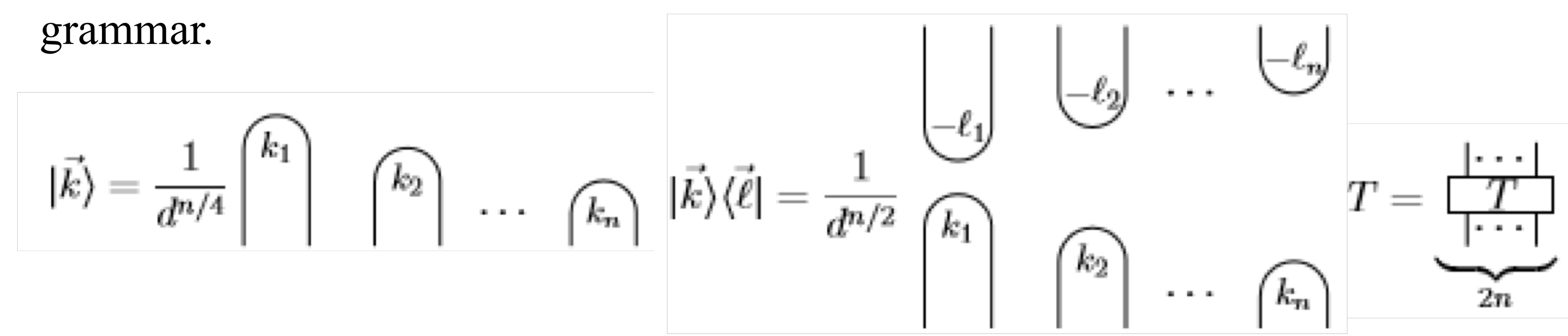
We propose two basic components to understanding that we call L and R. Here L denotes abstract concepts or language, while R stands for the concrete subjects or reality, which we desire to understand. We could also think of them as left and right. Simulation S represents a map from L to R. Our universe provides a great reservoir for ideas about the real world. We can consider this as R. We understand these ideas through abstraction, including theories of mathematics, physics, chemistry, and biology.



To deal with these real concepts, one often requires virtual concepts that have no meaning in the real world. These virtual concepts may not have an immediate real meaning, yet they may provide key insight to understanding the real structures. In the familiar example of a simple pendulum, classical mechanics uses a virtual space of coordinates (phase space) to model the trajectory of objects in real space over time. Virtual space is a mathematical abstraction that yields mathematical insights – the symmetries of paths in virtual space capture key information about the physical system at hand.

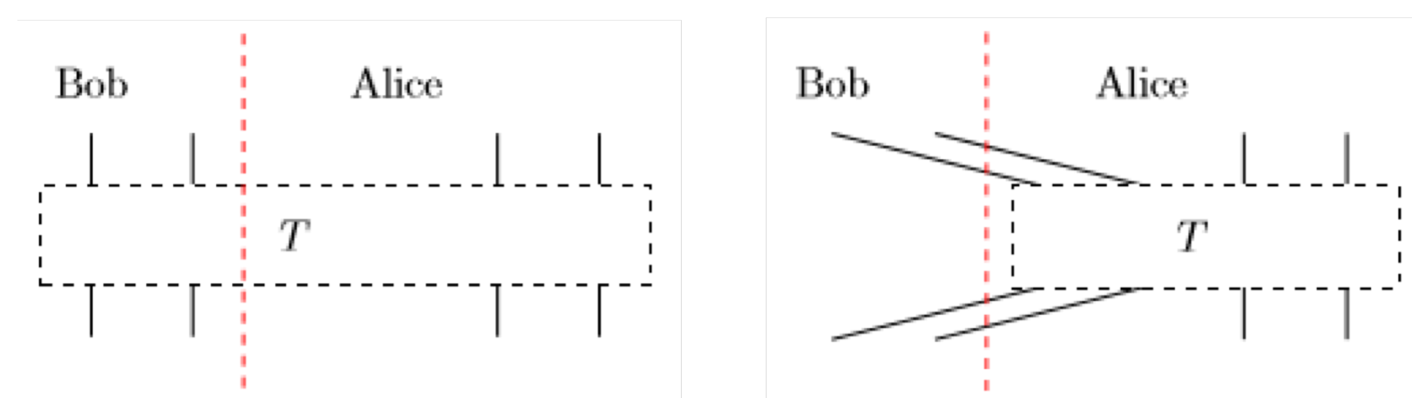


We are especially interested in the case that L is a picture language. In picture language, one can represent the concepts in L (such as discrete combinatorial data) by pictures which play the role of logical words, with axioms as their grammar.



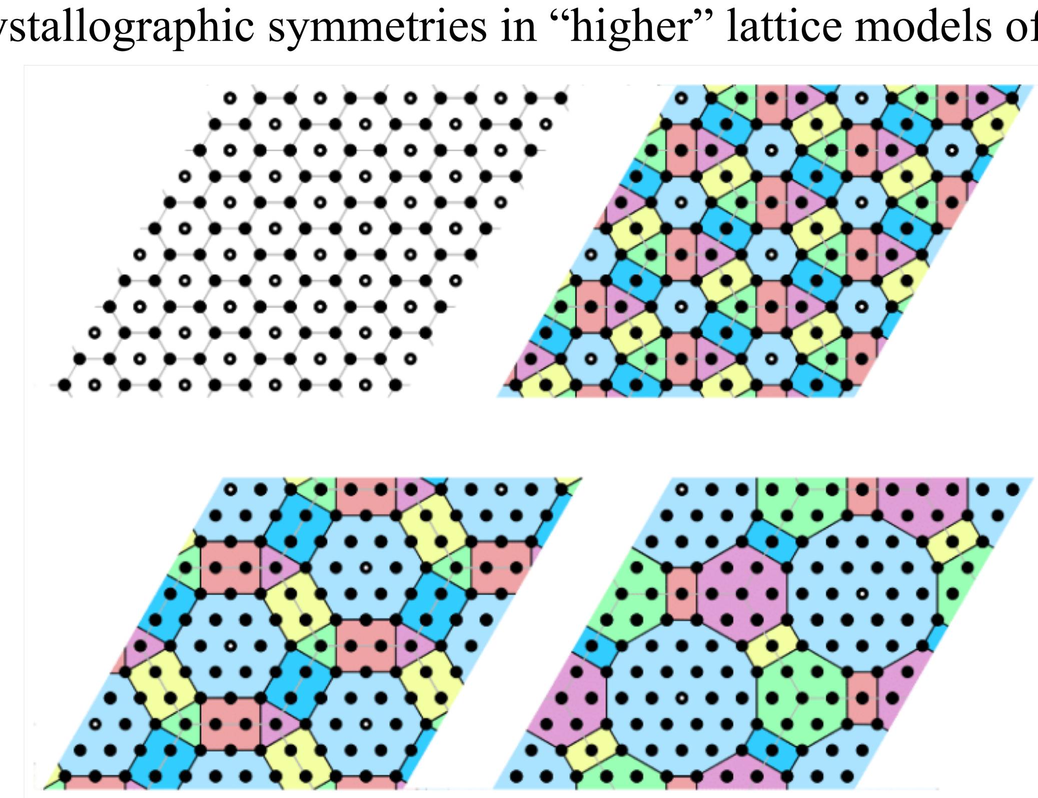
The axioms should be compatible with pictorial intuition. The Quon language extends the approach given by qudit isotopy to three dimensions, graphically encoding algebraic identities by means of the topological properties of pictures.

In communication we aim to change the location of quantum information, without changing the data. We simulate this process in a purely topological manner, that we call topological simulation. When we simulate the bipartite teleportation, we recover in a natural way the resource state, measurement, and Pauli matrices —as well as the protocol of Bennett and coworkers. When we simulate more complicated processes—by following topological intuition—then we find that our framework also leads to something new.



Symmetry

An elementary characteristic that transcends topology is shape. Shape goes in the direction of encapsulating the geometry of the picture. The geometry can indicate the presence of additional symmetry represented by the pictures. With the shape of a picture as an additional tool, mathematical concepts can be graphically encoded. In lattice models of statistical physics, people often study square lattices and honeycomb lattices, which capture additional symmetry in two and three directions respectively. Adrian Ocneanu is investigating how to extend these and additional crystallographic symmetries in “higher” lattice models of quantum field theory.



Some big picture questions

(1) How far can one understand mathematical duality in terms of pictorial duality? For example, to what extent can one understand further properties of Fourier duality or mirror symmetry?

(2) Many modern picture languages concern discrete combinatorial or topological data. A big question is how to construct a continuum theory from those pictures? Then one can ask how one can understand continuous symmetries, such as rotation invariance, in terms of picture language.

(3) Many people have studied pictures from the point of view of topology and algebra. How far can one go to understand a different aspect: the analysis of pictures?

Arthur Jaffe and Zhengwei Liu, A Mathematical Picture Language Program, to appear in the *Proceedings of the National Academy of Sciences* 10.1073/pnas.1710707114

Arthur Jaffe, Zhengwei Liu, Alex Wozniakowski, Constructive Simulation and Topological Design of Protocols, *New Journal of Physics* 19 063016.

The Picture Language Project at Harvard has been made possible by a generous grant from the Templeton Religious Trust.