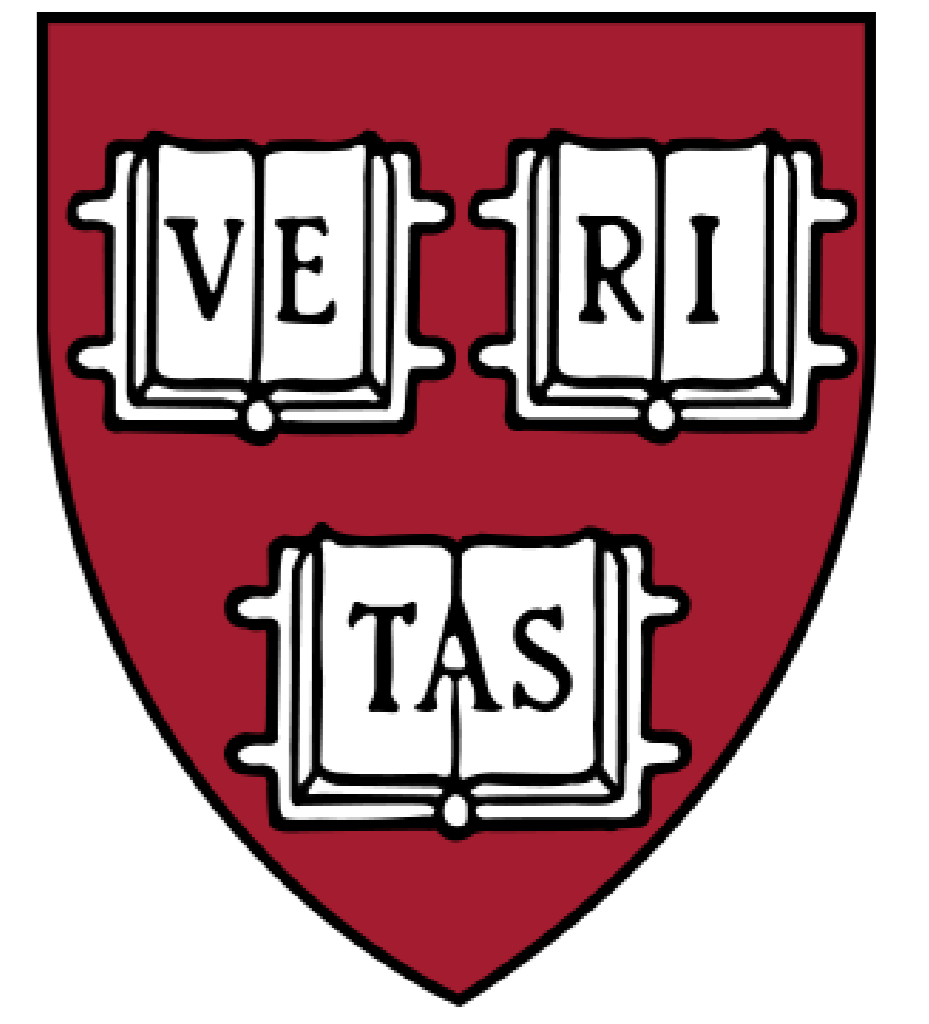


Constructing Lattices in Buildings

with Applications to Surface Group Amalgams



William Norledge

Abstract

Beginning with the work of Serre on the building for SL_2 over a non-Archimedean local field, orbihedra have been used to model the quotients of buildings by their lattices. This is made possible by the fact that buildings can be geometrically realized as non-positively curved polyhedral complexes. Developing the work of Tits on chamber systems [2] and group amalgams [3], we introduce covering theory of buildings which deals directly with the combinatorial structure of buildings, rather than their geometric realizations. We also give an example of a construction of lattices in so-called Bourdon buildings.

Coxeter Groups

A **Coxeter matrix** $M = (m_{st})_{s,t \in S}$ on a set S is a symmetric matrix whose entries are positive integers and ∞ , with the property that $m_{ss} = 1$ for all $s \in S$. The **Coxeter group** W of type M is the group with presentation,

$$W = \langle S \mid (st)^{m_{st}} = 1 : s, t \in S \rangle.$$

Amongst Coxeter groups are the Weyl groups, which act on root systems. In fact, every Coxeter group has a natural description as a group of reflections.

Buildings

Let W be a Coxeter group. A **building** (Δ, δ) of type W is a set of **chambers** Δ equipped with a ' W -valued metric' $\delta : \Delta \times \Delta \rightarrow W$ (the Bruhat order of W is used for the triangle inequality). To be a building, Δ must also satisfy an axiom which roughly says that Δ contains lots of isometrically embedded copies of W , called **apartments**.

Every building can be geometricly realized as a polyhedral complex in which the chambers become the maximal cells. Chambers are adjacent in this complex if the distance between them is a generator of W .

Generalized Chamber Systems

A **generalized chamber system** \mathcal{W} of type M is a directed multi-graph whose edges are labelled over S such that for each $s \in S$, the edges labelled by s have the structure of a groupoid; i.e. edges with the same label can be composed and have inverses.

- A **gallery** is a sequence of adjacent edges. A **gallery cycle** is a gallery which starts and finishes at the same vertex.
- The **type** of a gallery is the word over S which is 'spelt' by the sequence of its labelled edges.
- The **W -length** of a gallery is the element of W which is represented by its type.
- A **geodesic** is a gallery whose type is a representative of minimal length of its W -length.
- For $J \subseteq S$, a **J -residue** is a connected component in the graph which has the same vertices as \mathcal{W} , but whose edges are only those labelled by elements of J . A J -residue is called **spherical** if the subgroup of W generated by J is finite.

Weyl Data

Weyl data \mathcal{W} of type M is a generalized chamber system of type M such that for each pair $(s, t) \in S^2$ with $s \neq t$ and $m_{st} \neq \infty$, we have a collection of gallery cycles in \mathcal{W} called **suites**, each of which has type,

$$\underbrace{stst \dots st}_{2m_{st}}.$$

We also require that every 'half-suite', i.e. gallery of type,

$$\underbrace{stst \dots st}_{m_{st}}$$

is contained in a suite. The suites are used to tell us what (strict) homotopies of galleries are permitted in \mathcal{W} .

Homotopy of Galleries in Weyl Data

Strict Homotopy

An **elementary strict homotopy** is an alteration of a gallery which replaces a subgallery that travels halfway around a suite by the gallery which travels around the suite in the other direction (see Figure 1). Two galleries are called **strictly homotopic** if one can be transformed into the other via a sequence of elementary strict homotopies.

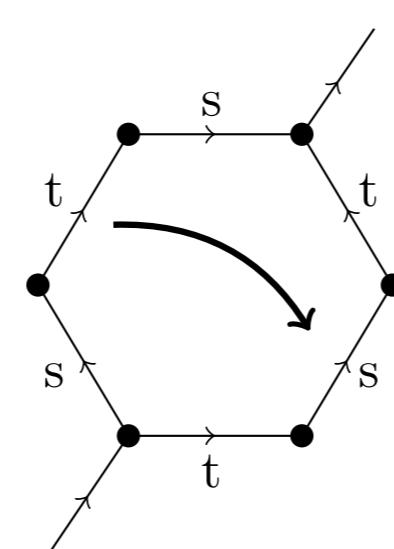


Figure 1: An elementary strict homotopy where $m_{st} = 3$

Homotopy

Two galleries are called **homotopic** if one can be transformed into the other via a sequence of any of the following:

1. elementary strict homotopies
2. adding or deleting a subgallery consisting of an edge followed by its inverse
3. replacing a subgallery of type ' ss ' with the subgallery of type ' s ' which is its composition, or visa versa.

Weyl Graphs

- Weyl data has property (W) if homotopic geodesics have the same W -length.
- Weyl data has property (C) if homotopic geodesics are strictly homotopic.
- Weyl data has property (H) if J -residues for $|J| = 2$ have property (W), and strictly homotopic geodesics of the same type are equal.

Theorem 1. The properties (W), (C), and (H) are equivalent.

A **Weyl graph** is Weyl data with any of these equivalent properties. The following generalizes a local to global theorem of Tits to Weyl graphs:

Theorem 2. Weyl data is a Weyl graph if and only if the spherical J -residues, for $|J| \in \{2, 3\}$, are Weyl graphs.

Covering Theory of Weyl Graphs

- The **fundamental group** $\pi_1(\mathcal{W})$ of a Weyl graph is the set of homotopy equivalence classes of its galleries which start and finish at the same fixed vertex, equipped with the binary operation induced by the concatenation of galleries. A Weyl graph is called **simply connected** if its fundamental group is trivial.
- A **covering** of Weyl graphs is a Weyl graph morphism with respect to which homotopies lift, and galleries lift uniquely (up to a choice of base-point).

Theorem 3. For any Weyl graph \mathcal{W} , there exists a universal covering $\widetilde{\mathcal{W}} \rightarrow \mathcal{W}$. Moreover, a covering $\mathcal{W}' \rightarrow \mathcal{W}$ is universal if and only if \mathcal{W}' is simply connected.

A **building** is equivalently a simply connected Weyl graph; one gets a W -valued metric from a simply connected Weyl graph by letting the W -distance between vertices be the W -length of the geodesics between them.

- The coverings of a fixed Weyl graph are in bijection with the conjugacy classes of subgroups of its fundamental group.
- Associated to the free (on vertices) action of a group G on a Weyl graph \mathcal{W} is the **quotient Weyl graph** $G \backslash \mathcal{W}$, and a covering $\mathcal{W} \rightarrow G \backslash \mathcal{W}$. If \mathcal{W} is a building, then G is isomorphic to $\pi_1(G \backslash \mathcal{W})$. Conversely, let \mathcal{W} again be any Weyl graph. Then $\pi_1(\mathcal{W})$ acts freely on the building $\widetilde{\mathcal{W}}$ such that \mathcal{W} is isomorphic to $\pi_1(\mathcal{W}) \backslash \widetilde{\mathcal{W}}$.

Surface Group Amalgams

Let $S = \{x_1, y_1, \dots, x_m, y_m\}$ label the vertices of a $2m$ -gon, $m \geq 2$ (see Figure 2), and let M be the Coxeter matrix on S with $m_{st} = 2$ if $s, t \in S$ are connected by an edge, and ∞ otherwise.

Let $p, q \geq 1$, and let $X_{p,q}$ denote the Weyl graph of type M whose vertices are those of a $p \times q$ grid, and with m many pairs of mutually inverse edges between vertices in the same row or column. The vertical edges between two fixed vertices are labelled by being put in bijection with the x_i 's, and similarly the horizontal edges are labeled by the y_j 's. All of the groupoids are simply connected, and the suites are all the gallery cycles spelling out the required alternating word.

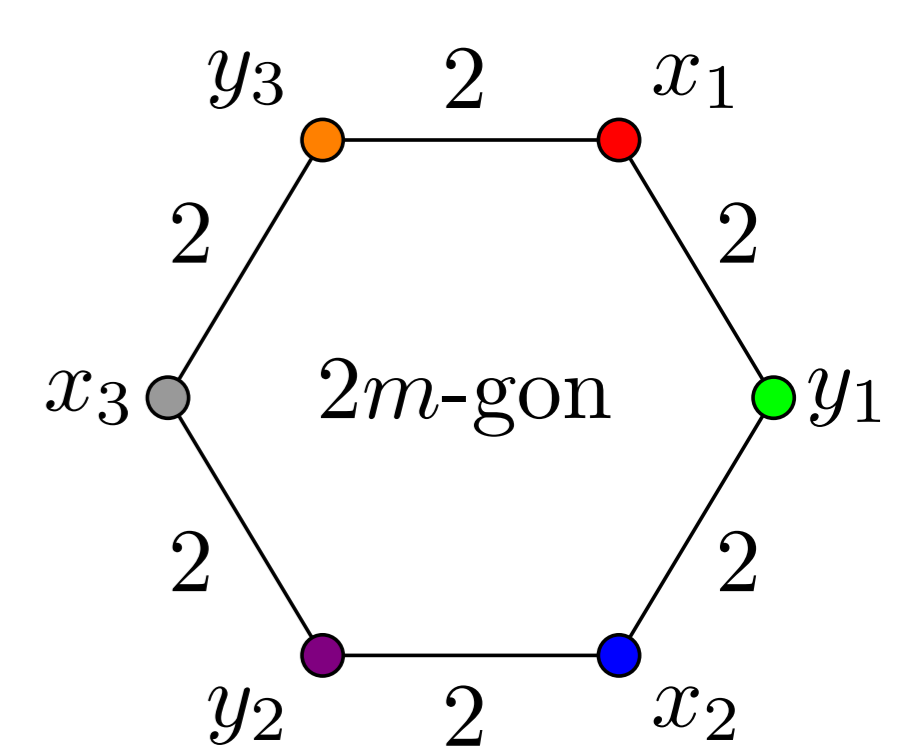


Figure 2: Coxeter matrix M where $m = 3$

Figure 3 shows $X_{3,2}$ for the Coxeter matrix in Figure 2. Mutually inverse edges have been drawn as single undirected edges. If one picks any two adjacent colors in Figure 2, then any 4-circuit in Figure 3 consisting of these two colors alternating will be a suite.

One can show that $\pi_1(X_{2,2})$ is the surface group S_{m-1} of genus $m - 1$, and $\pi_1(X_{1,2})$ is the free group F_{m-1} of rank $m - 1$. Since the 'grid' $X_{p,q}$ can be constructed by amalgamating copies of the 'squares' $X_{2,2}$ along 'edges' $X_{1,2}$, by the van Kampen theorem for Weyl graphs, we have the following:

Theorem 4. $\pi_1(X_{p,q})$ is an amalgam of $(p-1)(q-1)$ many genus $m - 1$ surface groups along rank $m - 1$ free groups.

The universal cover of $X_{p,q}$ is the Bourdon building whose geometric realization is the unique simply connected polyhedral complex such that all faces are regular right-angled $2m$ -gons and the link at each vertex is the complete bipartite graph $K_{p,q}$ (studied in [1]). Thus, the surface group amalgam $\pi_1(X_{p,q})$ is a torsion-free uniform lattice of this polyhedral complex.

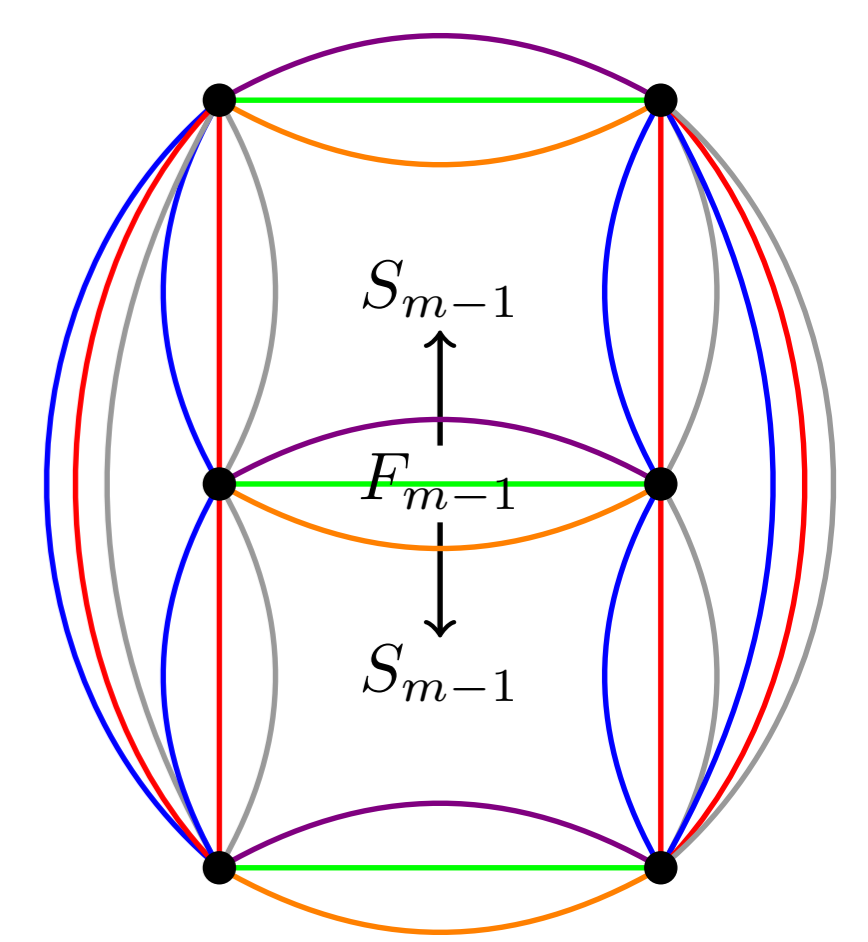


Figure 3: $X_{p,q}$

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