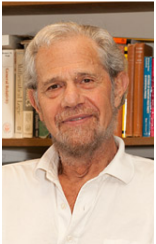
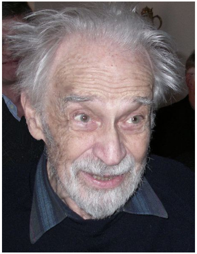
# Kochen, Specker, Bell, and Others

–

with a sketch of: *ETH* and the Fourth Pillar of QM



S. Kochen



E. Specker

J. Bell

*JF*

# Summary

I will start my lecture by telling you a story about *Ernst Specker* that I learned from *Raoul Bott*, to continue with a few comments on Specker's little known, but significant 1960 paper about the "logics of not simultaneously decidable statements".

As my main task in this talk, I will then attempt to explain to you

*the Kochen-Specker Theorem*

which says that there does **not** exist a **hidden variables theory** reproducing the contents of Quantum Mechanics (QM). The mathematics underlying the Kochen-Specker theorem is related to *Gleason's theorem*. I will mention an extension of Gleason's theorem to general von Neumann algebras. I will conclude the talk with an outline of recent ideas on how to complete the mathematical structure of QM and unravel its message – put differently, I will briefly describe what I call

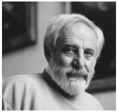
*the fourth pillar of Quantum Mechanics*

namely the "*ETH* Approach to QM".

# A story about Specker's stay at the Institute for Advanced Study in Princeton

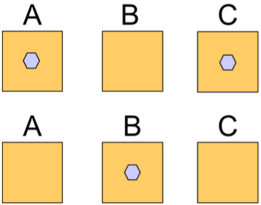
In the first half of the $20^{th}$ Century, thanks to *Heinz Hopf*, ETH Zurich became a world centre of a rather new field in mathematics that *Henri Poincaré* had named **"Analysis situs"**, nowadays called *"algebraic topology"*. – After the completion of his PhD under the supervison of Hopf, Specker spent more than a year at the IAS in Princeton. He knew everything of relevance in algebraic topology. At the Institute he met the young *Raoul Bott*, an electrical engineer turned into a mathematician with an untamed curiosity in algebraic topology who would come up with a new conjecture almost every day and would then try it out on Specker. After he had succeeded in disproving several of Bott's conjectures, Specker proposed a *bet* to him: He would disprove *everyone* of Bott's conjectures within five minutes. For a while, Specker won the bet; but, after some time, the situation changed, and Bott became a famous topologist.

Here is another story I learned from Bott: *Specker and the fire flies ...*



# King Asarhaddon's wise man from Ninive who taught at the school for prophets, named Arba'ilu

*Ernst Specker's* parable of the suitors of the wise man's daughter:
2 out of 3 boxes, A, B, C, are *either* empty *or* contain, each, a gem.




| A | B | C |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

"Classical-like" prediction
("properties" exist in space-time
before mesurement):
$x \in \{1,0\}$, $y \in \{1,0\}$
$P(x, y \mid x \neq y) = 2/3$
$P(x, y \mid x = y) = 1/3$
whatever pair of boxes is opened

"Quantum-like" prediction
$P(x, y \mid x \neq y) = 1$
$P(x, y \mid x = y) = 0$
whatever pair of boxes is opened.

↑ *Contextuality*
for a "Bell pair"

*The Assyrian prophet's contest*

Illustration by *A. Suarez*

$$A_f \Rightarrow B_e \Rightarrow C_f, \quad \text{but:} \quad A_f \Rightarrow C_e, \ \text{etc.}$$

Yet, **not all** of $A, B, C$ can be verified simultaneously! $\rightarrow \cdots$

# “Die Logik nicht gleichzeitig entscheidbarer Aussagen”

Ernst Specker, 1960

*La logique est d’abord une science naturelle.* - F. Gonseth

*“Kann die Beschreibung eines quantenmechanischen Systems durch Einführung von zusätzlichen – fiktiven – Aussagen so erweitert werden, dass im erweiterten Bereich die klassische Aussagenlogik gilt ... ?* [meaning that all statements/results of experiments on the system could be embedded in a Boolean lattice.]

*Die Antwort auf diese Frage ist* **negativ***, ausser im Fall von Hilbertschen Räumen der Dimension 1 und 2. ... Ein elementargeometrisches Argument zeigt, dass eine solche Zuordnung* (such an embedding) **unmöglich** *ist, und dass daher über ein quanten-mechanisches System (von Ausnahmefällen abgesehen) keine konsistenten Prophezeiungen möglich sind.”*

In his paper, Specker does not present any details concerning the “elementargeometrische Argument”. They were provided in the famous paper by Kochen and Specker, seven years later, which I paraphrase next.

# "The Problem of Hidden Variables in Quantum Mechanics"

Simon Kochen and Ernst Specker, 1967

*Question:* $\exists$ a hidden-variables theory recovering the predictions of quantum mechanics; or, in other words, can the predictions of quantum mechanics be embedded in a Boolean lattice?

Let $S$ be a physical system to be described quantum-mechanically. Its Hilbert space of pure state vectors is denoted by $\mathfrak{H}$; ...
If the answer to the above question were *"yes"* this would imply that $\exists$ a measure space $(\Omega, \mathfrak{F})$ and maps $f$ and $\rho$,

$$f : A = A^* \in B(\mathfrak{H}) \mapsto f_A : \Omega \to \mathbb{R}, \; f_A \text{ is } \mathfrak{F}\text{-measurable} \qquad (1)$$
$$\rho : \Psi \in \mathfrak{H} \mapsto \rho_{[\Psi]} = \text{ probability measure on } (\Omega, \mathfrak{F}),$$

with the following properties.

(P1) Preservation of expectation values: For every $A = A^* \in B(\mathfrak{H})$,

$$\|\Psi\|^{-2} \langle \Psi, A\,\Psi \rangle = \int_\Omega f_A(\omega)\, d\rho_{[\Psi]}(\omega)$$

# Properties of a putative embedding in a Boolean lattice

(P2) If $u : \mathbb{R} \to \mathbb{R}$ is an arbitrary bounded measureable function then

$$f_{u(A)} = u \circ f_A .$$

Note: (P1) and (P2) are compatible with each other (check!); and (P1) and (P2) imply the following fact:

(P3) Given any abelian algebra $\mathfrak{M}$ of commuting self-adjoint operators acting on $\mathfrak{H}$, then

$$f : A \in \mathfrak{M} \mapsto f_A \in L^\infty(\Omega)$$

is an *algebra homomorphism*; i.e.,

$$f_{A_1 \cdot A_2} = f_{A_1} \cdot f_{A_2}, \quad \forall A_1, A_2 \text{ in } \mathfrak{M}.$$

(Easy to prove if $\dim(\mathfrak{H}) < \infty$!)

# The Kochen-Specker Theorem

As already noticed by Specker in 1960, a hidden-variables theory satisfying (P1) - (P3) exists if $\dim(\mathfrak{H}) = 1$ or 2, (QM of a spin-$\frac{1}{2}$ object – nowadays called "Qbit", which sounds more interesting).

*Theorem.* (Kochen & Specker, 1967)
If $\dim(\mathfrak{H}) \geq 3$, a hidden-variables theory satisfying (P1)-(P3) does **not** exist.

*Proof.* We consider a particle, whose spin degree of freedom is described by a vector operator, $\vec{S}$, acting on the Hilbert space $\mathfrak{H} = \mathbb{C}^3 \simeq \mathbb{R}^3 \otimes \mathbb{C}$, (i.e., the particle has spin 1). Let $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ be the standard orthonormal basis in $\mathbb{R}^3$, and set $S_j := \vec{S} \cdot \vec{n}_j$, $j = 1, 2, 3$. Then

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \ S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One thus observes that the operators $P_j := 1 - S_j^2$, $j = 1, 2, 3$, are three mutually commuting orthogonal projections of rank 1, with $\sum_{j=1}^{3} P_j = 1$.

# Arbitrary orthonormal bases in $\mathbb{R}^3$

More generally, for an arbitrary vector $\vec{e}$ in $S^2$, $P(\vec{e}) := 1 - (\vec{S} \cdot \vec{e})^2$ is an orthogonal projection projecting onto the one-dimensional subspace of $\mathfrak{H}$ spanned by $\vec{e}$. $\big[$Thus, the matrix elements of $P(\vec{e})$ in the basis $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ are given by $P(\vec{e})_{ij} = e_i e_j\,,\ \forall i, j\,.\big]$

For an *arbitrary* orthonormal basis, $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, one then finds that

$$\sum_{j=1}^{3} P(\vec{e}_j) = 1\,, \quad P(\vec{e}_i) \cdot P(\vec{e}_j) = \delta_{ij} P(\vec{e}_i)\,. \tag{2}$$

The projections $\{P(\vec{e}_j)\}_{j=1}^{3}$ are functions of a single self-adjoint operator

$$A := \sum_{j=1}^{3} \alpha_j P(\vec{e}_j)\,, \quad \alpha_1 < \alpha_2 < \alpha_3\,. \tag{3}$$

generating a maximally abelian subalgebra of $B(\mathfrak{H}) = \mathbb{M}_3(\mathbb{C})$.

# A fatal assumption

We now **assume** that $\exists$ a hidden-variables theory satisfying properties (P1), (P2) and (P3).

Since $P(\vec{e})^2 = P(\vec{e})$, it follows from (P2) that

$$P(\vec{e}) \mapsto f_{P(\vec{e})} =: \chi_{\vec{e}} \tag{4}$$

is a characteristic function on $\Omega$. Eq. (2) implies that, for an arbitrary orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$,

$$\sum_{j=1}^{3} \chi_{\vec{e}_j} = 1, \quad \text{on } \Omega\,. \tag{5}$$

(For simplicity, we assume here and in the following that $\Omega$ is a discrete set.) For any point $\omega \in \Omega$,

$$\varphi_\omega(\vec{e}) := \chi_{\vec{e}}(\omega) \tag{6}$$

defines a function on $S^2$ with the following properties:

# Strange functions on the unit sphere in $\mathbb{R}^3$

(i) It takes only the values 0 and 1, i.e.,

$$\varphi_\omega(\vec{e}) = 0 \text{ or } 1, \text{ for any unit vector } \vec{e} \in S^2.$$

(ii) If $\vec{e}$ belongs to any orthonormal basis $\{\vec{e}_1 \equiv \vec{e}, \vec{e}_2, \vec{e}_3\}$ of $\mathbb{R}^3$ then the value, $\varphi_\omega(\vec{e})$, of $\varphi_\omega$ on $\vec{e}$ should be **independent** of the choice of $\vec{e}_2$ and $\vec{e}_3$, and

$$\sum_{j=1}^{3} \varphi_\omega(\vec{e}_j) = 1\,.$$

This follows from Eqs. (5) and (6).

(iii) Properties (i) and (ii) imply that the function $\varphi_\omega$ is an *additive measure on the lattice of orthogonal projections* acting on $\mathbb{C}^3 = \mathbb{R}^3 \otimes \mathbb{C}$, $\forall\, \omega \in \Omega$.

# Das "elementargeometrische Argument"

The evaluation of a function $\varphi_\omega$ with properties (i) - (iii) on finitely many unit vectors in $\mathbb{R}^3$, which give rise to finitely many orthonormal bases in $\mathbb{R}^3$, leads to the *contradiction* that, for some unit vectors $\vec{e}$, $\varphi_\omega(\vec{e}) = 0$ **and** $\varphi_\omega(\vec{e}) = 1$, depending on which completion of $\vec{e}$ to an orthonormal basis of $\mathbb{R}^3$ is considered – *"contextuality"*.

Kochen and Specker have found an explicit construction of finitely many unit vectors in $S^2$ leading to this contradiction. By now the best variant of their construction appears to require only 18 unit vectors.

There is an abstract proof of the claim that functions $\varphi_\omega$ on $S^2$ with properties (i) - (iii) do **not** exist, which is based on *Gleason's* theorem:[1]

Property (iii) says that the function $\varphi_\omega$ is an additive measure on the lattice of projections, $\forall\, \omega \in \Omega$. Gleason's theorem then says that

$\exists$ a *density matrix* $\Phi_\omega > 0$, with $\mathrm{tr}(\Phi_\omega) = 1$, such that

$$\varphi_\omega(\vec{e}) = \mathrm{tr}\big(\Phi_\omega\, P(\vec{e})\big) = \langle \vec{e}, \Phi_\omega\, \vec{e} \rangle\, .$$

This shows that $\exists$ a unit vector $\vec{e}$ such that $0 < \varphi_\omega(\vec{e}) < 1$. But this **contradicts** property (i) !

[1] I am grateful to *N. Straumann* for having explained this argument to me.

# Connection to Kakutani's theorem

We note that Gleason's theorem apparently implies that the functions $\varphi_\omega(\vec{e})$ are **continuous** in $\vec{e}$.

Thus, let us consider an arbitrary real-valued, **continuous** function, $\varphi$, on the $n$-dimensional sphere $S^n$ in $\mathbb{R}^{n+1}$ centered at the origin $\mathcal{O}$. *Dyson's* variant of *Kakutani's* theorem says that $\exists\, n+1$ points, $x_1, x_2, \ldots, x_{n+1}$, on $S^n$ such that the $n+1$ unit vectors $\left\{\vec{e}_j := \overline{\mathcal{O}\, x_j} \mid j = 1, 2, \ldots, n+1\right\}$ are mutually orthogonal, and

$$\varphi(\vec{e}_1) = \varphi(\vec{e}_2) = \cdots = \varphi(\vec{e}_{n+1}).$$

For $n = 2$, this contradicts properties (i) and (ii) of the functions $\varphi_\omega$ ! See F. J. Dyson, Ann Math. **54**, 534-536 (1951)

*Remarks:*

1. Ultimately, all the theorems asserting that there does not exist a hidden-variables theory reproducing the predictions of quantum mechanics exploit, in one way or another, the obvious fact that there does not exist a homomorphism from a non-commutative algebra *into* an *abelian* algebra. *Mermin's* version of Kochen-Specker makes this particularly clear.

# Open problems

2. *Gleason's* theorem can be generalized as follows: Additive measures on the lattice of orthogonal projections of a *general* von Neumann algebra are given by *normal states* on the von Neumann algebra[2]. –

   Another, better known approach to the non-existence of hidden variables theories of QM is based on:

3. *Bell's inequalities*: Consider correlations between outcomes of some family of *commuting* measurements on two "causally independent" systems, $A$ and $B$. It turns out that if all **quantum-mechanical** correlations are shrunk by a constant ${K_G}^{-1} < 1$ then the resulting values lie inside the range of corresponding **classical** correlations; (Tsirelson's theorem). Here, $K_G \approx 1.782$ is the *Grothendieck constant* appearing in the theory of tensor products. – However, as Bell's inequalities show, the range of quantum-mechanical correlations is *strictly larger* than the range of corresponding classical correlations:
   We define $$\mathfrak{H} := \mathbb{C}^2_A \otimes \mathbb{C}^2_B \,,$$

[2] see, e.g., *L. J. Bunce* & *J. D. Maitland Wright*, BAMS, **26**, 288-293 (1992)

# Bell's Inequalities

i.e., $A$ and $B$ are two (space-like separated) "Qbits". A maximally entangled state of $A \vee B$ is given by the spin-singlet (or Bell) state

$$\Psi := \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right] \in \mathfrak{H}\,.$$

We define a correlation matrix, $E$, by setting

$$E(\vec{e}_1, \vec{e}_2) := \langle \Psi, A_{\vec{e}_1} \cdot B_{\vec{e}_2} \Psi\rangle \overset{\text{in QM}}{=} -\vec{e}_1 \cdot \vec{e}_2\,, \tag{7}$$

where

$$A_{\vec{e}} := \vec{\sigma}\cdot\vec{e} \otimes \mathbf{1}\,, \quad B_{\vec{e}} := \mathbf{1} \otimes \vec{\sigma}\cdot\vec{e}\,.$$

We consider the following combination of correlations (CHSH):

$$F(\vec{e}_1, \vec{e}_2, \vec{e'}_1, \vec{e'}_2) := E(\vec{e}_1, \vec{e}_2) + E(\vec{e}_1, \vec{e'}_2) + E(\vec{e'}_1, \vec{e}_2) - E(\vec{e'}_1, \vec{e'}_2)\,.$$

Existence of a *"local"* hidden variables version of QM would imply that

$$-2 \leq F(\vec{e}_1, \vec{e}_2, \vec{e'}_1, \vec{e'}_2) \leq 2\,, \tag{8}$$

$\forall$ choices of $\vec{e}_1$, $\vec{e}_2$, $\vec{e'}_1$ and $\vec{e'}_2$; (an easy exercise!). But, in QM, one can choose $\vec{e}_1$, $\vec{e}_2$, $\vec{e'}_1$ and $\vec{e'}_2$ such that

$$F(\vec{e}_1, \vec{e}_2, \vec{e'}_1, \vec{e'}_2) = 2\sqrt{2}\ ! \tag{9}$$

# The *four pillars* of full-fledged Quantum Mechanics

So far, we have considered a rudimentary torso of QM. For, we have not incorporated *"time"*, *"events"*, and *"evolution"* into our presentation of QM, yet. I conclude with a sketch of a way in which this can be done.

The Quantum Theory of an (isolated open) *physical system* rests on *four* pillars:

(i) An isolated open system, $S$, of QM is described by a descending filtration, $\{\mathcal{E}_{\geq t}\}_{t\in\mathbb{R}}$, of von Neumann algebras, $\mathcal{E}_{\geq t}$, satisfying what is called the *Principle of Diminishing Potentialities (PDP)*:

$$\mathcal{E}_{\geq t'} \subsetneqq \mathcal{E}_{\geq t}, \ \forall t' > t\,. \tag{10}$$

*"Potential events possibly happening at time t or later"* (↗R. Haag) are represented by the **lattice of orthogonal projections** in $\mathcal{E}_{\geq t}$.

(ii) *"States* of $S$ at *time* $t$" are additive measures on the lattice of projections in $\mathcal{E}_{\geq t}$. Thus, they are given by *normal states* on $\mathcal{E}_{\geq t}$, $t \in \mathbb{R}$; (↗generalized Gleason theorem!)

(iii) *Heisenberg-picture "time evolution"* of operators by conjugation with the unitary *propagator* of $S$ acts on $\{\mathcal{E}_{\geq t}\}_{t\in\mathbb{R}}$ by shift of $t$.

# Actual events and the "collapse axiom"

(iv) Given a state $\omega$ on $\mathcal{E}_{\geq t}$, an *"actual event"*, described by a partition of unity given by a family, $\mathfrak{P}_t$, of disjoint orthogonal projections in $\mathcal{E}_{\geq t}$, is *happening at time t or later*, provided that $\mathfrak{P}_t$ generates the *center*, $\mathcal{Z}_\omega(\mathcal{E}_{\geq t})$, of the *centralizer*, $\mathcal{C}_\omega(\mathcal{E}_{\geq t})$, of the state $\omega$ on the algebra $\mathcal{E}_{\geq t}$, and that **at least two** of the projections in $\mathfrak{P}_t$ have a strictly positive expectation value (Born probability) in $\omega$.

*Axiom* ("State collapse postulate"): The state of the system right *after* time $t$ is given by a state, $\omega_\pi$, on the algebra $\mathcal{E}_{\geq t}$ defined by

$$\omega_\pi(X) := [\omega(\pi)]^{-1}\,\omega(\pi\,X\,\pi), \quad \forall X \in \mathcal{E}_{\geq t}\,, \tag{11}$$

where $\pi$ is one of the projections in the family $\mathfrak{P}_t$ (describing the actual event happening at time $t$ or later) with the property that $\omega(\pi) > 0$. The *probability* that $\pi$ selects the state of the system right after time $t$ is given by *Born's Rule*, i.e., by $\omega(\pi)$. □

→ Fundamentally *stochastic time evolution of states*[3], which "branch" whenever an *"event"* happens; branching probs. given by *Born's Rule*.

[3] i.e., "keine konsistenten Prophezeiungen möglich ..."

# QM is intrinsically probabilistic & *irreversible*

This description of the *time evolution of states* introduces a natural dichotomy between past and future (see graphical illustration):

| Past = History of Events (Facts) / Future = Ensemble of Potentialities |
|---|

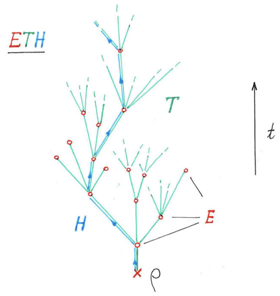
The particular completion of QM outlined here, in particular items (i) and (iv), is recent. It provides a solution of the so-called *"measurement problem"* and establishes a natural *"ontology"* for QM. It is known as the *ETH Approach* to QM and is the only consistent "Interpretation of Quantum Mechanics" known to this speaker.

*Remark:* In connection with item (iv), it would be interesting to extend *Huaxin Lin's* theorem on almost commuting self-adjoint operators to the setting of pairs, $(\omega, X)$, of a state, $\omega$, on a von Neumann algebra $\mathfrak{M}$ and an operator $X \in \mathfrak{M}$ with the property that $ad_X(\omega)$ has a tiny norm.

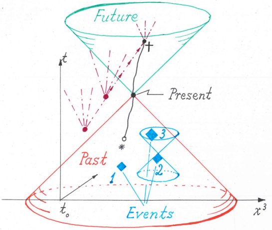
**QM remains a great source of interesting math problems!**

*Thank you!*

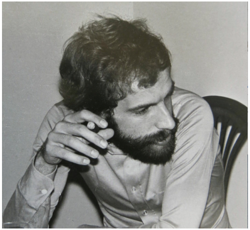
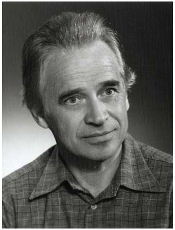
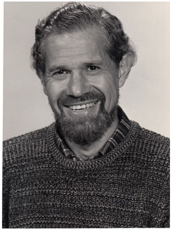
# Appendix – I. Metaphorical reps. of qm time evolution




*E*: "events" (proj. measnts.), *T*: "trees" (of states),
*H*: "histories" ; probs. of "histories" are det. by QM



# II. The Times They Are A-Changin'

$\Downarrow$

# III. Quotes from Specker's 1960 paper *"Die Logik nicht gleichzeitig entscheidbarer Aussagen"*

1. *"La logique est d'abord une science naturelle"* (Logics is primarily a natural science – an interesting claim worth to be debated.)

Das der Arbeit vorangestellte Motto ist der Untertitel des Kapitels *La physique de l'objet quelconque* aus dem Werk *Les mathématiques et la réalité* (by *Ferdinand Gonseth*).

2. Nach seiner Prophezeiung wurde nämlich jeder Freier vom Vater (the wise man of King Asarhaddon) aufgefordert, zwei Kästchen zu öffnen, welche jener (vorher) beide als leer oder beide als nicht leer bezeichnet hatte: es stellte sich stets heraus, dass das eine einen Edelstein enthielt und das andere nicht, und zwar lag der Edelstein bald im ersten, bald im zweiten der geöffneten Kästchen. ($\nearrow$ Measurements of z-components of spins of two (spin-$\frac{1}{2}$) Qbits prepared in a spin-singlet state.)

3. In einem gewissen Sinne gehören aber auch die scholastischen Spekulationen über die *"Infuturabilien"* hieher, das heisst die *Frage, ob sich die göttliche Allwissenheit auch auf Ereignisse erstrecke, die eingetreten wären, falls etwas geschehen wäre, was nicht geschehen ist.*