

# Equivalence of helicity and Euclidean self duality for gauge fields

Leonard Gross

Department of Mathematics  
Cornell University

Report on the paper  
<https://doi.org/10.1016/j.nuclphysb.2019.114685>

April 6, 2020

# 0-45 Experiment with polarized sunglasses



# 60-90 Experiment, continued



# How polarized sunglasses work

Maxwell's equations in empty space.

$$(\partial/\partial t)E = \text{curl } B$$

$$(\partial/\partial t)B = -\text{curl } E$$

$$\text{div } E = 0$$

$$\text{div } B = 0$$

Recall

$$\text{curl} = \nabla \times$$

(And if you prefer:  $\text{curl} = *d$  on 1-forms, where  $*$  is the three dimensional Hodge star operator.)

# Plane wave solutions

Suppose that  $a$  and  $b$  are real numbers, that  $\omega > 0$  and that  $e_1, e_2, e_3$  are the standard basis of  $\mathbb{R}^3$ . Then the vector fields

$$E(x, y, z, t) = a(\sin \psi)e_1 + b(\cos \psi)e_2, \quad \psi = \omega(t - z) \quad (1)$$

$$B(x, y, z, t) = -b(\cos \psi)e_1 + a(\sin \psi)e_2 \quad (2)$$

are solutions to Maxwell's equations. These are plane wave solutions. Power propagates in the direction of the Poynting vector

$$P \equiv E \times B = (a^2 + b^2)e_3 \quad (3)$$

One should think of this solution as a (very rough) approximation to a flashlight beam. The beam is pointing **upward** along the  $z$  axis.

# Linearly polarized light (for sunglasses)

For fixed  $x, y, z$  the endpoint of the vector  $E$  clearly moves around the ellipse

$$x^2/a^2 + y^2/b^2 = 1. \quad (4)$$

This really is an ellipse if  $a \neq 0$  and  $b \neq 0$ . But if  $b = 0$  then we see from (1) that  $E$  just moves back and forth along the  $x$  axis while  $B$  moves back and forth along the  $y$  axis. This is the prototype of *horizontally* polarized light.

Light from the sun that bounces off the curved roof of the car in front of you becomes horizontally polarized in this reflection. Fortunately, your polarized sunglasses allow only vertically polarized light thru, thereby cutting out this source of glare.

# Circularly polarized light (for us)

In case  $a = \pm b$  the tip of the electric vector moves around in a circle as time increases. So does the tip of the magnetic field vector. This is the case at any fixed point  $(x, y, z)$ . The direction of rotation depends on whether  $a = b$  or  $a = -b$ . Looking down on one of these circles from above the light beam is called **left circularly polarized** if the electric vector is moving counter clockwise and **right circularly polarized** if the electric vector is rotating clockwise. Looking down from above is the same as looking up the flashlight beam. Physicists usually refer to left circularly polarized light as having **positive helicity** and right circularly polarized light as having **negative helicity**. Here is some good news.

$$\text{curl } E = \omega B \quad \text{curl } B = \omega E \quad \text{positive helicity}$$

$$\text{curl } E = -\omega B \quad \text{curl } B = -\omega E \quad \text{negative helicity}$$

# Our problem

**Whereas**, Helicity in electromagnetism is defined in terms of plane wave expansions, and

**Whereas**, there are no plane wave expansions for the non-linear YM hyperbolic equations,

**Therefore**, we may ask whether helicity has any gauge invariant meaning in non-abelian gauge theories

**Method:** a. In EM show equivalence between the plane wave definition of helicity and (anti-)self duality of gauge fields.

b. In YM use the latter for a definition and then justify it.

# The canonical formalism: Quantitative version

Recall that there is a gauge potential

$$A(x, t) = \sum_{i=1}^3 A_i(x, t) dx^i, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}$$

such that

$$B(x, t) = \text{curl } A(x, t), \quad E(x, t) = -\dot{A}(x, t), \quad \text{div } A(x, t) = 0.$$

In terms of  $A$ , Maxwell's equations in empty space read

$$\left( (\partial/\partial t)^2 - \Delta \right) A(x, t) = 0, \quad \text{div } A(x, t) = 0$$

A general class of solutions, given as a plane wave expansion, is

$$A(x, t) = \mathcal{R}e \int_{\mathbb{R}^3} a(\mathbf{k}) e^{i(x \cdot \mathbf{k} - |\mathbf{k}|t)} d^3 \mathbf{k} / |\mathbf{k}|, \quad \mathbf{k} \cdot a(\mathbf{k}) = 0, \quad a(\mathbf{k}) \in \mathbb{C}^3$$

The  $q$ 's and  $p$ 's of the canonical formalism are given by  $A = A(\cdot, 0)$  and  $E = -\dot{A}(\cdot, 0)$ .

# The electromagnetic phase space

$A = \sum_{j=1}^3 A_j(x) dx^j$  will denote a 1-form on  $\mathbb{R}^3$ .

## Definition

(Sobolev spaces)

$$H_a = \{A : \|(-\Delta)^{a/2} A\|_{L^2(\mathbb{R}^3)} < \infty\} \quad (5)$$

$$\mathcal{C} = \{A : \operatorname{div} A = 0, \|A\|_{H_{1/2}} < \infty\} \quad (6)$$

Then

$$\mathcal{C}^* = \{1\text{-forms } E : \operatorname{div} E = 0, \|E\|_{H_{-1/2}} < \infty\} \quad (7)$$

in the pairing

$$\langle A, E \rangle = \int_{\mathbb{R}^3} A(x) \cdot E(x) d^3x \quad (8)$$

So we can identify

$$T^*(\mathcal{C}) = \{\text{pairs } A, E : \text{ a. } \operatorname{div} A = \operatorname{div} E = 0 \quad (9)$$

$$\text{ b. } \|A\|_{H_{1/2}}^2 + \|E\|_{H_{-1/2}}^2 < \infty\} \quad (10)$$

Note:

$$\|A\|_{H_{1/2}}^2 + \|E\|_{H_{-1/2}}^2 = \|B\|_{H_{-1/2}}^2 + \|E\|_{H_{-1/2}}^2 \quad (11)$$

## Theorem

*(Bargmann and Wigner, 1948) If  $A(x, t)$  is the unique solution to Maxwell's equations in empty space with initial data*

$$A(0) = A, \quad \dot{A}(0) = -E \quad (12)$$

*then any Lorentz transformation of the solution leaves (10) invariant. The norm square (10) is the unique norm with this property.*

# Characterization of helicity in EM.

## Facts about the operator curl.

1. The operator curl acts in  $\mathcal{C}$  as a self-adjoint operator. (Do an integration by parts.)
2. It has a zero nullspace in  $\mathcal{C}$ . (If  $\operatorname{div} A = 0$  and  $\operatorname{curl} A = 0$  then  $A = 0$ . (Size constraints are needed.))
3. Let  $\mathcal{C}_+$  denote the positive spectral subspace of curl in  $\mathcal{C}$  and let  $\mathcal{C}_-$  denote the negative spectral subspace. Then

$$\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_-. \quad (13)$$

$$T^*(\mathcal{C}) = T^*(\mathcal{C}_+) \oplus T^*(\mathcal{C}_-). \quad (14)$$

Otherwise said: Any pair  $\{A, E\}$  in phase space is a unique sum:

$$A = A_+ + A_-, \quad E = E_+ + E_- \quad (15)$$

with  $A_{\pm} \in \mathcal{C}_{\pm}$  and  $E_+$  a conjugate momentum to some element of  $\mathcal{C}_+$  and  $E_-$  a conjugate momentum to some element in  $\mathcal{C}_-$ .

## (Equivalence of helicity with *sgn curl*)

**Moral:** The operator curl decomposes configuration space into two subspaces, which automatically decompose phase space into two subspaces.

### Theorem

*(Equivalence of helicity with  $sgn\ curl$ )*

Suppose that  $A(x, t)$  is a solution to Maxwell's equations with

$$A(0) = A \quad \text{and} \quad -\dot{A}(0) = E. \quad (16)$$

Then its plane wave expansion is composed entirely of plane waves of positive helicity iff  $\{A, E\} \in T^*(\mathcal{C}_+)$ .

Its plane wave expansion is composed entirely of plane waves of negative helicity iff  $\{A, E\} \in T^*(\mathcal{C}_-)$ .

## Next step: Get rid of curl via Maxwell-Poisson equation

**Status:** We now have a characterization of helicity for normalizable solutions to Maxwell's equations

Let

$$a(x, s) = \sum_{j=1}^3 a_j(x, s) dx^j \quad x \in \mathbb{R}^3, \quad s \geq 0. \quad (17)$$

be continuous on  $[0, \infty)$  into  $L^2_{loc}(\mathbb{R}^3)$ . The curvature of  $a$  is

$$F(x, s) = ds \wedge a'(x, s) + da(x, s) \quad (18)$$

where  $d$  is the three dimensional exterior derivative.

**The Maxwell-Poisson equation with initial value  $A$**

$$a''(x, s) = d^* da(x, s), \quad s > 0 \quad \text{and} \quad a(0) = A. \quad (\text{M-P equation}) \quad (19)$$

Terminology: The solution  $a$  has **finite Poisson action** if

$$\int_{\mathbb{R}_+^4} |F(x, s)|_{\Lambda^2}^2 d^3x ds < \infty \quad (20)$$

### Theorem

*If  $A \in \mathcal{C}$  then the M-P equation has a unique solution with finite Poisson action. Moreover*

$$\|A\|_{\mathcal{C}}^2 \equiv \|A\|_{H_{1/2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}_+^4} |F(x, s)|_{\Lambda^2}^2 d^3x ds \quad (21)$$

### Definition

$a$  is **self dual** if  $*_e F = F$ .

$a$  is **anti-self dual** if  $*_e F = -F$ ,

where  $*_e$  is the four dimensional Euclidean Hodge star operator.

# Equivalence of helicity with Euclidean (anti-) self duality

## Theorem

If  $A \in \mathcal{C}$  and  $a(x, s)$  is its Poisson extension then

$A \in \mathcal{C}_-$  iff  $a$  is self dual.

$A \in \mathcal{C}_+$  iff  $a$  is anti-self dual.

## Proof.

Use the representation

$$a(s) = e^{-s|C|} A \quad (22)$$

for the solution to the Maxwell-Poisson equation. □

# Summary of equivalences in electromagnetism

Helicity of a normalizable solution in EM can be characterized by any of the following three equivalent ways.

- a. Plane wave expansion.
- b. Spectral decomposition of configuration space by curl.

$$\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_- \quad (23)$$

- c. Self duality or anti-self duality of the Maxwell-Poisson extension of a potential to a Euclidean half space.

# Yang-Mills theory

$K$ : compact connected Lie group with lie algebra  $\mathfrak{k}$ .

$A$ :  $\mathfrak{k}$  valued connection form on  $\mathbb{R}^3$

$$A = \sum_{j=1}^3 A_j(x) dx^j, \quad A_j(x) \in \mathfrak{k} \text{ for each } x \in \mathbb{R}^3 \quad (24)$$

$\mathcal{A}$ : a space of such  $\mathfrak{k}$  valued 1-forms on  $\mathbb{R}^3$  (sort of like  $H_{1/2}(\mathbb{R}^3)$  forms)

Gauge group:  $\mathcal{G} = \{g : \mathbb{R}^3 \rightarrow K\}$ , (sort of like  $H_{3/2}(\mathbb{R}^3; K)$ ).

$g$  acts on  $A$  by  $A^g = g^{-1}Ag + g^{-1}dg$ .

$$\text{Configuration space: } \mathcal{C} = \mathcal{A}/\mathcal{G}. \quad (25)$$

# Yang-Mills-Poisson equation

Let

$$a(x, s) = \sum_{j=1}^3 a_j(x, s) dx^j \quad x \in \mathbb{R}^3, \quad s \geq 0, \quad a(x, s) \in \mathfrak{k} \quad (26)$$

$$b(x, s) = da(x, s) + a(x, s) \wedge a(x, s) \quad (27)$$

$$F(x, s) = ds \wedge a'(x, s) + b(x, s) \quad (28)$$

$a$  is a 1-form on  $\mathbb{R}^3$  for each  $s$ . Equivalently, it is a 1-form on  $\mathbb{R}_+^4$  in temporal gauge.

$d$  is the 3 dimensional exterior derivative

$b$  is the 3 dimensional curvature of  $a$  for each  $s$ .

$F$  is the 4 dimensional curvature of  $a$  (regarded as a 1-form on  $\mathbb{R}_+^4$  in temporal gauge).

The Yang-Mills-Poisson equation on  $\mathbb{R}_+^4$  with initial value  $A$  is

$$a''(x, s) = d^*b(x, s) \quad x \in \mathbb{R}^3, \quad s > 0 \quad (\text{YMP equation}). \quad (29)$$

Initial condition  $a(0) = A$ .

### Definition

The **Poisson action** of  $A$  is

$$\mathcal{P}(A) = \int_{\mathbb{R}_+^4} |F(x, s)|_{\Lambda^2 \otimes \mathfrak{g}}^2 d^3x ds \quad (30)$$

Fact:  $\mathcal{P}(A^g) = \mathcal{P}(A)$ .

Therefore the Poisson action descends to a function on  $\mathcal{C}$ .

Examples: Any instanton on  $\mathbb{R}^4$  restricts to a solution on the half-space with finite Poisson action.

**Theorem** : For any given  $A \in H_{1/2}(\mathbb{R}^3)$  the YMP equation has a unique solution with finite action.

**Status of proof:** a. I don't have a proof.

b. But by conformally transforming the half space to a bounded region in  $\mathbb{R}^4$  the problem reduces almost to the Dirichlet problem for the elliptic Yang-Mills equation. The latter has been extensively studied by Antonella Marini et al. The distinction from our case is that we must allow a singularity at one point of the boundary.

See discussion in my paper

<https://doi.org/10.1016/j.nuclphysb.2019.114685>

and especially the reference

[54] Antonella Marini, Rachel Maitra and Vincent Moncrief, A euclidean signature semi-classical program, (2019), 74 pages, <http://arxiv.org/abs/1901.02380>.

# The helicity decomposition of configuration space

**Notation** Let

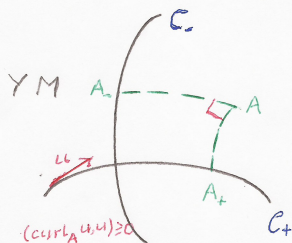
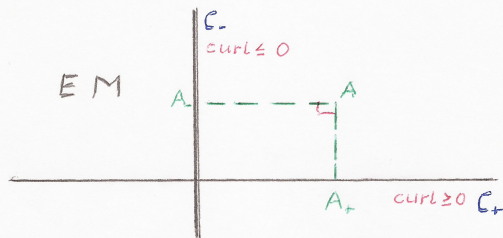
$$\mathcal{A}_+ = \{A : \mathfrak{a} \text{ is anti-self dual}\} \quad (31)$$

$$\mathcal{A}_- = \{A : \mathfrak{a} \text{ is self dual}\} \quad (32)$$

Fact:  $\mathcal{A}_\pm$  are each gauge invariant sets. The projection  $\pi : \mathcal{A} \rightarrow \mathcal{C}$  therefore defines two submanifolds  $\mathcal{C}_\pm$  of  $\mathcal{C}$ .

$$\mathcal{C}_\pm = \mathcal{A}_\pm / \mathcal{G} \quad (33)$$

# Desired theorem



## Theorem

*The picture in the previous frame holds. In particular there is a natural decomposition*

$$\mathcal{C} = \mathcal{C}_+ \times \mathcal{C}_- \quad (34)$$

*and  $\text{curl}_A$  has the right sign on tangent vectors to  $\mathcal{C}_\pm$ .*

# Outline of proof

In the abelian case, i.e. EM, the subspaces  $\mathcal{C}_\pm$  are the nullspaces of the operators  $C \mp |C|$ . Let's focus on  $\mathcal{C}_+$ . It's the null space of  $C - |C|$ . This operator is terribly big and negative on  $\mathcal{C}_-$ . Consequently

$$A_+ \equiv \lim_{t \rightarrow +\infty} e^{t(C-|C|)} A \in \mathcal{C}_+. \quad (35)$$

This functional analytic method is no good for our nonlinear configuration space. Let's change viewpoint a little. The exponential is the solution of the differential equation

$$dA(t)/dt = \{CA(t) - |C|A(t)\}, \quad A(0) = A \quad (36)$$

Now the function  $A \rightarrow CA$  is a vector field on  $\mathcal{C}$ . Does it have an analog in the non-abelian case? Sure, or I wouldn't be asking. Thus  $\text{curl } A =$  magnetic field  $B$ . So replace  $CA$  by the curvature  $B = dA + [A \wedge A]$ . [Actually one must insert the 3 dimensional Hodge  $*$  in front of the last expression. Let's ignore this.]

What about  $|C|A$ ? You remember that the solution to the Maxwell-Poisson equation is  $a(s) = e^{-s|C|}A$ . So  $a'(0) = -|C|A$ . So let's replace  $-|C|A$  by  $a'(0)$  where  $a$  is now the solution to the Yang-Mills-Poisson equation. Both  $B$  and  $a'(0)$  depend non-linearly on  $A$  as does the vector field

$$h(A) = *B - a'(0) \quad (37)$$

$h(A)$  is a  $\mathfrak{k}$  valued 1-form for each  $A$  and is therefore a tangent vector to  $\mathcal{A}$  at  $A$ .  $h$  is a gauge covariant vector field on  $\mathcal{A}$  and therefore descends to a vector field on  $\mathcal{C}$ .

Replace the linear vector field  $curl A - |curl| A$  by  $h(A)$ .

**Theorem:**  $\lim_{t \rightarrow +\infty} \exp(th)(A)$  exists and lies in  $\mathcal{C}_+$ .

Hint of proof: Use  $\mathcal{P}(A)$  as a Liapounov function.

# Unanswered questions

1. Existence and uniqueness of solutions to the Y-M-P equation with  $H_{1/2}$  initial data. [Marini et al]

2. Other gauge invariant action functionals besides Poisson. These may come from Euclidean QFT or expansion of ground state, [Moncrief et al].

Goals:

a. Lorentz invariance of such an action functional.

b. Foliation induced by an action functional should have nice properties.

E.g.  $(\text{curl}_A u, u)$  has the right sign for tangent vectors to the foliation.

$$[\text{curl}_A u = *\{du + adA \wedge u\}]$$

3. Other foliations induced by the Poisson action or other actions. Desired property:  $(\text{curl}_A u, u)$  has the right sign for tangent vectors to the foliation.

4. (A long range goal) Stochastic independence of helicity manifolds for a given foliation wrt ground state measure. (This holds in EM.)