Kitaev Model as a error correcting code

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arxiv: 1908.02829.

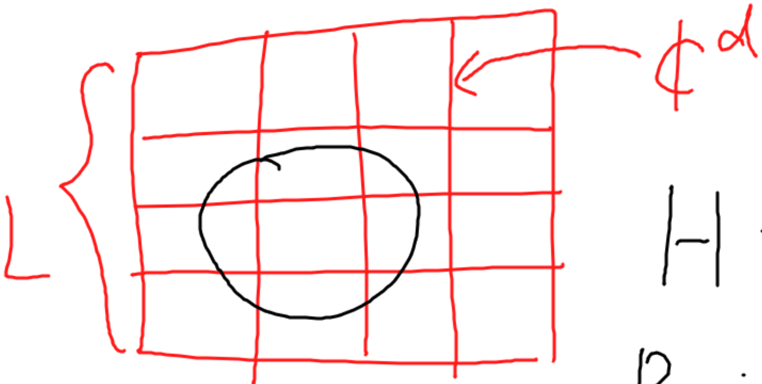
Kitaev model: spin lattice model, e.g. toric code

→ topological quantum order (TQO)
(top phase of matter)

→ Gapped spectrum, anyons, braid anyons,
stable under perturbations

→ top quantum computing

" folklore " Goal: Mathematically justify



$$\mathcal{H} = \bigotimes_{\text{edges}} \mathbb{C}^d$$

$$H = -\sum_i P_i$$

P_i : local projector, $P_i^2 = P_i$

$$[P_i, P_j] = 0$$

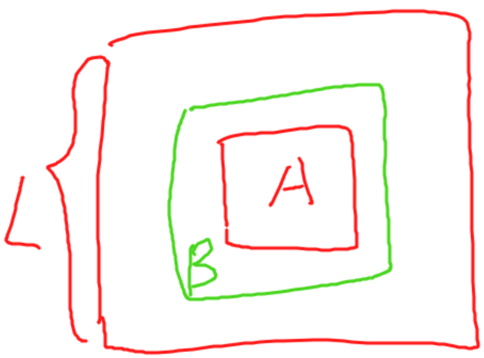
H frustration-free, $V_{g.s} = \{ |\psi\rangle \in \mathcal{H} : P_i|\psi\rangle = |\psi\rangle \ \forall i \}$

$$P = \prod P_i : \mathcal{H} \longrightarrow V_{g.s}$$

Def (Bravyi, Hastings, Michalakis).

A spin system is a TQO if

TQO1: $\exists\ \alpha > 0$, $\forall$ sublattice A of size $\leq L^{\alpha}$, then $\forall\ O_A$ acting on A,



$$P O_A P = cP$$ scalar

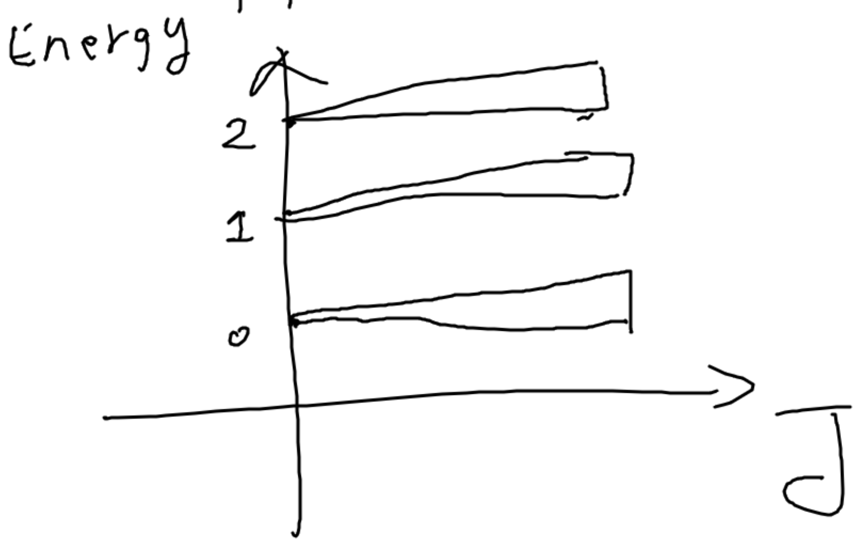
$\iff V_{g.s}$ corrects errors in A.

TQO2: $P = \prod P_i$, $P_B = \prod_{\mathrm{supp}(P_i) \subseteq B} P_i$

$\mathrm{Tr}_{A^c}(P)$ and $\mathrm{Tr}_{A^c}(P_B)$ have the same kernel.

(BHM) If H_0 is a TQO, then its spectral gap is stable under local perturbation.

$$H = H_0 + V, \quad \|V\| \leq J$$



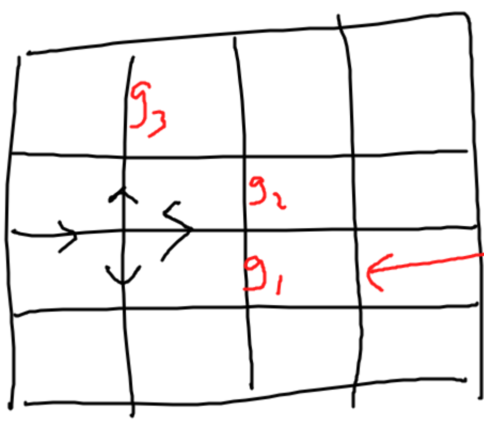
<u>Goal</u>: Kitaev model is a TQO.

Kitaev model

$G = \{g_1, g_2, \cdots\}$ finite group

Σ : a closed surface, e.g. torus

$\mathcal{L}$: a lattice on Σ. $\mathcal{L} = (\underset{\text{vertices}}{V}, \underset{\text{edges}}{E}, \underset{\text{faces}}{F})$



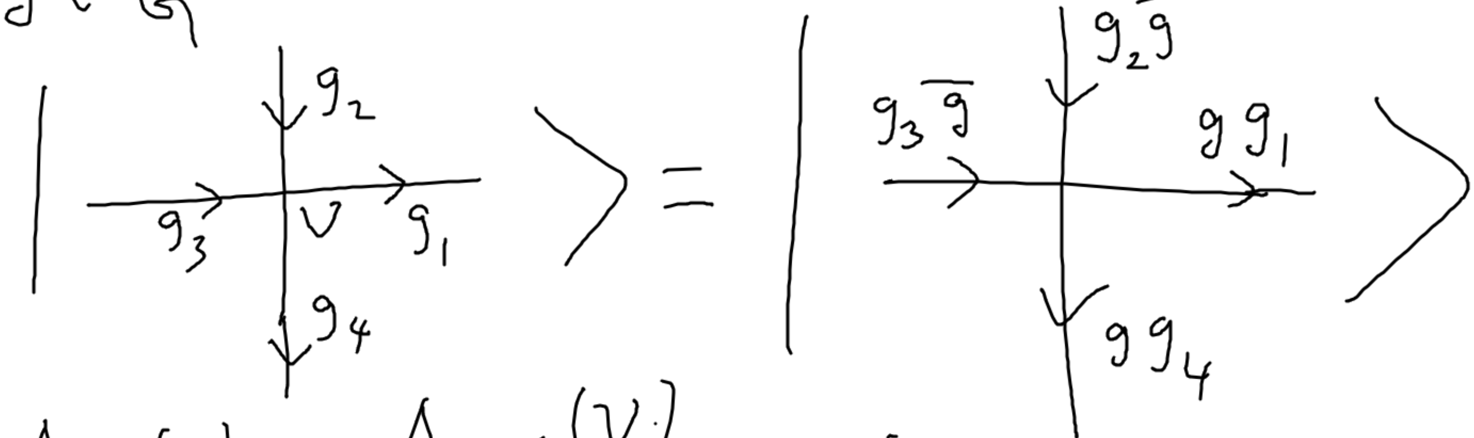
orient edges arbitrarily.

$$\mathbb{C}[G] = \mathrm{span}_{\mathbb{C}}\{g \mid g \in G\}$$

$$\mathcal{H} = \bigotimes_{e \in E} \mathbb{C}[G]$$

$$|g_{e_1}, g_{e_2}, \cdots\rangle = \bigotimes_{e \in E} g_e$$

1) $\forall\, v \in V,\ g \in G$

$A_g(v)$ 

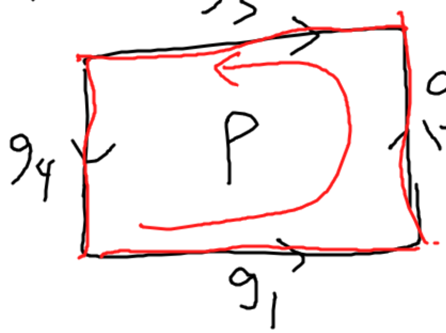
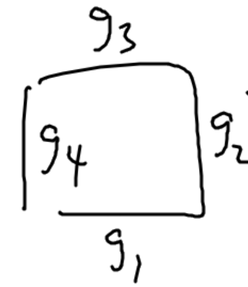
gauge transf.

- $A_g(v)\, A_{g'}(v) = A_{gg'}(v)$
- $v \neq v'$, $[A_g(v), A_{g'}(v')] = 0$

$$A(v) \equiv \frac{1}{|G|} \sum_g A_g(v)$$

- $A(v)^2 = A(v)$

2) $\forall p \in F$

$B(p) \Big|$  $\Big\rangle = \delta_{g_1 g_2 \bar{g}_3 g_4, 1} \Big|$  $\Big\rangle$

$\cdot\ B(p)^2 = B(p)$

B(p) picks out states with trivial holonomy.

A(v) picks out states which is gauge invariant.

$\cdot\ \{ A(v), B(p) \mid v \in V, p \in F \}$ pairwise commute

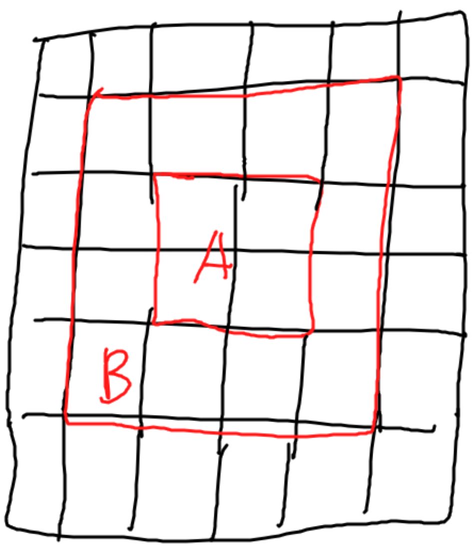
$$H = -\sum_v A(v) - \sum_p B(p)$$

$$V_{g.s} = \left\{ |\psi\rangle \in \mathcal{H} : \begin{array}{l} A(v)|\psi\rangle = |\psi\rangle \\ B(p)|\psi\rangle = |\psi\rangle \end{array}, \forall v, p \right\}$$

<u>Facts</u>: ① $G = \mathbb{Z}_2 \rightsquigarrow$ toric code.

② $\dim V_{g.s} = \left| \mathrm{Hom}(\pi_1(\Sigma), G) / \text{conjugation} \right|$

Thm (CDHPRRZ, '19)



Let $A \subset B \subset \mathcal{L}$ sublattice, connected, <u>contractible</u>,

$$V_{g.s}^{loc} = \left\{ |\psi\rangle \in \mathcal{H} : \begin{array}{l} A(v)|\psi\rangle = |\psi\rangle = B(p)|\psi\rangle, \\ v \in B^{int}, \ p \subset B \end{array} \right\}$$

Then, $\forall |\psi\rangle \in V_{g.s}^{loc}$, $\langle\psi|\psi\rangle = 1$, we have

$\mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$ is independent of $|\psi\rangle$.

Thm $\Rightarrow$ TQO :

basis $\{|\psi_i\rangle\}$ of $V_{g.s} \subset V_{g.s}^{loc}$

$Tr_{A^c} |\psi_i\rangle\langle\psi_i| \equiv \rho$, $Tr_{A^c} |\psi_i\rangle\langle\psi_j| = 0$, $i \neq j$

$P = \sum_i |\psi_i\rangle\langle\psi_i|$, $P O_A P = Tr(\rho O_A)\, P$

$\Rightarrow$ TQO $\underline{1}$.

$Tr_{A^c} P = \dim V_{g.s}\ \rho$

$Tr_{A^c} P_B = \dim V_{g.s}^{loc}\ \rho$

$\Rightarrow$ TQO 2 .

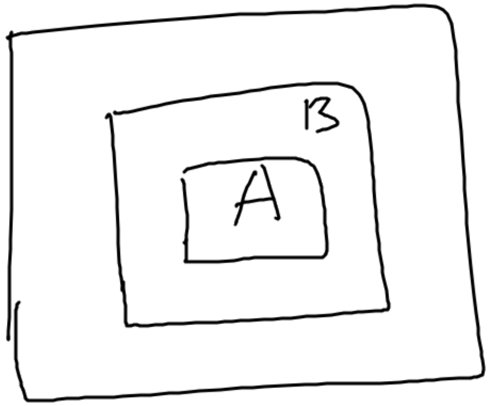
Rem: ① Toric code case is easy

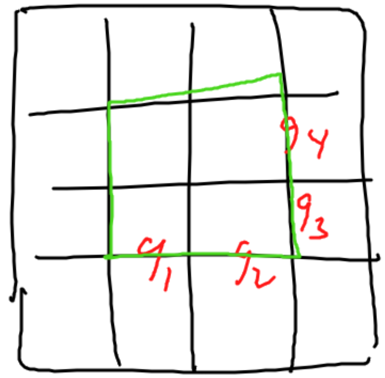
② P Naaijkens (thesis) Kitaev model on infinite plane.

③ Y. Qiu, Z. Wang

Levin-Wen is TQO

Sketch:





$|g\rangle = \bigotimes_{e \in E} |g_e\rangle$

Given a loop γ, $\mathrm{hol}(\gamma) = g_1 g_2 g_3 \cdots$

$B(p)|g\rangle = |g\rangle$ iff $\mathrm{hol}(\partial p) = 1$

$B(p)|g\rangle = |g\rangle, \forall p \in B$ iff $\mathrm{hol}(\gamma) = 1$, $\forall$ loop γ in B, ($\because$ B contractible)

$$|\psi\rangle \in V^{loc}_{g.s} = \sum_{\substack{g \text{ trivial} \\ \text{hol in } A}} c_g \underbrace{|g\rangle_A}_{A} \underbrace{|\tilde{g}\rangle_{A^c}}_{A^c}$$

$$= \sum_{\substack{g \text{ trivial} \\ \text{hol}}} |g\rangle_A \underbrace{|\phi(g)\rangle_{A^c}}_{\text{un-normalized}}$$

Claim: $\forall\, g, g'$ trivial hol in A, $g|_{\partial A} = g'_{\partial A}$, then $\exists$ gauge transf U acting on A^{int}, s.t.

$$U_{int}|g\rangle = |g'\rangle$$

$$\Rightarrow |\phi(g)\rangle_{A^c} = |\phi(g')\rangle_{A^c} \equiv |\phi(g_{|\partial A})\rangle_{A^c}$$

For $h: \partial A \to G$, trivial hol

$$|g_h\rangle_A := \sum_{\substack{g_{|\partial A} = h \\ g \text{ trivial hol}}} |g\rangle_A$$

$$|\Psi\rangle = \sum_h |g_h\rangle_A |\phi(h)\rangle_{A^c}$$

<u>Claim</u>: $\{|\phi(h)\rangle_{A^c}\}$ ortho, have the same norm

$$\mathrm{Tr}_{AC} |\psi\rangle\langle\psi| \sim \sum_{n} |g_n\rangle\langle g_n| \qquad \blacksquare$$