Quantum Fourier Analysis

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$$\hat{f}(y) = \int e^{2\pi i x y} f(x)\, dx \qquad f \in C^\infty(\mathbb{R})$$

Fourier transform

multiplication ⟷ convolution

$$\|f\|_2^2 = \int |f(x)|^2 dx$$

$$\|\hat{f}\|_2 = \|f\|.$$

$$P \xrightarrow{F} Q$$

position momentum

Classical Symmetry : group

Classical Fourier Analysis

Finite Group G.

A model :

$L^2(G)$

multiplication

convolution

Haar measure

B model :

$\mathbb{C}(G) \curvearrowright L^2(G)$

matrix multiplication

Hadamard product

$([a_{ij}]*[b_{ij}]=[a_{ij}b_{ij}])$

Trace

(1)

Young's inequality

$f, g \in L^2(G)$

$\|f*g\|_r \le \|f\|_p \|g\|_q$

$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad \infty \ge p,q,r \ge 1$

$\|f\|_p = \left| \int_{g\in G} |f|^p \, dg \right|^{\frac{1}{p}}$

Young's inequality

$F, G \in \mathbb{C}(G)$

$\|F*G\|_r \le \|F\|_p \|G\|_q$

$\|F\|_p = \| |F| \|_p$

$|F| = (F^* F)^{\frac{1}{2}}$

$\| |F| \| = Tr(|f|^p)^{\frac{1}{p}}$

(2) $f, g \geq 0$ $\quad F, G \geq 0$

$f * g \geq 0$ $\quad F * G \geq 0$

Convolution Positivity. $\quad$ Schur Product Theorem

(3) Sum set estimate

Two subset of G,

S and T

$S+T = \{s+t : s \in S,\ t \in T\}$

$\#(S+T) \geq \max\{\#S,\ \#T\}$

$1_S,\ 1_T$

P, Q projection in $\mathcal{L}(G)$.

$\int \text{Range}(1_S * 1_T) =: \text{Supp}(1_S \times 1_T)$

$\text{Supp}(1_S * 1_T) \geq \text{Supp}(1_S)$, $\quad \underline{\text{Supp}(P*Q) \geq \text{Supp}(P)}$

new

Part II. quantum Fourier analysis

quantum sysnetry beyond group

A family of quantum symetries

Given a finite group G,

Irr Rep (G),

Example $G = S_3$.

Irr Rep $= \{1, \sigma, \gamma\}$

dim $1, 1, 2$

Tensor Product: $\sigma^2 = 1, \ \sigma\gamma = \gamma,$

$\gamma^2 = 1 \oplus \sigma \oplus \gamma$

Fusion Ring: $\{\chi_i\} \quad \chi_1 = 1.$

(Lusztig) $\chi_i \chi_j = \sum_k N_{ij}^k \chi_k \quad N_{ij}^k \in \mathbb{N}$

$i \xrightarrow{\text{dual}} i^*$

$N_{i\bar{i}}^1 = \begin{cases} 1 & i = \bar{j}^* \end{cases}$

(Pavel Etingof)

Question: QFA holds on Fusion Ring?

Quantum Symetry

L-Wu-Palcoux, Yes.

Fusion Ring R,

A model:

$L^2(R)$

multiplication (diamond)

Convolution

Haar measure

($\exists 1$ positive character)

Perron-Frobenius theorem

$d(x_i)d(x_j) = \sum_k N_{ij}^k d(x_0)$

$d(x_i) > 0$.

δ_{x_i}

$x_i \diamond x_j = \delta_{ij} x_i \cdot d(x_i)^{-1}$

B model

$L(R) \curvearrowright L^2(R)$

multiplication

convolution

Theorem :	A	Fusion ring B	$\mathrm{Rep}(G_1)$
Young,	✓	✗	✓
Convolution Positivity	✓	✗	✓
Sunset Estimate	✓	✗	✓

$\mathrm{Irr}\,\mathrm{Rep}(S_3) = \{1, \sigma, \gamma\}$

$\sigma^2 = 1, \quad \sigma\gamma = \gamma, \quad \gamma^2 = 1 \oplus \sigma + \gamma.$

$1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \sigma = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\sigma \circledast \gamma = 0 \qquad \gamma \circledast \gamma = \frac{1}{2}\gamma$

$*$ $\qquad$ $*$

In general $*$ on matrices may NOT be Hadamard product.

Theorem: Fusion Ring R is the Grothendieck ring of a unitary fusion category, then Young, CP Sumset hold on R.

on both A model & B model.

Turaev Viro 3D TQFT.

Levin-Wen model

Unitary Categorification

Fusion Ring ⟶ UFC.

QFA ⟶ analytic obstructions!

Suppprisingly Efficient.

S. Palcoux 17 integral Fusion Ring

15/17 can not be V. Lat.

Part II. Generalizations

① Quantum Symetry: Subfactor (Jones)

Examples ahead are special, interesting, examples of subfactors.

Finite ⟶ Infinite.

Kac algebras

Locally Compact Quantum Groups.

② Inequalities in families.

Brascamp-Lieb.

↓

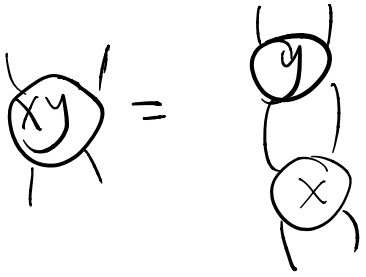
topological / pictorial.

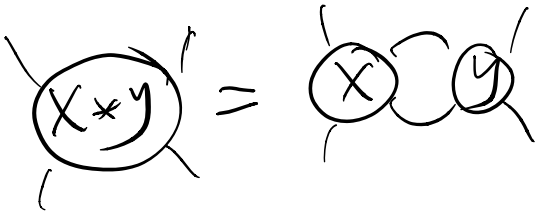
Part IV pictures.

planar algebras (Jones)

$L^2(G)$

$L(G)$



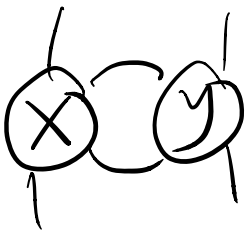


$tr(X)$: 

$\Phi \longrightarrow$ 90° rotation.

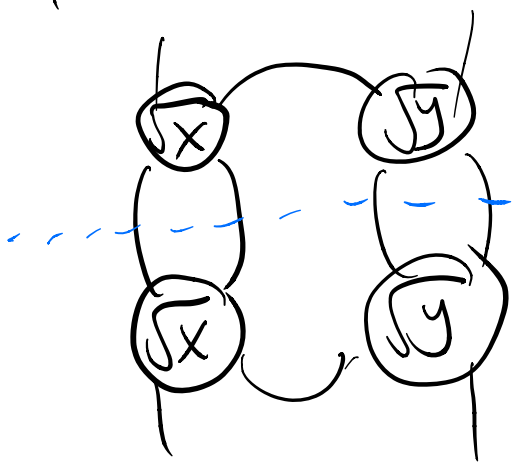
(Осталь)

$x, y \geq 0$



Drinfeld Center.

$=$



≥ 0

Part V. Entropy.

Shannon $-t \log t \, dt$

von Neumann $\mathrm{Tr}(-T \log T)$

Rényi

Relative entropy.

Entanglement entropy.

Uncertainty Principle for entropies for Fourier duality over Quantum Symmetries.

$\mathbb{Z}_2$ $\qquad\qquad$ $\mathcal{F}$

$|0\rangle\langle 0| + |1\rangle\langle 1| \longleftrightarrow |0\rangle\langle 0|$

$H(|X|^2) + H(|\hat{X}|^2) \geq c > 0$

$|00\rangle + |11\rangle \longleftrightarrow |00\rangle$

→ entanglement entropy.

Fourier transform

Topological $\mathcal{F}$ in TQFT

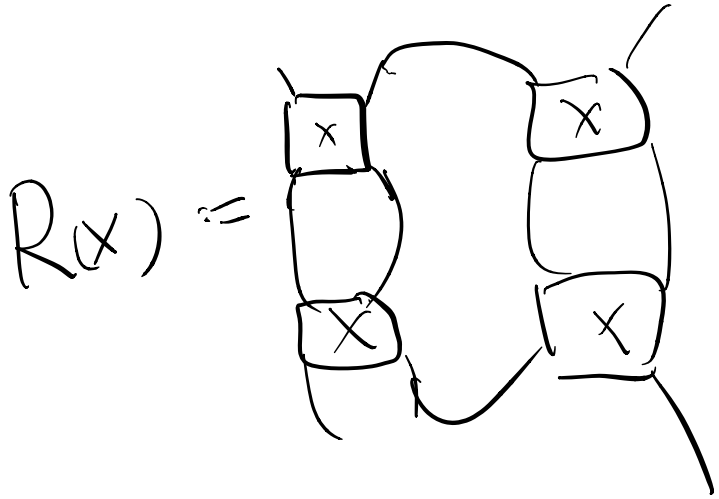
on higher-genus

surface.

bishifts of bipprojections

Gaussian functions

2b central limit theorem.



$x \geq 0$

$\sqrt{x} \geq 0$.

$\lim_{n \to \infty} R^n(x)$ ⟶ bipprojections

⟵ invariant.

$f \in C^\infty(\mathbb{R}^k)$

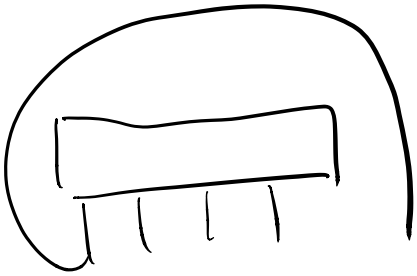
$\lim_{n\to\infty} \frac{1}{k} R^n(f) \longrightarrow$ Gaussian

↑ numerically fast

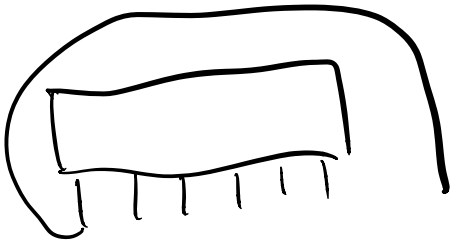
↑ Conjecture.

Surface algebra : Quon Language

→ a single qubit



$F^4 = 1$



$F^6 = 1$

$S_{i,j} =$ 

't Hooft