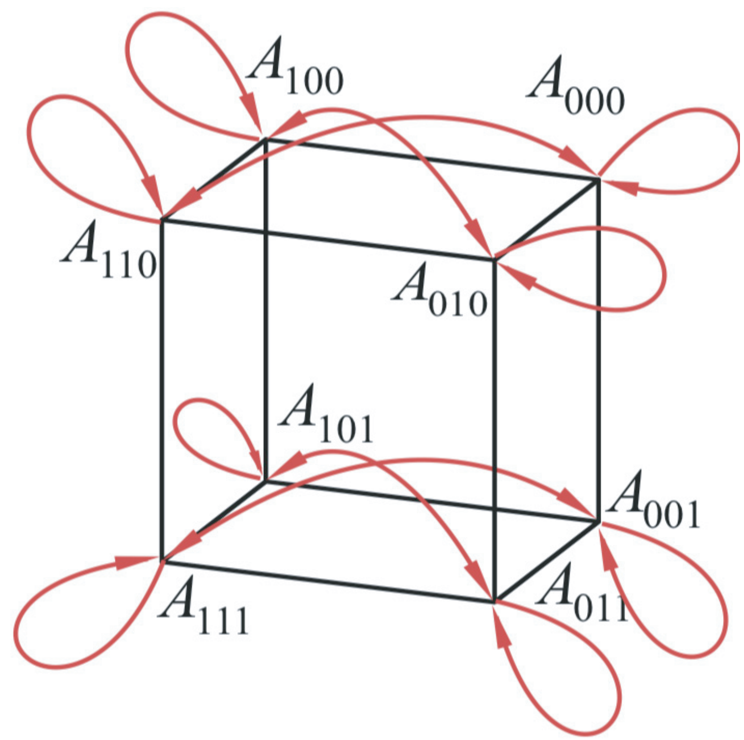
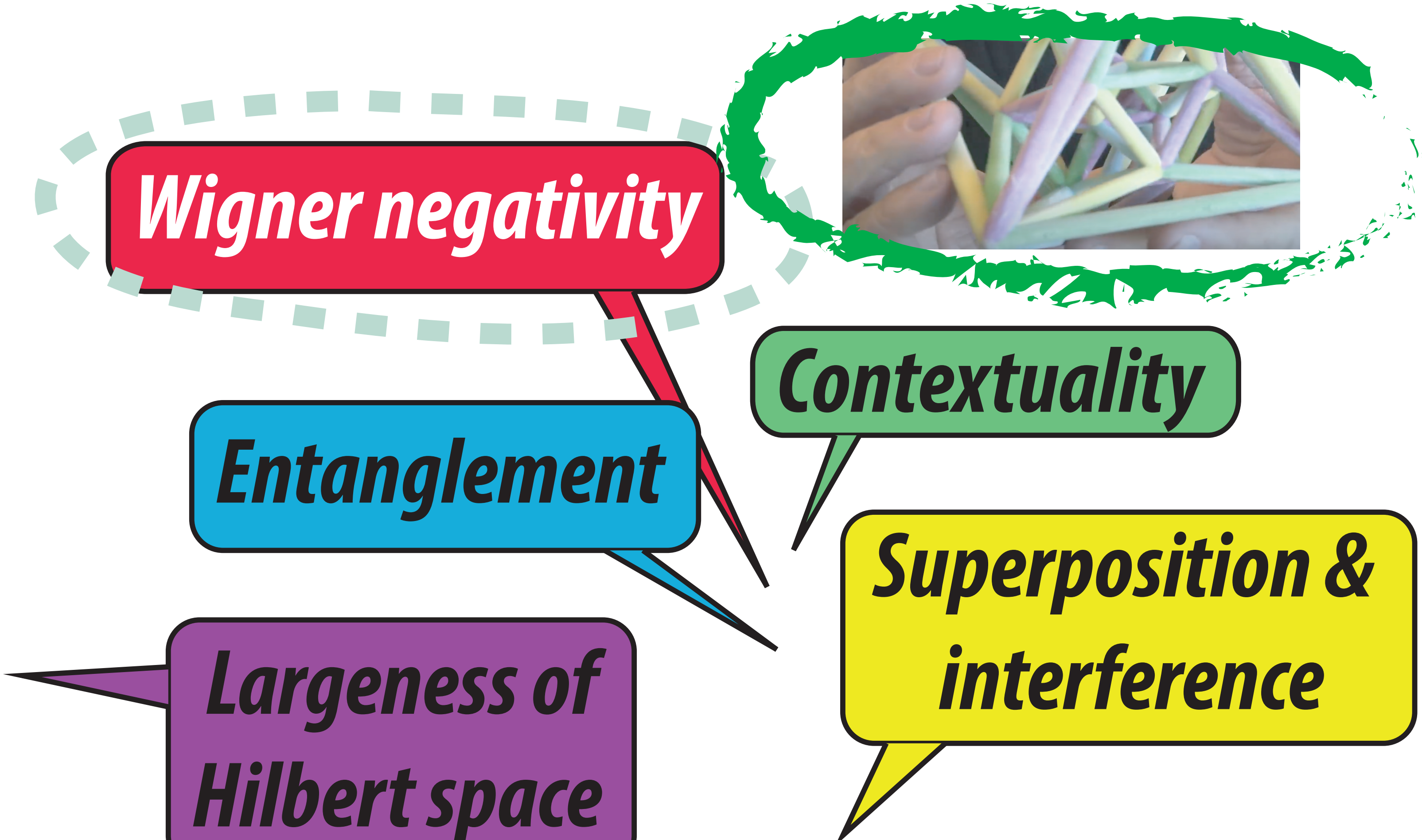
A hidden variable model for universal quantum computation with magic states on qubits



Robert Raussendorf, UBC Vancouver

Harvard Mathematical Picture Language Seminar, June 2020

Joint work with Michael Zurel and Cihan Okay, arXiv:2004.01992



What makes quantum computing work?

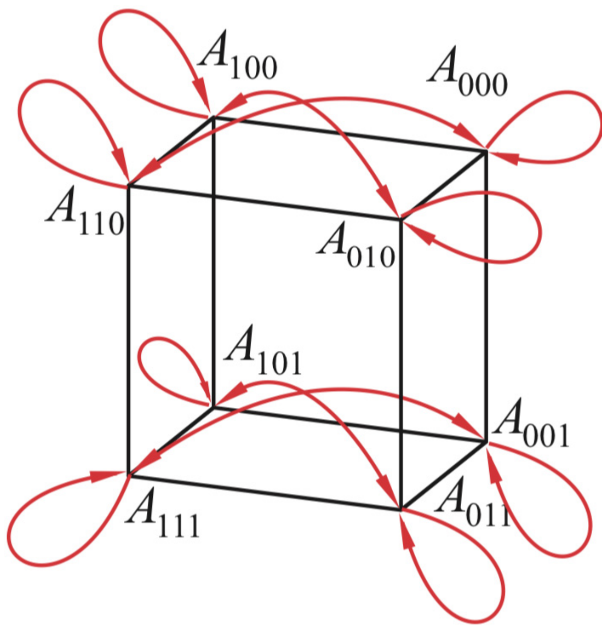
It is often pointed out that the fundamental objects in quantum mechanics are amplitudes, not probabilities [1], [2]. This fact notwithstanding, here we construct a description of universal quantum computation—and hence of all quantum mechanics in finite-dimensional Hilbert spaces—in terms of Bayesian update of a probability distribution. In this formulation, quantum algorithms look structurally akin to classical diffusion problems.

M. Zurel, C. Okay and R. Raussendorf, arXiv:2004.01992

Summary of the result

We have constructed a hidden variable model with positive representation for

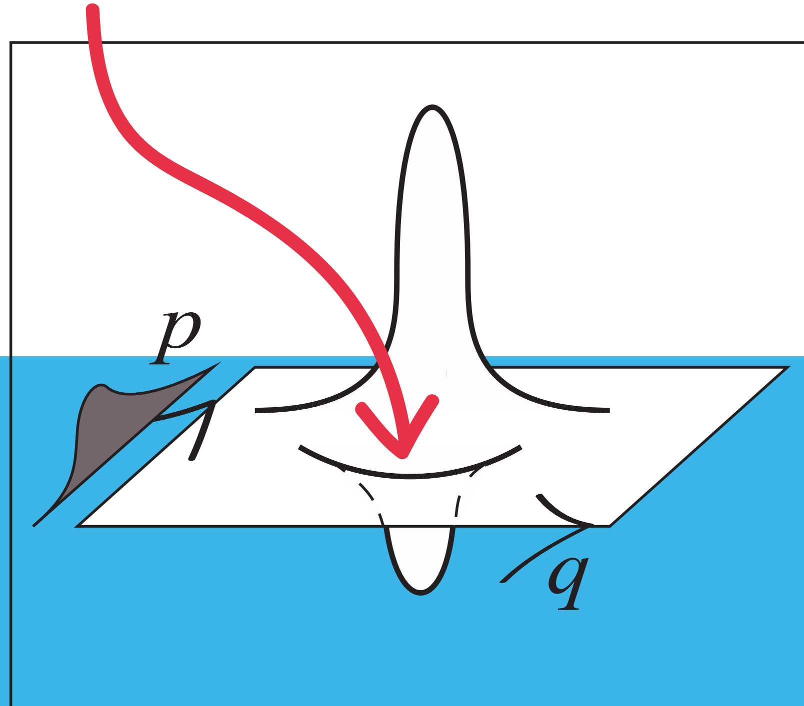
- All quantum states
- Clifford unitary gates
- Pauli measurements



Those operations suffice for universal quantum computation.

A bit of subject history

Negativity of the Wigner function
is an indicator of quantumness *



*: This even holds in quantum computation

Earlier results

Theorem[1]–[3]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

Negativity in the Wigner function
is necessary
for quantum computation[1] - [3]

[1] Qudits in odd d: V. Veitch *et al.*, New J. Phys. 14, 113011 (2012).
[2] Rebits: N. Delfosse *et al.*, Phys. Rev. X 5, 021003 (2015).
[3] Qubits: R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

Earlier results ... and a recent counterpoint

Theorem[1]–[3]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

Negativity in the Wigner function
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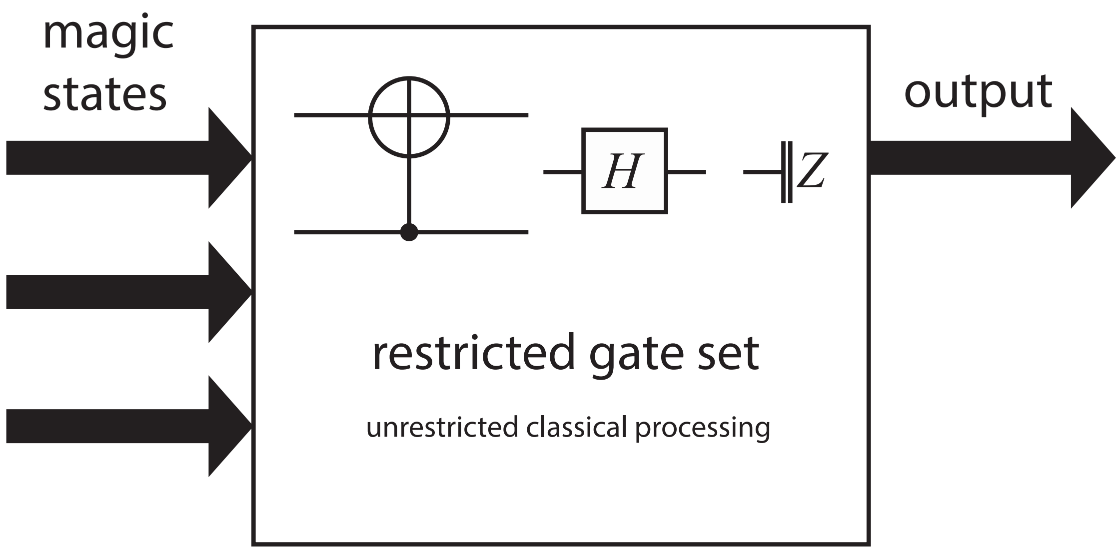
Counterpoint: The above theorem hinges on the precise definition of the Wigner function. *Quantum states and their update under measurement can be represented positively.*

[4] M. Zurel, C. Okay and T. Raussendorf, arXiv:2004.01992

Outline

1. Magic states, quantum computation, Wigner functions
   (a) Quantum computation with magic states
   (b) The role of Wigner functions and their negativity
   (c) The trouble with qubits
2. A hidden variable model for QM in finite dimensions
   (a) Model and result
   (b) Compatibility with the PBR theorem
   (c) Compatibility with Gleason's theorem
   (d) What does this mean for Wigner function negativity?

Quantum Computation with magic states

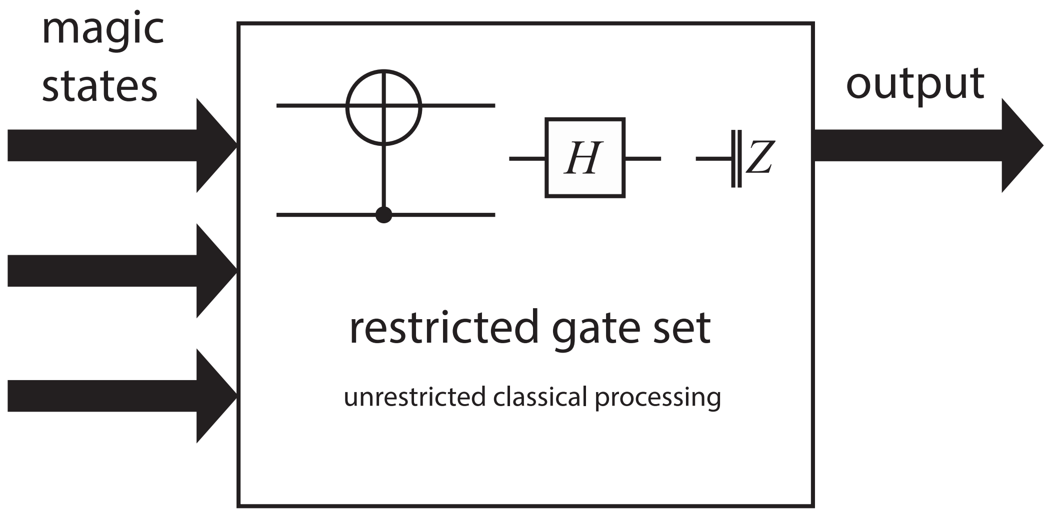


- Non-universal restricted gate set: *e.g. Clifford gates*.
- Universality reached through injection of *magic states*.

\+ *As of now, leading scheme for fault-tolerant QC.*

Computational power is shifted from gates to states

A question



Which properties must the magic states have to enable a speedup?

A: Wigner function negativity

Quantum speedup requires $W < 0$

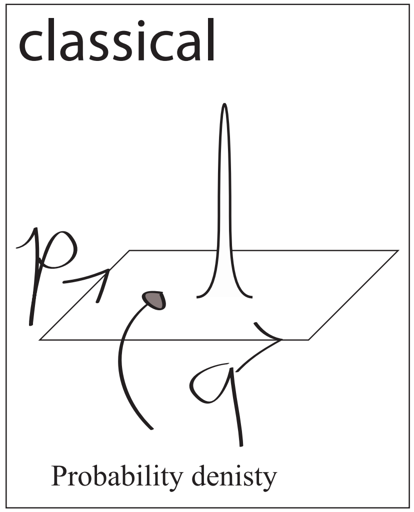
In the case of odd local Hilbert space dimension:

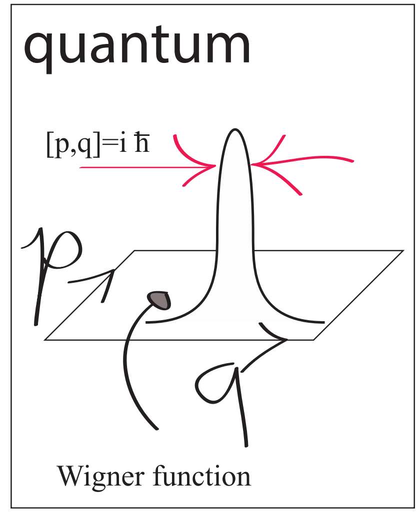
Theorem [*]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

In other words: If the Wigner function of the initial magic states is positive, then the corresponding quantum computation can be efficiently classically simulated.

*: V. Veitch *et al.*, New J. Phys 14 (2012).

[quantum] mechanics in phase space





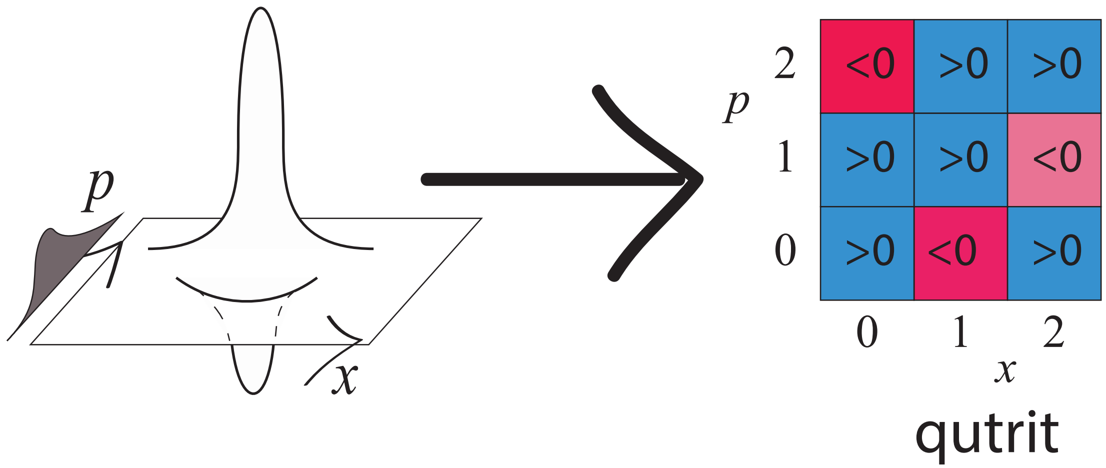
- The Wigner function

$$W_\psi(p,q) = \frac{1}{\pi}\int d\xi\, e^{-2\pi i \xi p} \psi^\dagger(q-\xi/2)\psi(q+\xi/2).$$

is a quasi-probability distribution.

It is the closest quantum counterpart to the classical probability distribution over phase space.

Wigner functions for qudits



Wigner functions can be adapted to finite-dimensional state spaces.

- The Wigner function W can assume negative values.
- The marginals of W are probability distributions.
- The phase space for W on n qu$d$its is a $d^n \times d^n$ grid.

Properties of the qudit Wigner function

(1) Covariance:

If and only if the local Hilbert space dimension is odd, then:

- The n-qudit Wigner function transforms *covariantly* under all Clifford unitaries.

  For any given Clifford unitary U it holds that

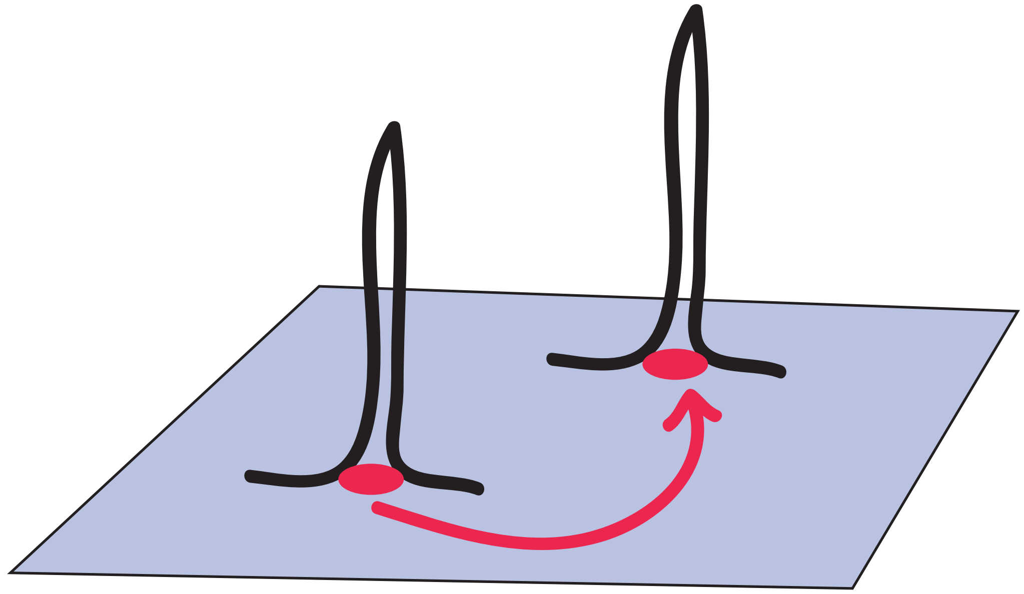
$$\boxed{W_{U\rho U^\dagger}(\mathbf{v}) = W_\rho(L_U \mathbf{v}), \ \ \forall \rho}$$

  with L_U an affine transformation on the phase space V.

D. Gross, PhD thesis, Imperial College London (2005).

Update under Clifford unitaries

For any given Clifford unitary U: $W_{U\rho U^\dagger}(\mathbf{v}) = W_\rho(L_U\mathbf{v}), \ \forall\rho.$



Properties of the qudit Wigner function

(2) Positivity preservation under measurement:

If and only if the local Hilbert space dimension is odd, then:

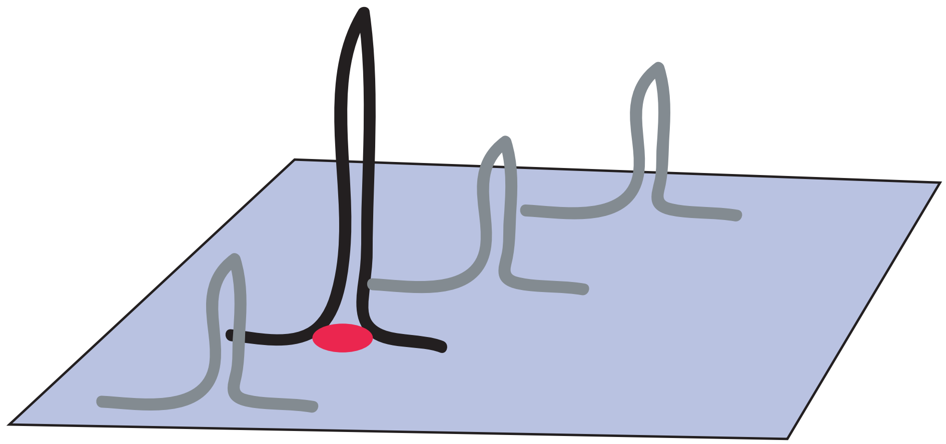
- The n-qudit Wigner function *preserves* positivity under all Pauli measurements.

  Denote $P_{\mathbf{a},s}$ the projector corrresponding to the measurement of the observable $T_{\mathbf{a}}$ with eigenvalue ω^s. Then,

$$\boxed{W_\rho \geq 0 \Rightarrow W_{P_{\mathbf{a},s}\rho P_{\mathbf{a},s}} \geq 0, \;\; \forall \mathbf{a}, \forall s.}$$

Update under Pauli measurements

deterministic outcome for all Pauli measurements,
probabilistic post-measurement state

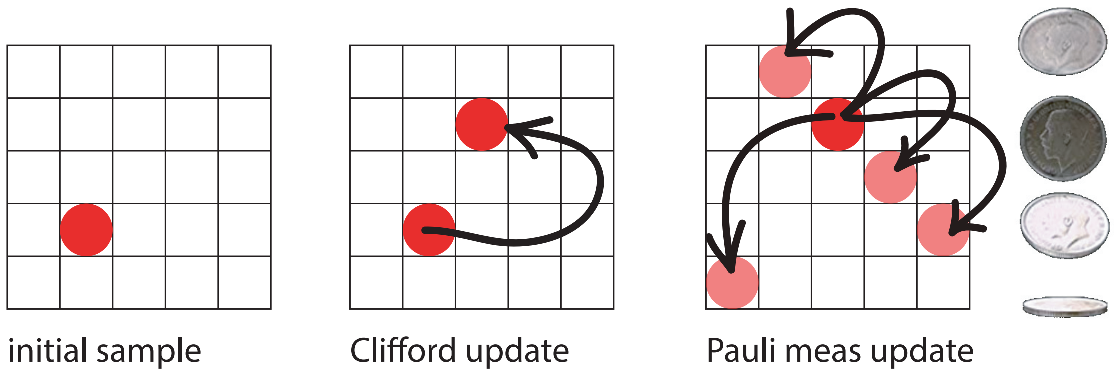


The result of Pauli measurement on a peaked Wigner function is a ridge.

Theorem – proof idea

We will show that:

If $W_{\rho_{\text{magic}}} \geq 0 \Rightarrow$ efficiently classical simulation $\Rightarrow$ no speedup.



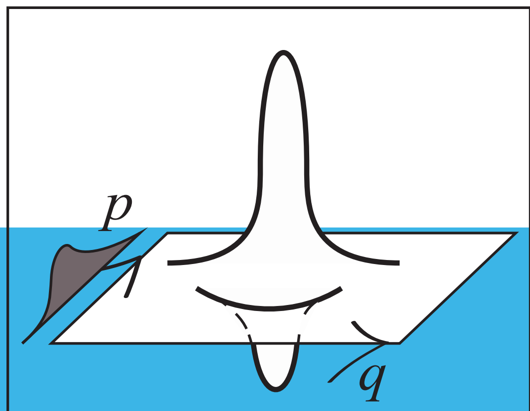
Simulation algorithm:

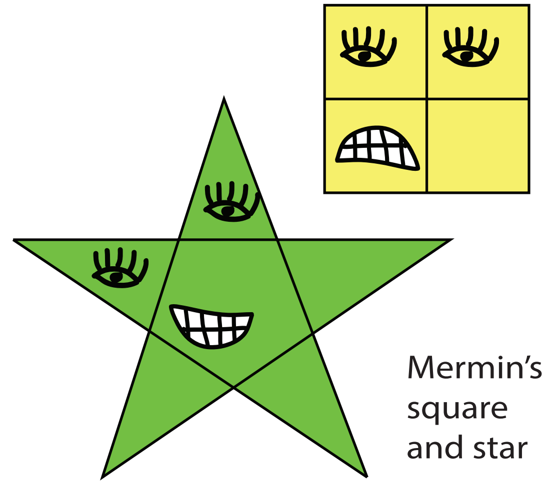
1. $W_{\rho_{\text{magic}}} \geq 0$ is a probability distribution. $\longrightarrow$ Sample from it! Each sample is a point in phase space.

2. Update the phase space points under Clifford gates and Pauli measurement.

The trouble with qubits

quantum optics:
Hilbert space dimension infinite





Odd: all nice & safe

quantum computation:
Hilbert space dimension finite

Even: monsters lurking

The trouble with qubits

Consider a Wigner function W such that for all states ρ

$$\rho = \sum_{\mathbf{v}} W_\rho(\mathbf{v}) A_{\mathbf{v}}.$$

Phase point operators $\{A_{\mathbf{v}}\}$ span the space of density matrices.

Theorem [*] If $\{A_{\mathbf{v}}\}$ is an operator basis then W cannot be Clifford covariant.

Theorem [**] If $\{A_{\mathbf{v}}\}$ is an operator basis then W cannot represent Pauli measurement positively.

Theorem [***] A memory of $O(n^2)$ bits is required for simulating contextuality on n-qubit systems.

Lesson: Base Wigner functions on over-complete sets $\{A_{\mathbf{v}}\}$.

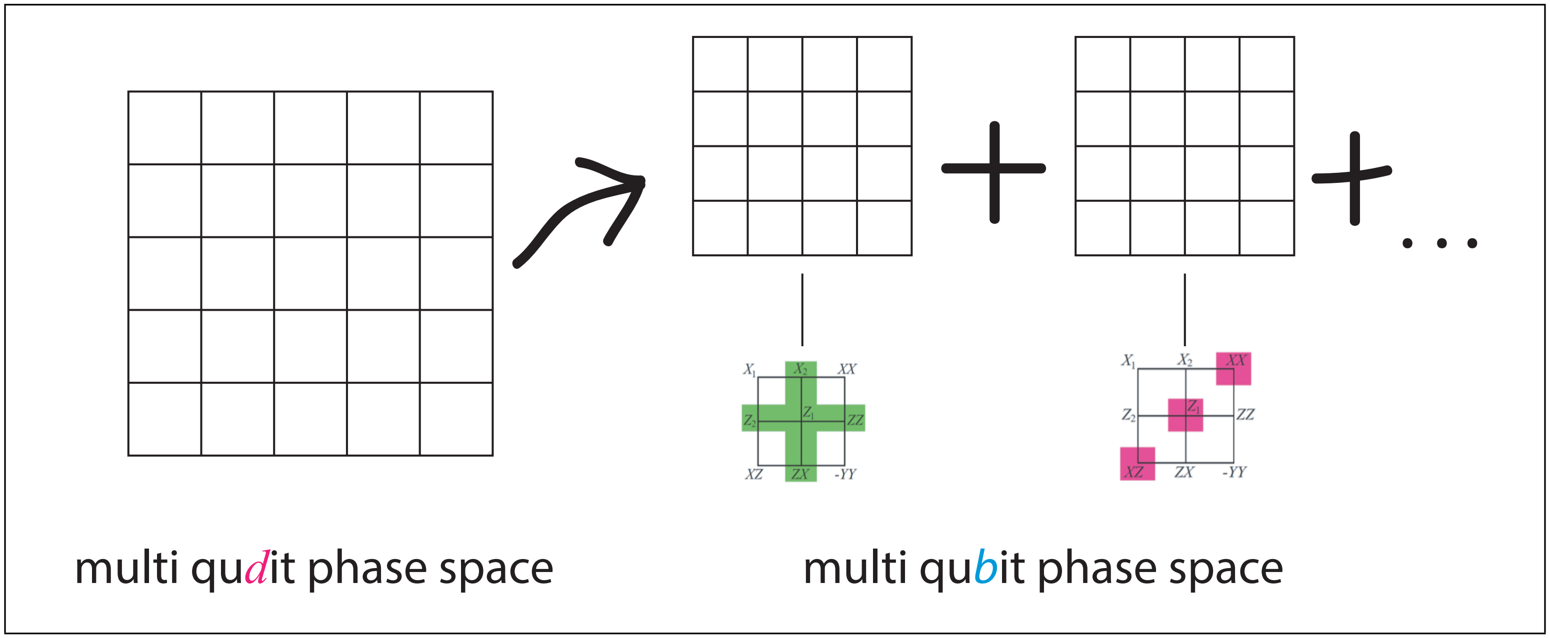
[*] H. Zhu, Phys. Rev. Lett. **116**, 040501 (2016).

[**] R. Raussendorf *et al.* Phys. Rev. A **95**, 052334 (2017).

[***] A. Karanjai *et al.*, arXiv:1802.07744.

Resolution for qubits

Lesson: Base Wigner functions on over-complete sets $\{A_{\mathbf{v}}\}$.



R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020),

also see: W.M. Kirby and P.J. Love, Phys. Rev. Lett. 123, 200501 (2019).

Resolution for qubits

Theorem: Quantum computation with magic states can have a quantum speedup only if the quasiprobability distribution $\tilde{W}_\rho$ of the initial magic state ρ is negative.*

Price to pay:

- Quasiprobability function $\tilde{W}$ is not unique.
- Phase space way more complicated than for qudits and rebits.

*: Btw, that is not the result of this talk.

R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

More on the role of negativity

Denote the robustness $\mathfrak{R}(\rho)$ of a state ρ by

$$\boxed{\mathfrak{R}(\rho) = \min_{\tilde{W}_\rho} \sum_{v \in \mathcal{V}} \left|\tilde{W}_\rho(v)\right|.}$$

Robustness is a measure of negativity.

In <u>direct analogy</u> with qudits, we have for the qubit case:

Theorem.* The classical simulation cost C for quantum computation with a magic state ρ by sampling, up to error ϵ, is

$$C \sim \frac{\mathfrak{R}(\rho)^2}{\epsilon^2}.$$

Qudits: V. Veitch *et al.*, New J. Phys. 14, 113011 (2012).

General framework: H. Pashayan *et al.*, Phys. Rev. Lett. 115, 070501 (2015)

Qubits: R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

*: Again, this is not the main result of this talk.

A hidden variable model

for multi-qubit states, Clifford gates and Pauli measurements

Model and result

•

Compatibility with the PBR theorem

•

Compatibility with Gleason's theorem

•

What does this mean for Wigner function negativity?

The HVM state space

Looking back one more time at [PRA 101, 012350 (2020)], the central result in it is ...

Theorem 2 *For any $n \in \mathbb{N}$, the set $\mathcal{P}_n$ of positively representable n-qubit quantum states is closed under Pauli measurement.*

Question: *For any number n of qubits, what is the largest set Λ_n of operators that is closed under Pauli measurement?*

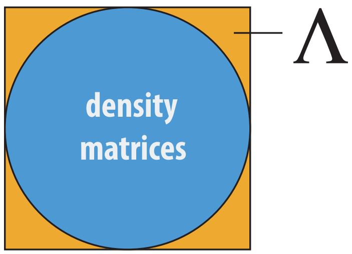
The HVM state space

Question: *For any number n of qubits, what is the largest state space Λ_n that is closed under Pauli measurement?*

Tinkering with this question brings up the following construct:

Definition: For any $n \in \mathbb{N}$, denote by $\mathcal{S}_n$ the set of pure n-qubit stabilizer states. Then, the polytope Λ_n is the set of operators $O \in \text{Op}(\mathbb{C}^{2^n})$ with the properties

1. O is Hermitian.
2. $\text{Tr}(O) = 1$.
3. $\text{Tr}(O\,|\sigma\rangle\langle\sigma|) \geq 0$, for all $|\sigma\rangle \in \mathcal{S}_n$.



- We can describe the state polytope Λ_n by its extremal vertices $\{A_\alpha,\ \alpha \in \mathcal{V}_n\}$. $\mathcal{V}_n$ is the generalized HVM phase space.
- $\mathcal{V}_n$ is finite for all n.

Everything is positive

Theorem 1. $\mathcal{V}_n$ has the following properties.

(i) *Positive representation.* All quantum states ρ are represented by a probability function $p : \mathcal{V}_n \to \mathbb{R}_{\geq 0}$,

$$\rho = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) A_\alpha.$$

(ii) *Positivity preservation.* Denote by $\Pi_{a,s}$ the projection corresponding to the measurement of the Pauli observable T_a with outcome s. Then,

$$\Pi_{a,s} A_\alpha \Pi_{a,s} = \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta,$$

where the $q_{a,\alpha}$ are probability functions.

(iii) *Born rule.* Denote by $P_{\rho,a}(s)$ the probability of obtaining outcome s in the measurement of the Pauli observable T_a on the state ρ. Then,

$$P_{\rho,a}(s) = \mathrm{Tr}(\Pi_{a,s}\rho) = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) Q_a(s|\alpha),$$

where all Q_a are conditional probability functions.

Proof of Theorem 1 – Part (i)

Statement:

(i) *Positive representation.* All quantum states ρ are represented by a probability function $p : \mathcal{V}_n \to \mathbb{R}_{\geq 0}$,

$$\rho = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) A_\alpha. \tag{1}$$

Proof: For every valid quantum state ρ it holds that

- ρ is Hermitian.
- $\mathrm{Tr}(\rho) = 1$.
- $\mathrm{Tr}(\rho\,|\sigma\rangle\langle\sigma|) \geq 0$, for all stabilizer states $|\sigma\rangle \in \mathcal{S}_n$.

Therefore, by the definition of Λ_n, $\rho \in \Lambda_n$ for all density matrices. Therefore, Eq. (1) holds. □

Proof of Theorem 1 – Part (ii)

Statement:

(ii) *Positivity preservation.* Denote by $\Pi_{a,s}$ the projection corresponding to the measurement of the Pauli observable T_a with outcome s. Then,

$$\Pi_{a,s} A_\alpha \Pi_{a,s} = \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta, \tag{2}$$

where the $q_{a,\alpha}$ are probability functions.

Proof: Recall from the definition of Λ_n that $\text{Tr}\left(A_\alpha \left|\sigma\right\rangle\left\langle\sigma\right|\right) \geq 0$, $\forall |\sigma\rangle \in \mathcal{S}_n$. This property is inherited under Pauli projection.

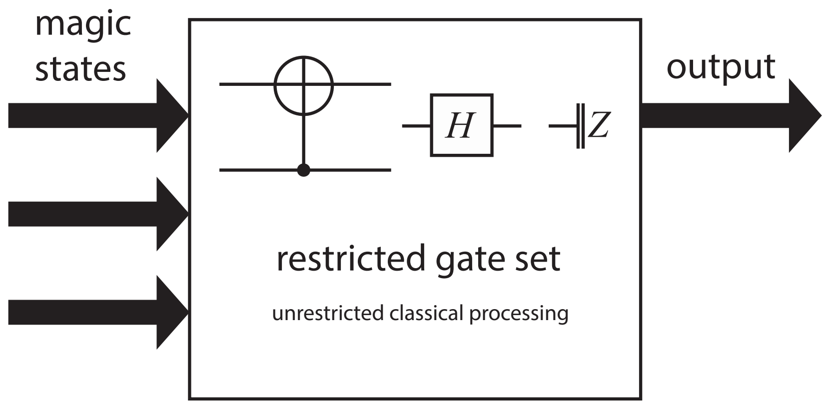
$$\begin{aligned} \text{Tr}\left(\Pi_{a,s} A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right|\right) &= \text{Tr}\left(A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right| \Pi_{a,s}\right) \\ &= \text{Tr}\left(A_\alpha (\Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right| \Pi_{a,s})\right) \\ &= (c \geq 0) \cdot \text{Tr}\left(A_\alpha |\sigma'\rangle\langle\sigma'|\right), \\ &= \geq 0. \end{aligned}$$

Main case: $\text{Tr}(\Pi_{a,s} A_\alpha) > 0$. $\frac{\Pi_{a,s} A_\alpha \Pi_{a,s}}{\text{Tr}(\Pi_{a,s} A_\alpha)} \in \Lambda_n$ & Eq. (2) holds.

Other case: $\text{Tr}(\Pi_{a,s} A_\alpha) = 0$. Then, $\text{Tr}\left(\Pi_{a,s} A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right|\right) = 0$ for all $|\sigma\rangle \in \mathcal{S}_n$, hence $\Pi_{a,s} A_\alpha \Pi_{a,s} = 0$, and Eq. (2) holds with $q_{\alpha,a} \equiv 0$. $\square$

Application to QC with magic states

Bottom line so far: all quantum states and their update under Pauli measurement can be positively represented.



Recall: QC with magic states consists of magic states, Clifford gates and Pauli measurement.

Clifford gates

Lemma. If $X \in \Lambda_n$ then $UXU^\dagger \in \Lambda_n$, for all Clifford unitaries U.

Proof. For any $X \in \Lambda_n$, U Clifford and all stabilizer states $|\sigma\rangle$ it holds that

$$\begin{aligned}
\mathrm{Tr}(UXU^\dagger|\sigma\rangle\langle\sigma|) &= \mathrm{Tr}(X\,U^\dagger|\sigma\rangle\langle\sigma|U) \\
&= \mathrm{Tr}(X\,|\sigma'\rangle\langle\sigma'|) \\
&\geq 0,
\end{aligned}$$

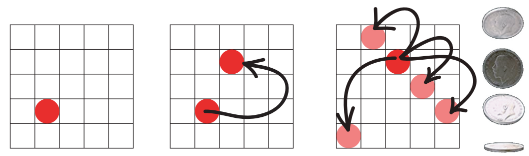
Hence, $UXU^\dagger \in \Lambda_n$. $\square$

Consequence: $UA_\alpha U^\dagger = \sum_{\beta\in\mathcal{V}_n} q_{\alpha,U}(\beta)A_\beta$, with all $q_{\alpha,U}(\beta) \geq 0$.

[In fact, we have Clifford covariance: $UA_\alpha U^\dagger = A_{U\alpha}$.]

Application to QC with magic states

Theorem 2. Universal quantum computation by Clifford unitaries and Pauli measurements on magic states can be described by iterated sampling from probability functions.



This is about universal QC, hence all quantum mechanics in finite Hilbert space dimension.

Both the states and the operations are positively represented.

We do not claim to efficiently simulate universal quantum computation.

Consistency

and consequences

Consistency with PBR

The Pusey-Barrett-Rudolph theorem states that

Theorem. Any model in which a quantum state represents mere information about an underlying physical state of the system [HVM], *and in which systems that are prepared independently have independent physical states*, must make predictions which contradict those of quantum theory.

The condition in italics is called "preparation independence". Our HVM does not satisfy it,

$$A_\alpha \otimes A_\beta \neq A_\gamma.$$

Thus no contradiction with PBR.

Consistency with Gleason

Theorem (Gleason). In Hilbert spaces $\mathcal{H}$ of dimension ≥ 3, the only way to consistently assign probabilities $p(h)$ to subspaces $h \subset \mathcal{H}$, represented by the projection Π_h, is via

$$p(h) = \mathrm{Tr}(\Pi_h \, \rho), \tag{3}$$

for some valid density matrix ρ.

- We do not contradict Eq. (3), but we rather reproduce it.

- Informal version of Gleason: "Density operators are for QM, probabilities for classical statistical mechanics"
  $\Rightarrow$ We object to that. We describe all QM states ρ by probability functions p_ρ.

- Possible because of restriction to Pauli measurement. Does not affect computational universality, though.

Consequence: Wigner negativity

- For *suitably defined* Wigner functions on qudit, rebit and qubit systems, negativity remains a precondition for quantumness. (The prior results are not invalidated.)

- However, it is possible to define a probability function that has the same scope of classical simulability, without the need for any negativity.

Negativity as indicator of quantumness is an artifact of a special choice.

Conclusion

- We have described a hidden variable model for universal quantum computation where all states, operations, and measurements are represented by classical probabilities.

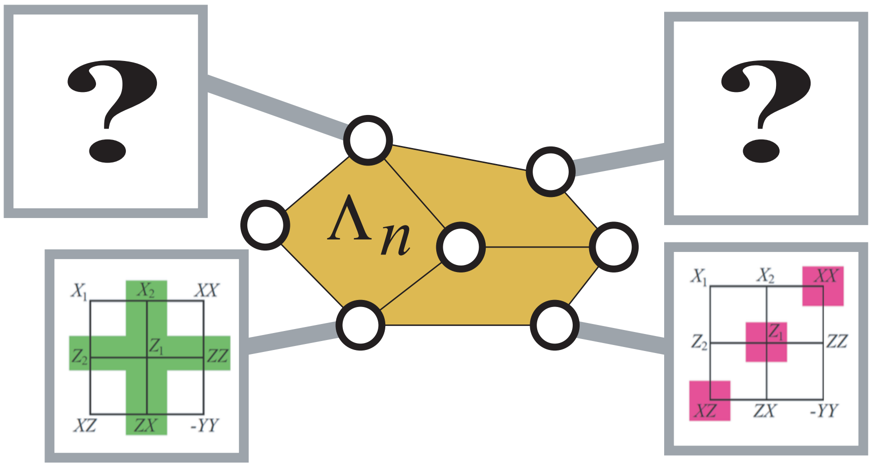
- No negativity is required anywhere.

- The classical simulation algorithm is not necessarily efficient.

arXiv:2004.01992

Outlook

From the perspective of quantum computation, the interesting objects are the extremal vertices A_α of the state polytope Λ_n.



Can those vertices be fully classified?

Where and how is quantumness hiding in them?

arXiv:2004.01992