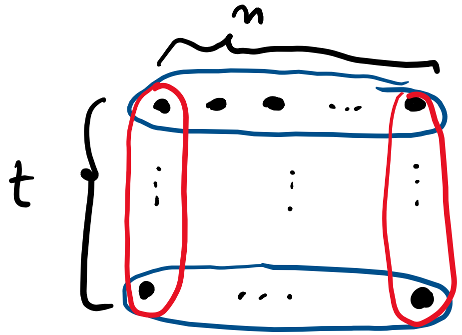
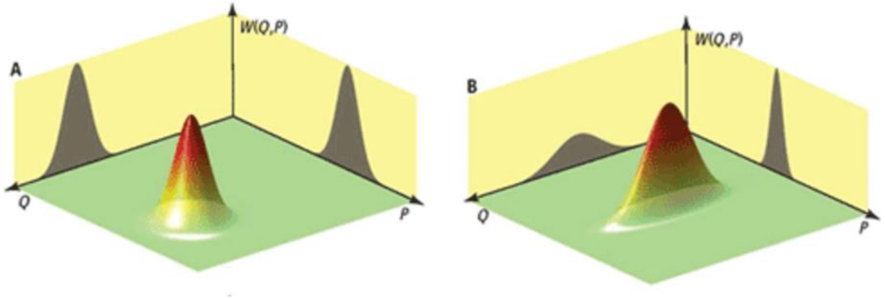
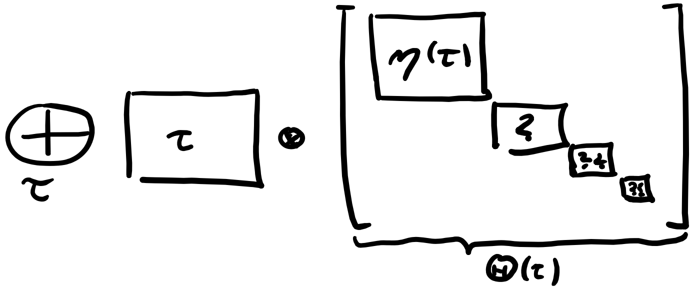
The Representation Theory of the Clifford Group
with
Applications in Quantum Information





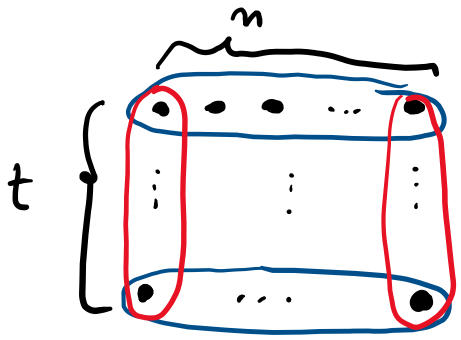


David Gross, University of Cologne

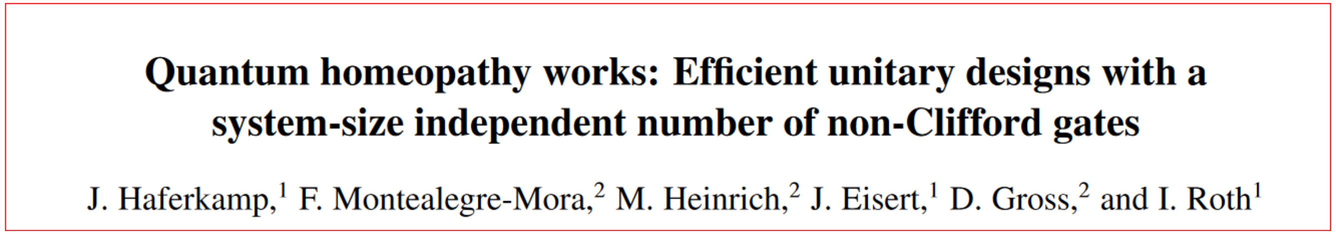
Mathematical Picture Language Project

Work with: Sepehr Nezami, Michael Walter, Felipe Montealegre, Huangjun Zhu, Markus Heinrich, Jonas Haferkamp, Ingo Roth, Jens Eisert (that’s it, though)

Outline

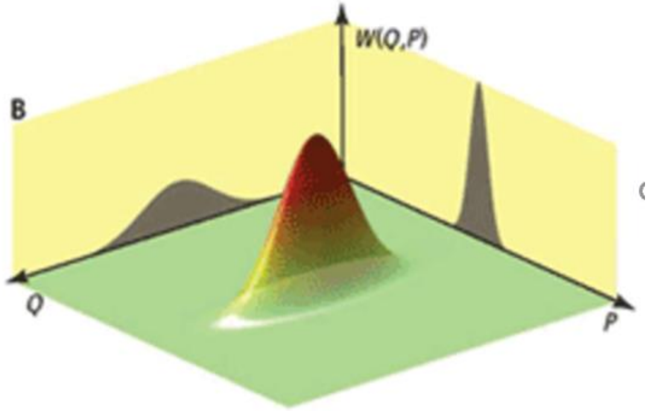


- Clifford?
- Some representation theory of the Clifford group
- Applications
- [More representation theory]



Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates

J. Haferkamp,[1] F. Montealegre-Mora,[2] M. Heinrich,[2] J. Eisert,[1] D. Gross,[2] and I. Roth[1]



Clifford?

Set up dictionary:

Continuous variables *many-body QM*	*Discrete variables* *Quantum information*
Gaussian states	Stabilizer states
CCR, Weyl operators	Generalized Pauli operators / Weyl operators
Canonical transformations	Clifford group
	Stabilizer codes

"Physics is that subset of human experience which can be reduced to coupled harmonic oscillators"

-- Michael Peskin

"Quantum information is that subset of physics which can be reduced to Clifford actions on stabilizers"

-- David Gross

Continuous variables: Gaussian states

- Hilbert space: $L^2(\mathbb{R}^n)$
- Gaussian wave functions

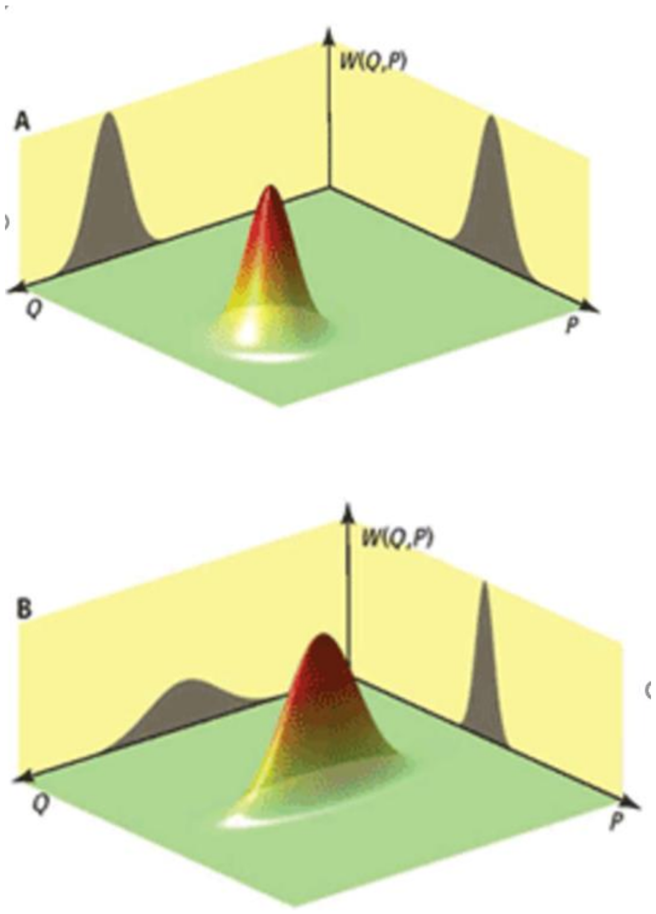
$$\psi(x) \simeq \exp(-\,x\;C\;x + ix\;b)\,, \qquad \vec{x} \in \mathbb{R}^n, C \in \mathbb{C}^{n\times n}$$

- Ground states of Hamiltonians quadratic in X_i, P_j

$$H = \begin{bmatrix} P_1 \\ \vdots \\ P_n \\ Q_1 \\ \vdots \\ Q_n \end{bmatrix} \begin{bmatrix} X_{11} & \cdots & X_{1\,2n} \\ \vdots & \ddots & \vdots \\ X_{2n\,1} & \cdots & X_{2n\,2n} \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_n \\ Q_1 \\ \vdots \\ Q_n \end{bmatrix}$$

- Characterized by $(2n)^2$ values of *covariance matrix*

$$\langle\psi|X_i P_j|\psi\rangle$$



Continuous variables: Symmmetries 1/2

- Position & momentum shifts: Weyl operators

$$w(p,q) = e^{i\,(p\,\hat{Q} - q\,\hat{P})}, \qquad (p,q) \in \mathbb{R}^2$$

$$w(p,0) \mathrel{\hat{=}} \begin{bmatrix} \ddots & & \\ & e^{ipx} & \\ & & \ddots \end{bmatrix}$$

- Group law involves *symplectic form*

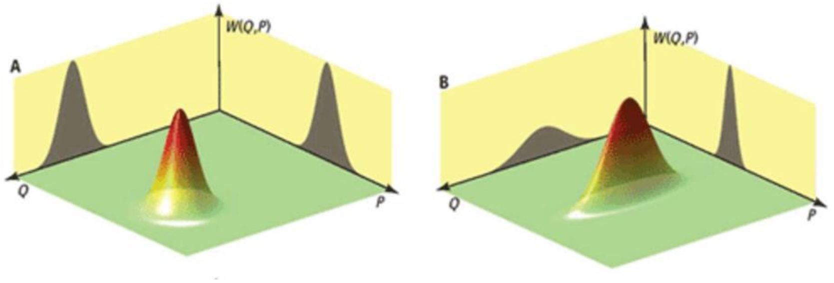
$$w(p,q)w(p',q') = w(p+p',q+q')e^{i\,\pi\,(p\,q' - q\,p')}$$

$$w(0,q) \mathrel{\hat{=}} \overset{q\,\updownarrow}{} \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

- *Canonic transformations*: Unitaries preserving group law

$$U\,w(p,q)\;U^\dagger = w\big(S(p,q)\big) \qquad S \in \mathrm{Sp}(\mathbb{R}^{2n})$$

- …represent *symplectic group* (up to phases).
  - Metaplectic representation, oscillator representation, Bosonic Bogoliubov transformation, canonic transformations…

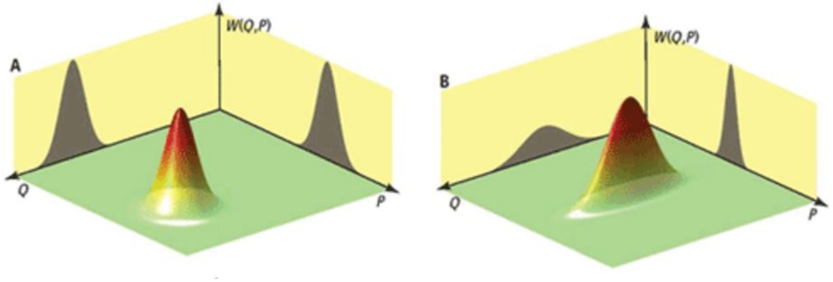


Continuous variables: Symmmetries 2/2

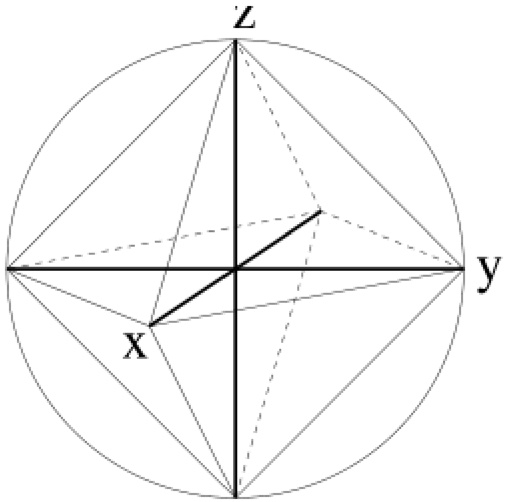
(Simple) Fact:

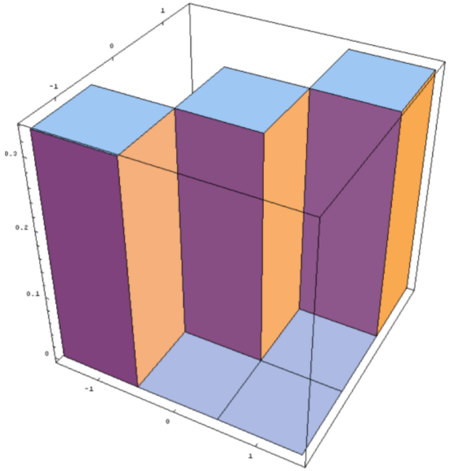
Gaussian wave functions form orbit under affine symplectic group $\mathrm{Sp}(2n) \ltimes \mathbb{R}^{2n}$.

$$\psi(\vec{x}) \simeq \exp(-\vec{x}\,C\,\vec{x} + i\vec{x}b)\,, \qquad \vec{x} \in \mathbb{R}^n$$



Discrete variables: stabilizer states 1 / 2





- Hilbert space:

$$\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d \simeq L^2(\mathbb{Z}_d^n)$$

- *Stabilizer states* include *discrete Gaussians…*

$$\psi(x) \simeq \exp\left(\frac{2\pi\, i}{d}(-x\, C\, x + xb)\right)$$

$$x, b \in \mathbb{Z}_d^n, C \in \mathbb{Z}_d^{n\times n}$$

- …and a bit more.

[Disclaimer: Will sweep some technicalities in even characteristic ($d = 2$) under the rug]

Symmetries: Discrete

- Position & momentum shifts by $(p, q) \in \mathbb{Z}_d^{2n}$:

$$X: |x\rangle \mapsto |x+1\rangle, \quad Z: |x\rangle \mapsto e^{i\frac{2\pi}{d}x}|x\rangle, \quad W(p,q) = Z^p X^q.$$

$$\begin{bmatrix} \omega^{1p} & & & \\ & \omega^{2p} & & \\ & & \ddots & \\ & & & \omega^{(d-1)p} \end{bmatrix}$$

$$\begin{bmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{bmatrix}$$

- Group law involves *symplectic form*

$$w(p,q)w(p',q') = w(p+p', q+q')e^{i\,2\pi/d\,(p\,q'-q\,p')}$$

- *Clifford transformations*: Unitaries preserving group law

$$U\,w(p,q)U^\dagger = w\big(S(p,q)\big) \qquad S \in \mathrm{Sp}\big(\mathbb{Z}_d^{2n}\big).$$

- Generated by *phase gate, Hadamard, controlled-Z*

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$C\text{-}Z = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

Discrete variables: stabilizer states 2 / 2

$$w(p,q)w(p',q') = w(p+p',q+q')e^{i\,2\,\pi/d\,(p\,q'-q\,p')}$$

- Weyl operators commute ⇔ symplectic form vanishes $p\,q' - q\,p' = 0.$

Definition

- Subspace $M \subset \mathbb{R}^{2n}$ is *isotropic* if symplectic form vanishes on M
- Abelian group $w(M)$ is a *stabilizer group*
- Joint $+1$ subspace is *stabilizer code*
- If $\dim M = n$, subspace is one-dimensional ⇒ *stabilizer state*
- Result: Stabilizer states are one affine symplectic orbit; all Gaussians arise this way

$$X \otimes X \left(|00\rangle + |11\rangle \right) = |00\rangle + |11\rangle$$

$$Z \otimes Z \left(|00\rangle + |11\rangle \right) = |00\rangle + |11\rangle$$

Application: Quantum Fault Tolerance

- Noise challenges quantum computers

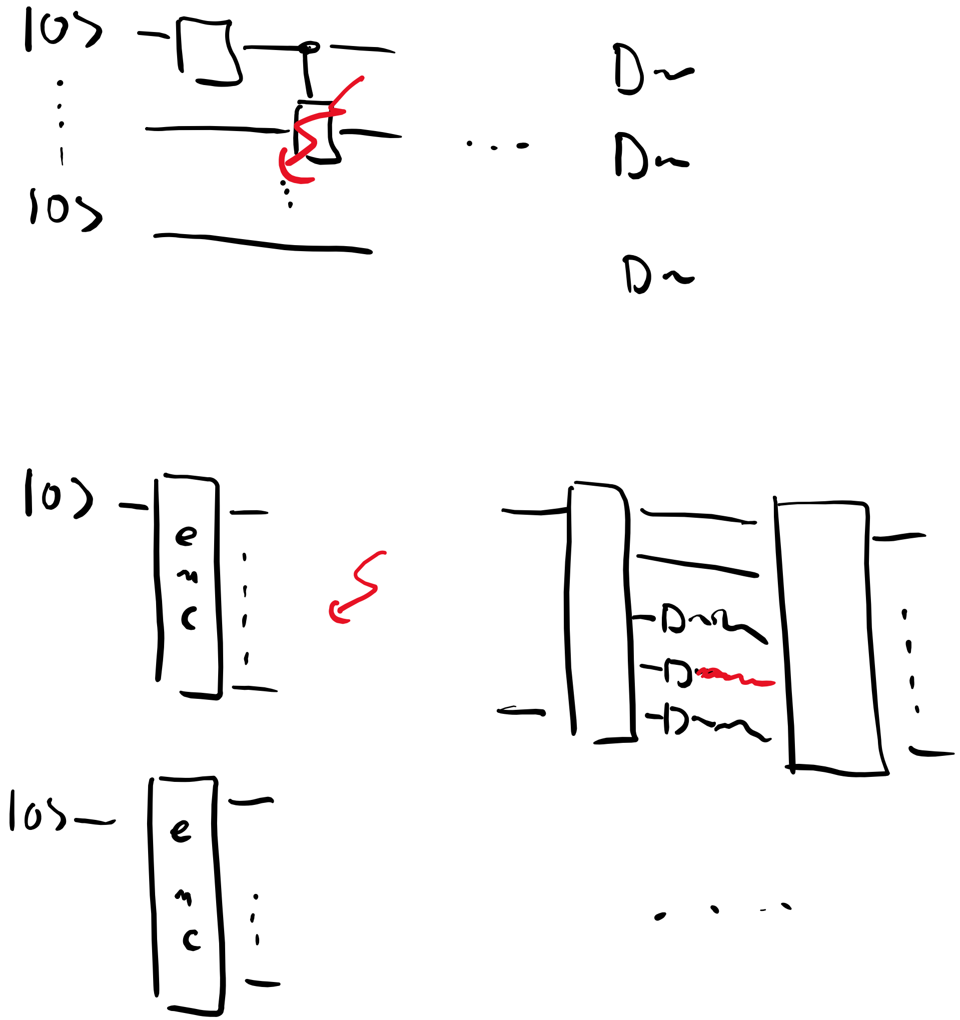
Fault tolerance:

- Encode each *logical qubit* into may physical qubits
- …s.t. few-qubit errors can be detected & corrected

Fact:

- all known encodings are stabilizer codes
- all robust gates are Clifford gates

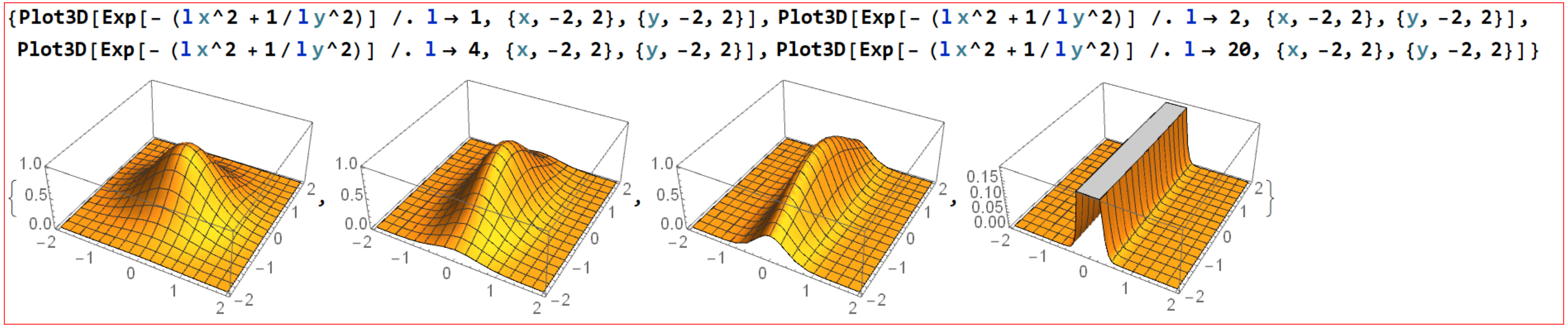
(all = virtually all ☺)



Discrete variables: stabilizer states 3 / 2

$$\psi(x) \simeq \exp\left(\frac{2\pi\, i}{d}(-x\, C\, x + xb)\right) \qquad\qquad \psi(x) \simeq \exp(-\, x\, C\, x + ix\, b)$$

How are covariance matrices related to stabilizer groups?



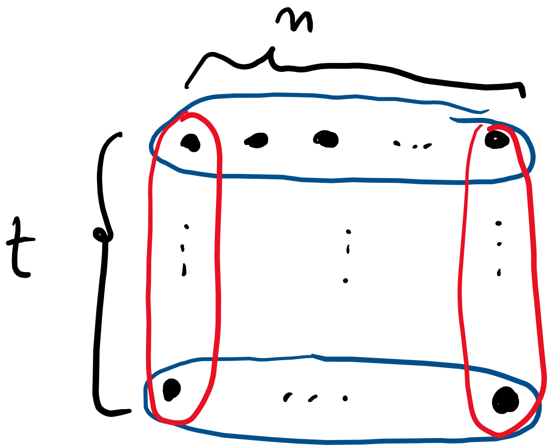
```
{Plot3D[Exp[- (l x^2 + 1 / l y^2)] /. l → 1, {x, -2, 2}, {y, -2, 2}], Plot3D[Exp[- (l x^2 + 1 / l y^2)] /. l → 2, {x, -2, 2}, {y, -2, 2}],
 Plot3D[Exp[- (l x^2 + 1 / l y^2)] /. l → 4, {x, -2, 2}, {y, -2, 2}], Plot3D[Exp[- (l x^2 + 1 / l y^2)] /. l → 20, {x, -2, 2}, {y, -2, 2}]}
```

Stabilizer group = invariant direction of “infinitely squeezed states”.

Clifford?

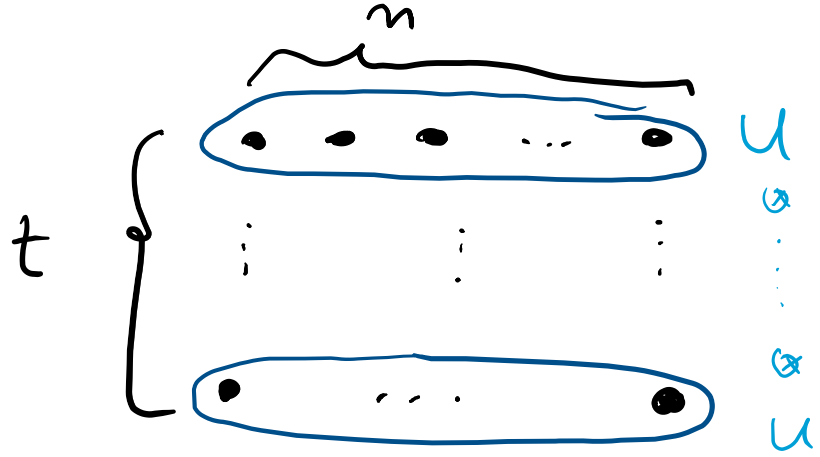
Continuous variables *many-body QM*	*Discrete variables* *Quantum information*
Gaussian states	Stabilizer states
CCR, Weyl operators	Generalized Pauli operators / Weyl operators
Canonical transformations	Clifford group
	Stabilizer codes

- Rich enough to capture interesting phenomena
- Well-behaved enough to allow explicit analysis



Some representation theory of the Clifford group

Central question



$\bullet = L^2(\mathbb{R})$

$\bullet = \mathbb{C}^d$

Want to understand:

Representation theory of t-fold tensor powers $U^{\otimes t}$ of Clifford unitaries U.

But why should I care?

Quantum Info: Representation theory of *t*-th tensor powers used in, e.g.:

- Randomized benchmarking
- Decoupling technique
- Non-malleable quantum one-time pads

$t=2$

- Variance bounds for randomized benchmarking
- Stabilizer POVM optimal state-independent measurement for pure states

$t=4$

- Simulating quantum circuits via low-rank stabilizer decompositions

$t=5$

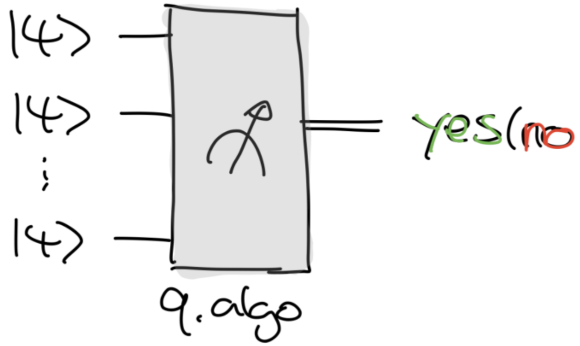
- Efficient spherical/unitary designs
- Connected to *discrete Howe-Kashiwara-Vergne duality*.

$t \in \mathbb{N}$

Stabilizer testing (Guess whether Gauss)

Guess whether Gauss:

- Given t copies of unknown state $\psi^{\otimes t}$ decide:
  - ψ is a stabilizer state, or
  - ψ is far away from the set of stabilizer states?



Problem [Montanaro, de Wolf]:

- Possible for dimension-independent t?

Guess whether Gauss

Ansatz

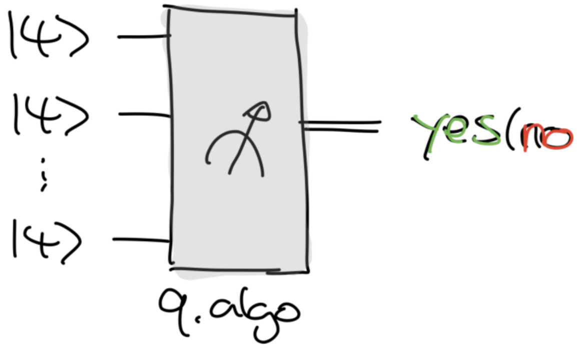
- Fix some t.
- Let P_{stab} project onto span of t-th tensor power of stabilizer states
- Use POVM $\{P_{\text{stab}}, I - P_{\text{stab}}\}$

Rep-theory: P_{stab} is in commutant of Cliff^t.

Negative results: P_{stab} = P_{Sym} for

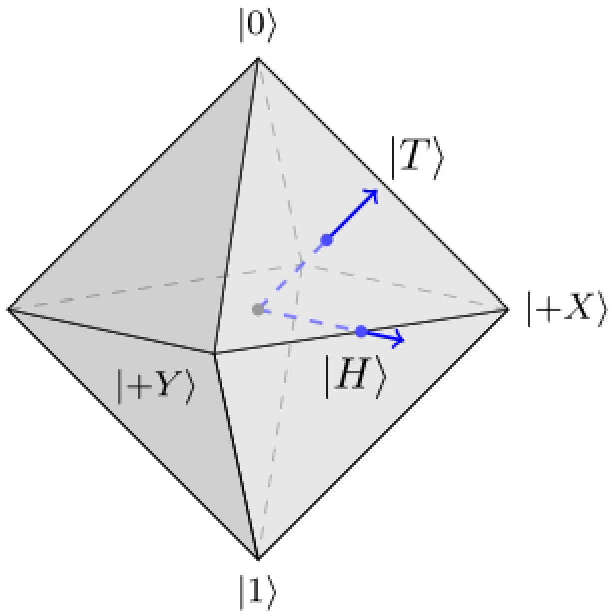
- $t = 1$ [trivial]
- $t = 2$ [DiVincenzo, Leung, Terhal 2002]
- $t = 3$ [Webb; Zhu; Kueng, DG 2015]
- $t = 4$ [Zhu, Grassl, Kueng, DG 2016]
- $t = 5$ [Walter, Nezami, DG 2018]

Later: Optimal solution for $t = 6$.

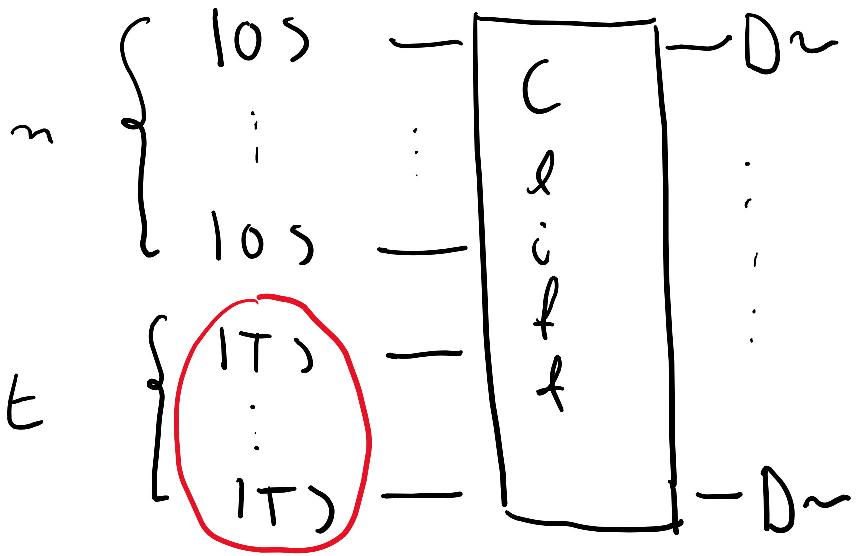


Lemon to Lemonade: Stabilizer rank

- [Gottesman-Knill]
  Cliffords on stabs are efficiently classically simulable
- [Bravyi-Kitaev]
  Cliffords on non-stabilizers are universal.

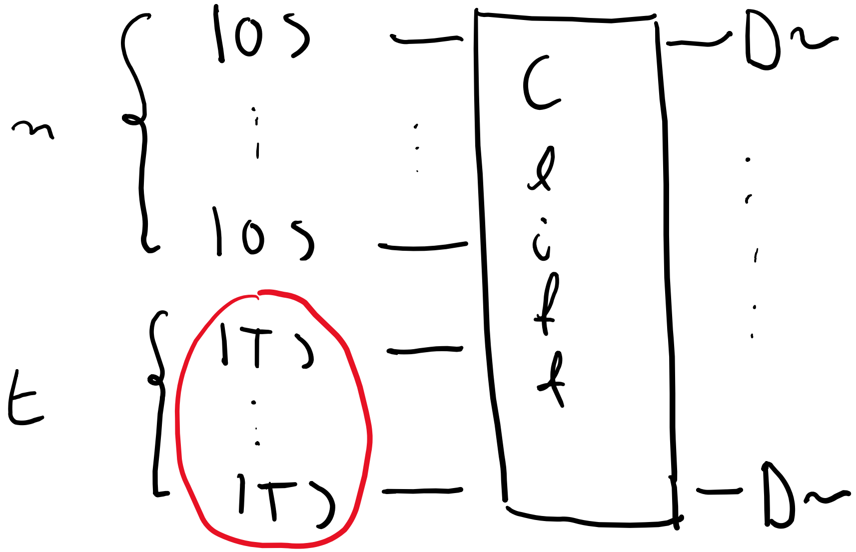


- Magic state model:



Q: What's best exponent *c* such that one can simulate in $O(2^{c\,t})$ time?

Stabilizer rank



- Expand magic states into superposition of stabilizers…

$$|T\rangle^{\otimes t} = \sum_{i=1}^{r} c_i \, |s_i\rangle$$

- …efficient algorithm if $r \ll 2^t$.

- Maybe best current classical simulation algorithm.

Q: Can one bound stabilizer rank?

[Bravyi, Brown, Calpin, Campbell, Gosset, Howard `18]

Lemon to Lemonade: Stabilizer rank

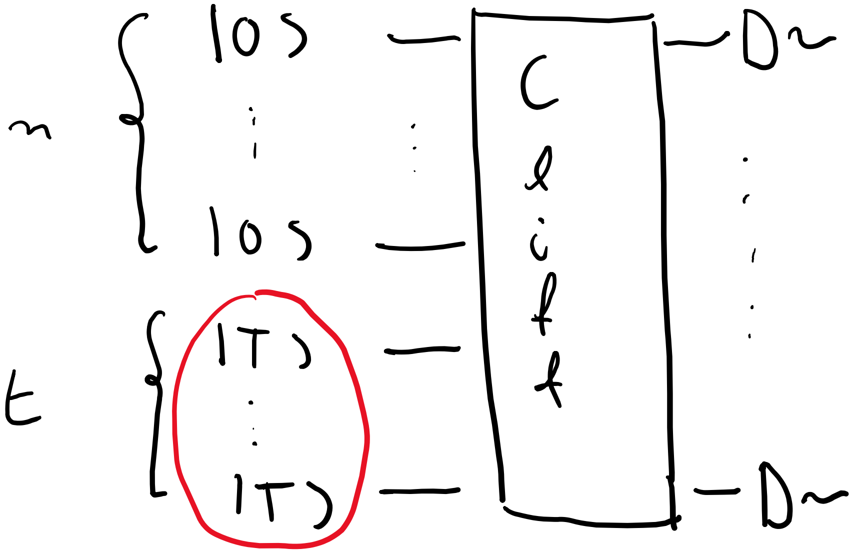
Theorem [Nezami, Walter, DG 18; Zhu, Grassl, Kueng, DG 16]

- For $t \leq 5$, the powers $|S\rangle^{\otimes t}$ of stabilizer states span symmetric space $\mathrm{Sym}^t(\mathbb{C}^{2^n})$.

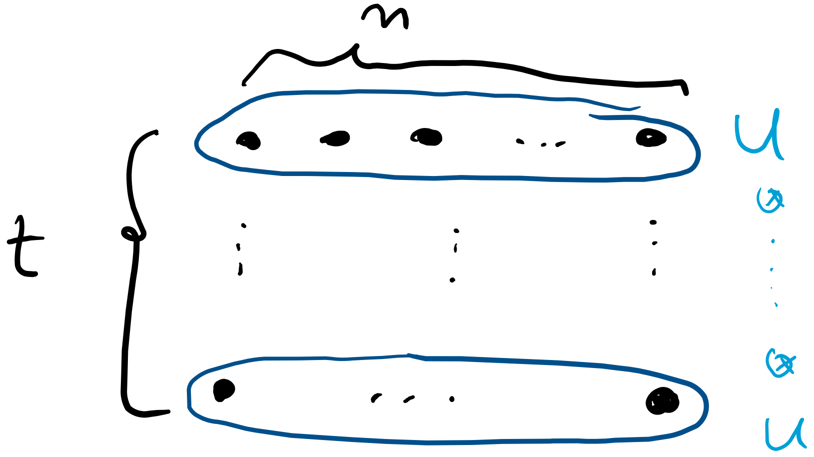
- For powers of single qubit states:

$$\mathrm{stabrank}(|\psi\rangle^{\otimes 5}) \leq \dim \mathrm{Sym}^5(\mathbb{C}^2) = 6 \ll 2^5 = 32.$$

- ⇒ Best-known general bound on stabilizer rank.



Central question



$\bullet = L^2(\mathbb{R})$

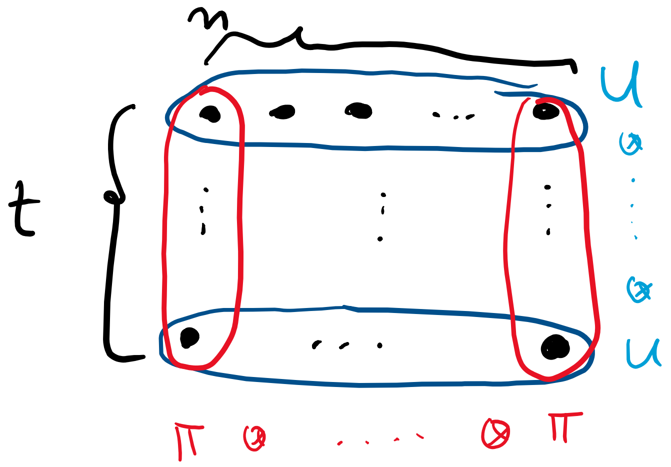
$\bullet = \mathbb{C}^d$

Want to understand:

Representation theory of t-fold tensor powers $U^{\otimes t}$ of Clifford unitaries U.

What would qualify as "understanding"?

Schur-Weyl duality 1



- On t-th tensor power $\mathcal{H}^{\otimes t}$ of a Hilbert space $\mathcal{H}$, commuting actions:

$$U(\mathcal{H}) \ni U \mapsto U \otimes \cdots \otimes U,$$

$$S_t \ni \pi\colon\ |\psi_1\rangle \otimes \cdots \otimes |\psi_t\rangle \mapsto |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_t}\rangle.$$

- Operator A commutes with $U^{\otimes t}$ iff

$$A = \sum_{\pi \in S_t} c_\pi \, \pi$$

and vice versa.

[Nezami, Walter, DG 18]

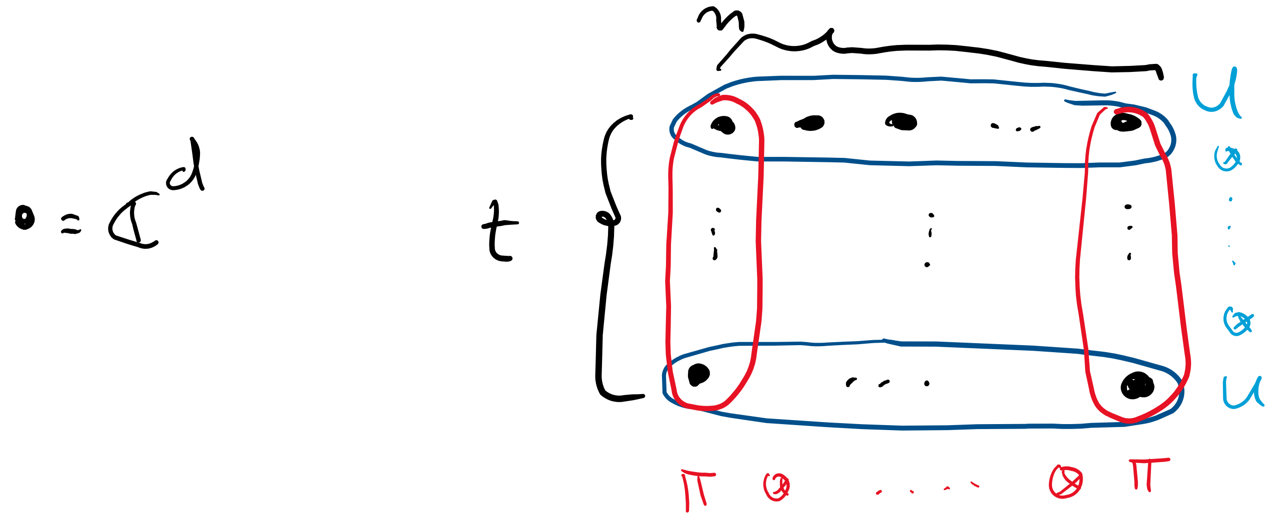
- Under action of $S_t \times U(\mathcal{H})$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_\lambda S_\lambda \otimes U_\lambda$$

- S_λ irrep of S_t, U_λ irrep of $U(\mathcal{H})$.

[Montealegre-Mora, DG 19]

Schur-Weyl duality 2: Transversality

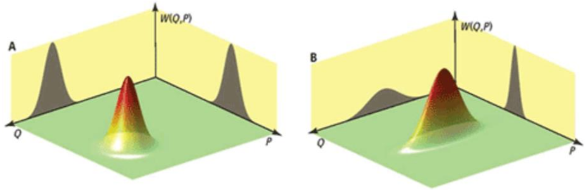


Both algebras

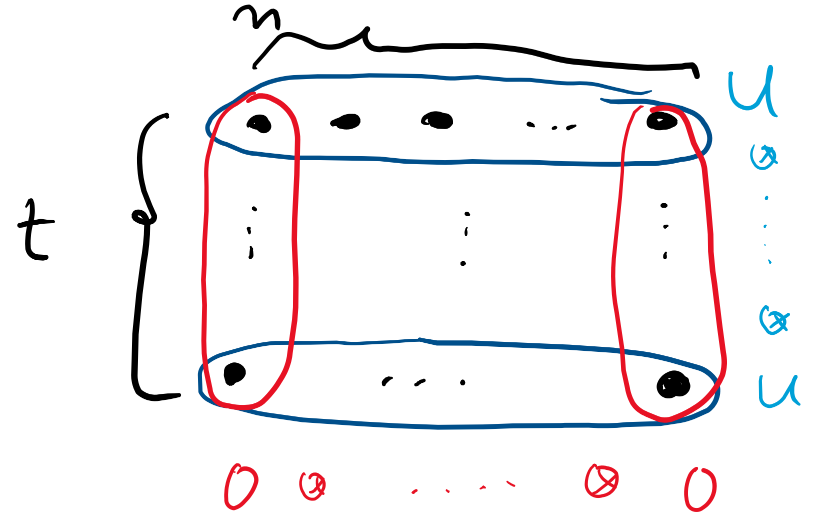
- Generated *uniformly* by *tensor powers*!
- Which are also *unitary*!

⇒ Often, results are *independent of n!*

CV: Howe-Kashiwara-Vergne Duality

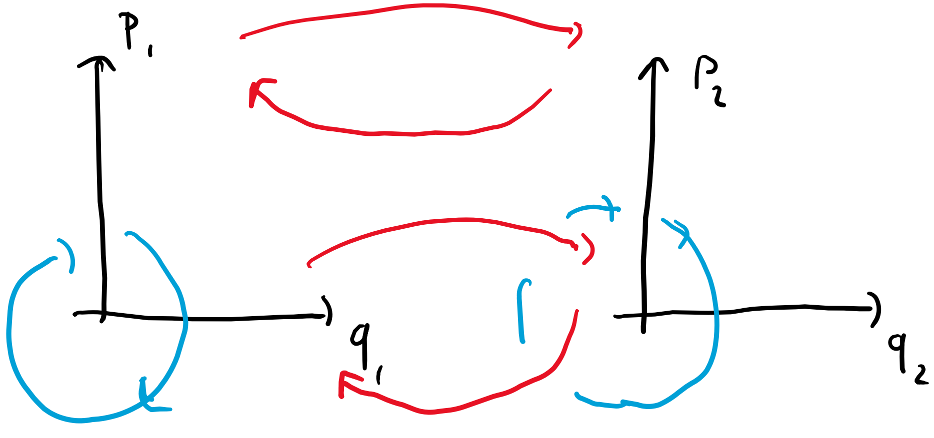


- On $L^2(\mathbb{R}^n)$, what's commutant of canonical trafos?
- Smaller group ⇒ larger commutant…
- …turns out $O(t) \supset S_t$ does the job.

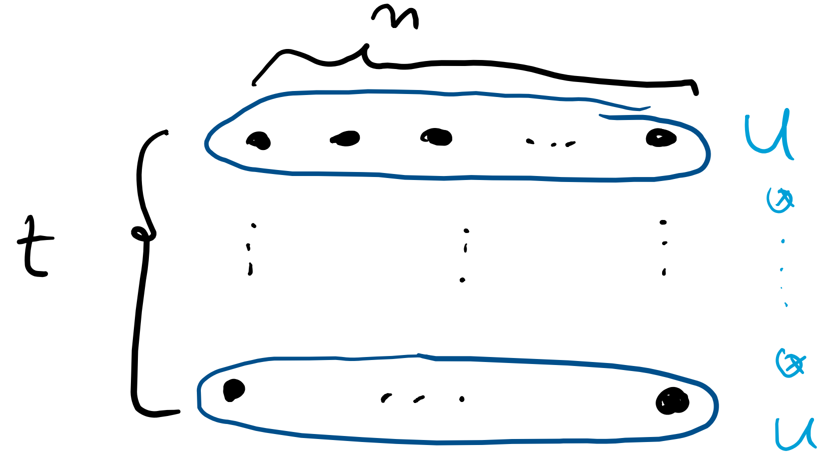


Ex (n=t=2):

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & & \\ \sin\alpha & \cos\alpha & & 0 \\ & & \cos\alpha & -\sin\alpha \\ 0 & & \sin\alpha & \cos\alpha \end{bmatrix}, \quad \begin{bmatrix} \cos\beta\,\mathbb{1} & -\sin\beta\,\mathbb{1} \\ \sin\beta\,\mathbb{1} & \cos\mathbb{1} \end{bmatrix}$$



Central question



$\bullet = L^2(\mathbb{R})$

$\bullet = \mathbb{C}^d$

Want to understand:

Representation theory of t-fold tensor powers $U^{\otimes t}$ of Clifford unitaries U.

OK, let’s go!

For real, now.

Theorem [Nezami, Walter, DG 18]

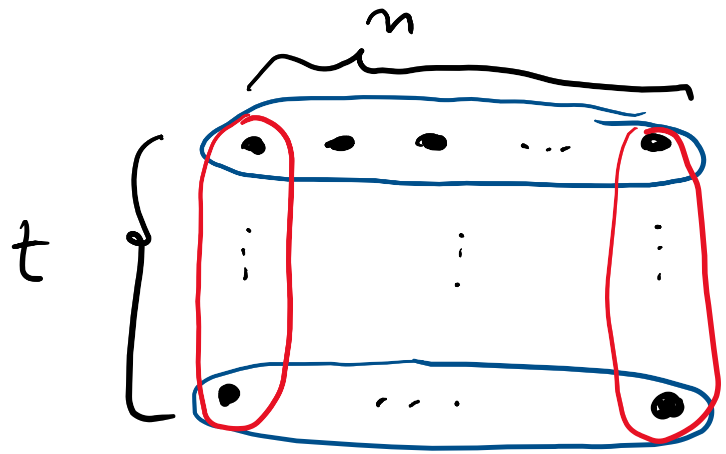
The commutant algebra of t-th tensor powers of the Clifford group over d^n is generated by t-th tensor powers of:

- Discrete orthogonal transformations
- Self-orthogonal CSS code projections

Discrete orthogonal transformations

A $t \times t$ matrix O, entries in $\mathbb{Z}_d$, is *orthogonal* if

$$O^T O = \mathrm{Id} \quad \mathrm{mod\ d}$$



$\bullet = \mathbb{C}^d$

$\Downarrow$

$|F\rangle \qquad F \in \mathbb{Z}_d^{t \times n}$

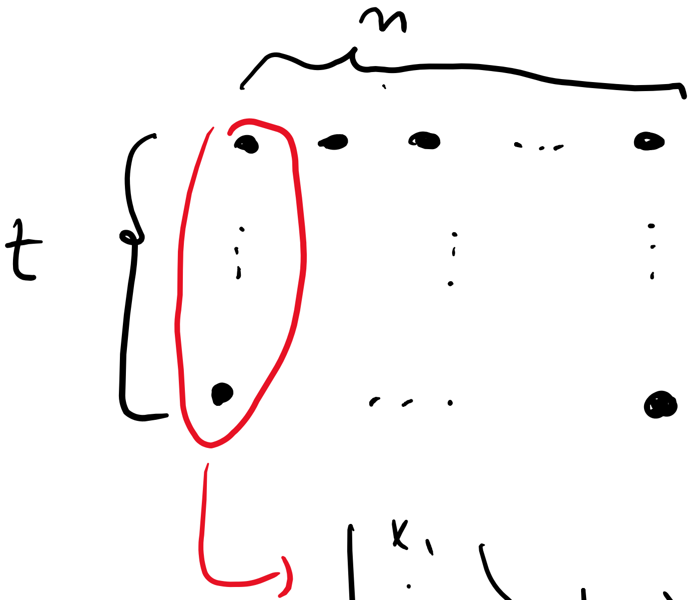
$O : |F\rangle \mapsto |OF\rangle$

Discrete orthogonal transformations

Example: **Anti-permutations**

- Binary complement of permutation matrices

$$\overline{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



$$\begin{pmatrix} x_1 \\ \vdots \\ x_6 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} - x_1 \\ \vdots \\ \bar{x} - x_6 \end{pmatrix} \hat{=}$$

flip bits
if parity
is odd

Calderbank-Shor-Steane codes

Let $N \subset \mathbb{Z}_d^t$ be *self-orthogonal*:

$$\sum_i u_i v_i = 0 \mod d, \qquad\qquad u, v \in N.$$

$$\left.\begin{array}{l} X^u := X_1^{u_1} \otimes \cdots \otimes X_t^{u_t} \\[2em] Z^v := X_1^{v_1} \otimes \cdots \otimes Z_t^{v_t} \end{array}\right\} \Rightarrow \quad X^u Z^v = Z^v X^u$$

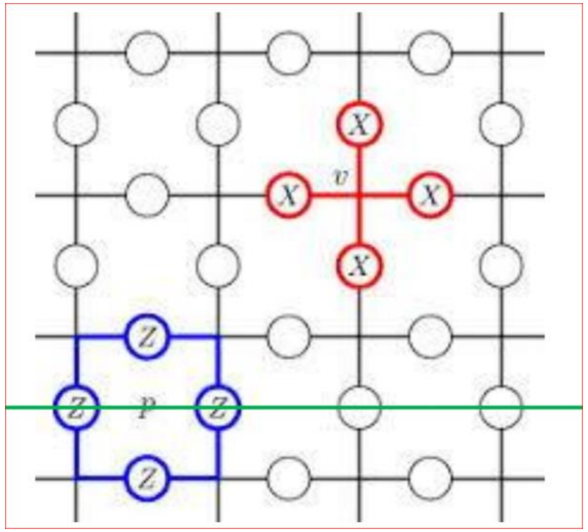
A *self-orthogonal CSS code* is the common eigenspace of these commuting Weyl operators.

Calderbank-Shor-Steane codes

Ex.:

$X \otimes X \left(|00\rangle + |11\rangle \right) = |00\rangle + |11\rangle$

$Z \otimes Z \left(|00\rangle + |11\rangle \right) = |00\rangle + |11\rangle$



toric code (not quite self-dual)

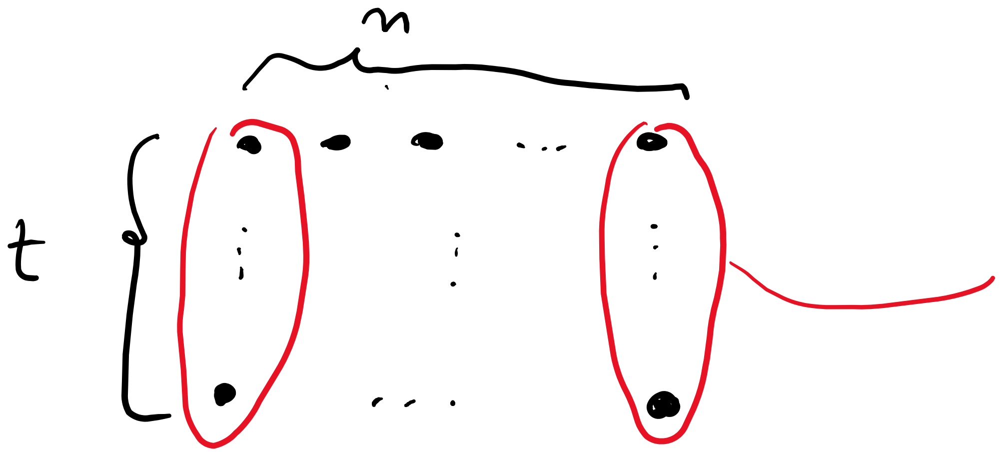
The commutant

Theorem [Nezami, Walter, DG 18]

Commutant generated by tensor powers of:

- Finite orthogonal transformations
- Self-orthogonal CSS code projections

- Transversal! ☺
- Product generators not *quite* a group.
  ⇒ (mild) failure of finite *Howe duality*

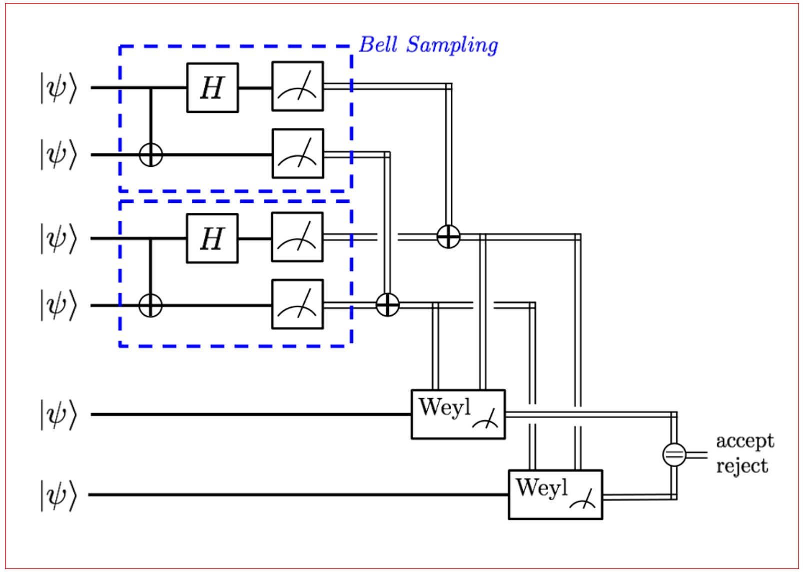


orthogonal $O^{\otimes n}$

or

CSS code projector $P^{\otimes n}$.

Back to applications

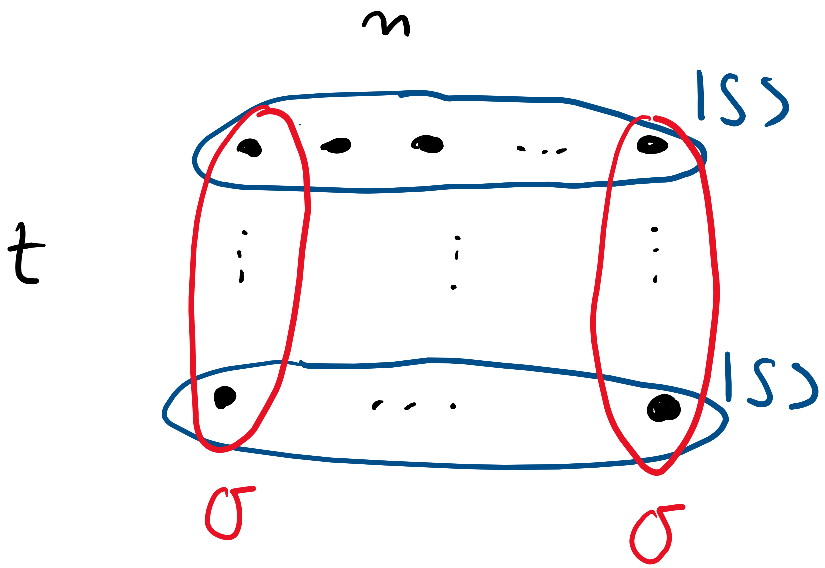


Stabilizer states have additional symmetry

Consider stabilizer state $|s\rangle$ on n qudits...

...and its t-th tensor power.

Tensor powers of stabilizer states are invariant under the stochastic orthogonal group.



Proof: True for $|S\rangle = |0,\dots,0\rangle$:

$$\left|\sigma \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right\rangle = \left|\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\right\rangle$$

$$\Rightarrow \sigma^{\otimes n} |S\rangle^{\otimes t} = \sigma^{\otimes n} U^{\otimes t} |\underline{0}\rangle^{\otimes t}$$

$$= U^{\otimes t} \sigma^{\otimes n} |\underline{0}\rangle^{\otimes t} = |S\rangle^{\otimes t}$$

$\square$

Guess whether Gauss

Thm. [Nezami, Walter, DG 18]

Let ψ be state on *n* qubits.

Measure projection on $(+1)$-eigenspace of anti-identity on $\psi^{\otimes 6}$.

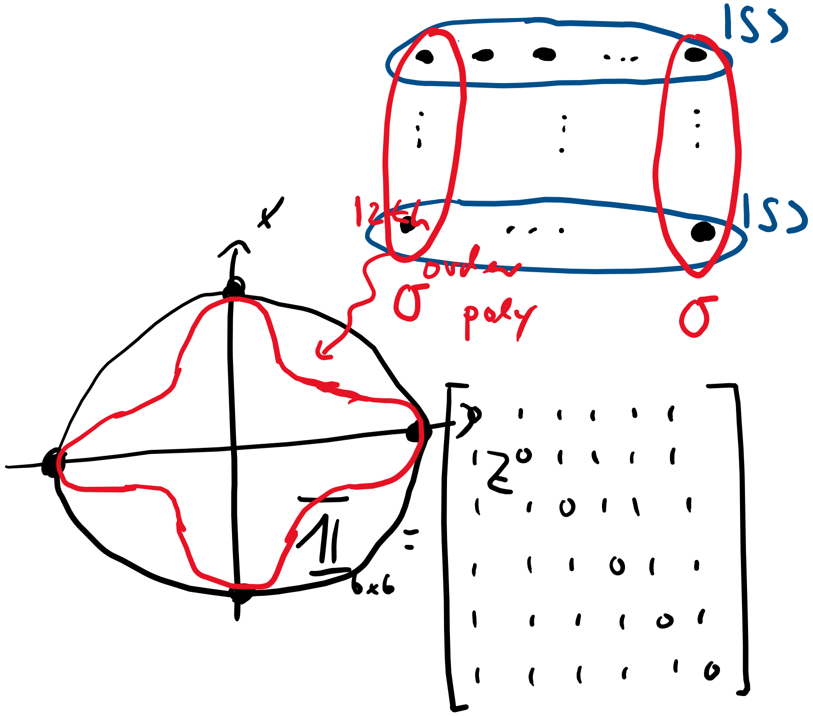
If ψ is stabilizer, will accept with $p = 1$.

If

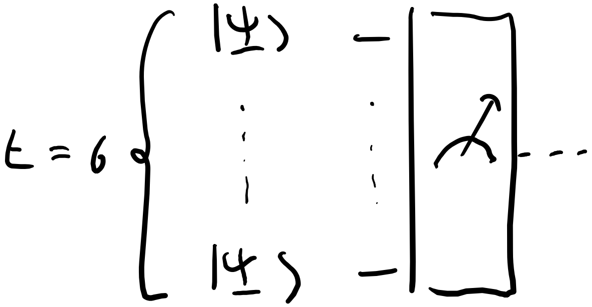
$$\max_S |\langle\psi|S\rangle|^2 \leq 1 - \epsilon,$$

accepts with $p \leq 1 - 4\epsilon$.

(Solves open pro. in q. property testing).

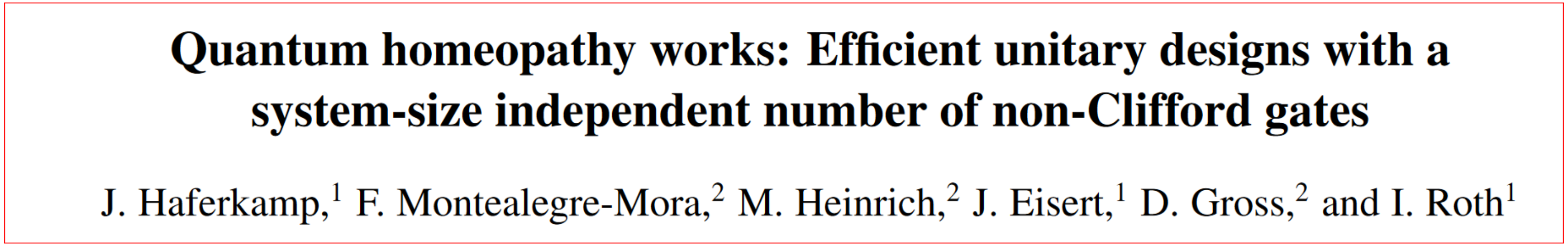


$$\bar{\mathbb{1}}_{6\times 6} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$



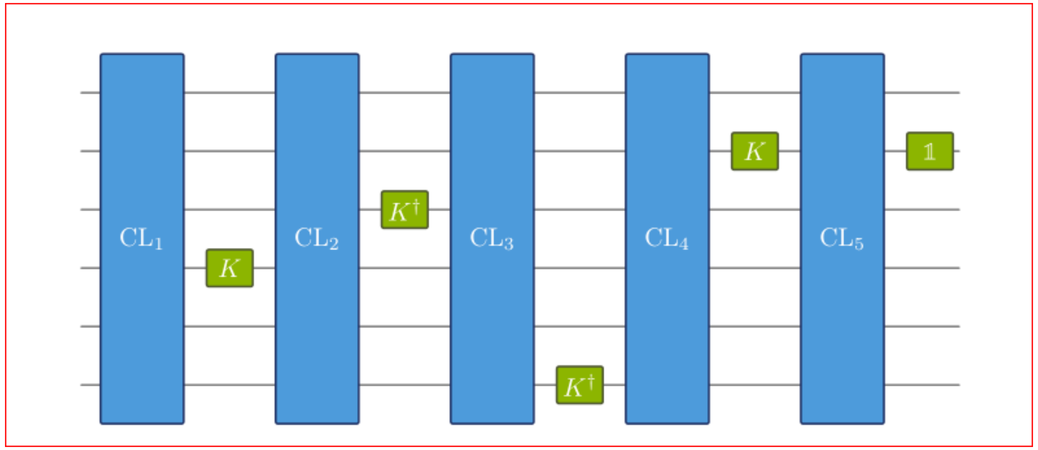
accept if find +1-eigen space of $\bar{\mathbb{1}}^{\otimes n}$

Application: Homeopathy



Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates

J. Haferkamp,[1] F. Montealegre-Mora,[2] M. Heinrich,[2] J. Eisert,[1] D. Gross,[2] and I. Roth[1]



Random walk on groups

Old question:

- Take group G and
- set of generators $\{g_1, \ldots, g_k\}$.

How big does m have to be such that the distribution of random words

$$g_{i_1} \cdots g_{i_m}$$

approximates Haar measure on G?

Famouns example [Diaconis]:

To shuffle 52 cards, $m = 7$ necessary and sufficient.

Quantum shuffling

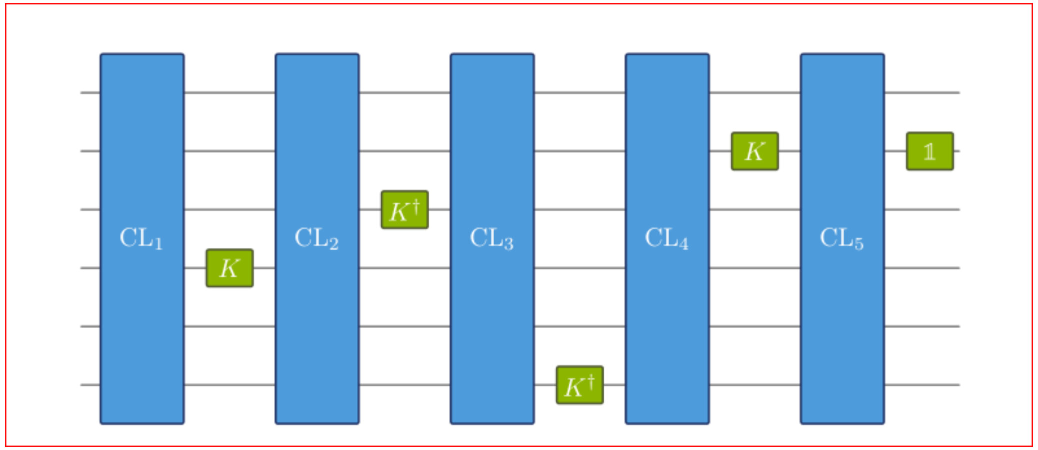
New instance of old question:

- Take n qubits and
- set of gates $\{g_1, \ldots, g_k\}$.

How big does m have to be such that the distribution of circuit

$$g_{i_1} \cdots g_{i_m}$$

approximates Haar measure on $U(2^n)$?



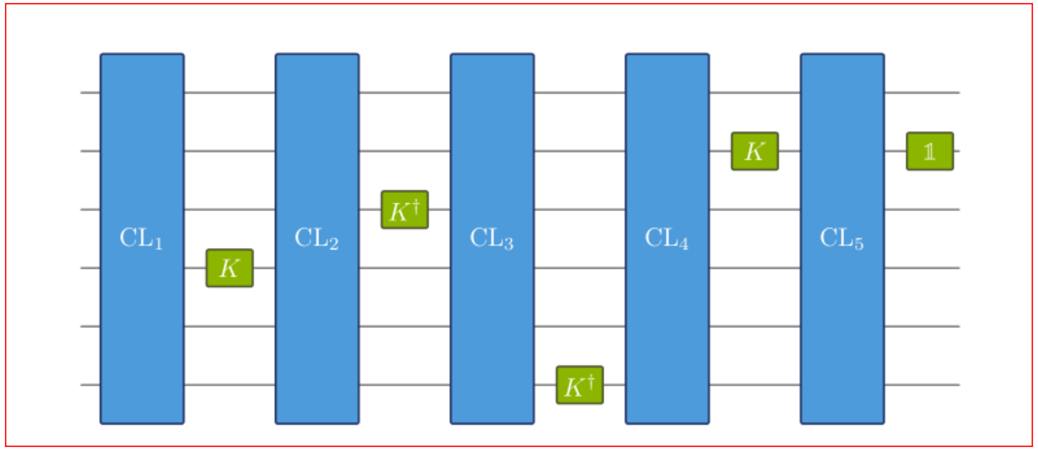
Quantum shuffling: Applications

Exciting applications, like

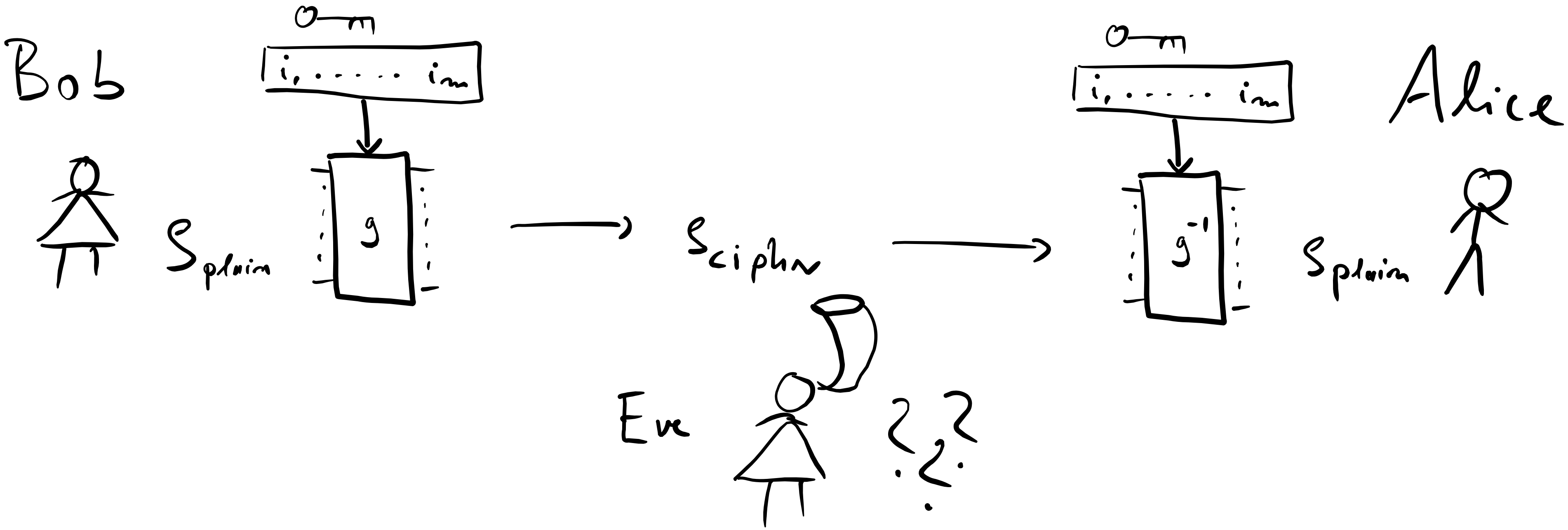
- Black hole information paradox!

Boring applications, like

- Quantum encryption



Quantum shuffling: Crypto application



Need:

- Short key length m
- Easy, fault-tolerant implementation, ideally Clifford-based.

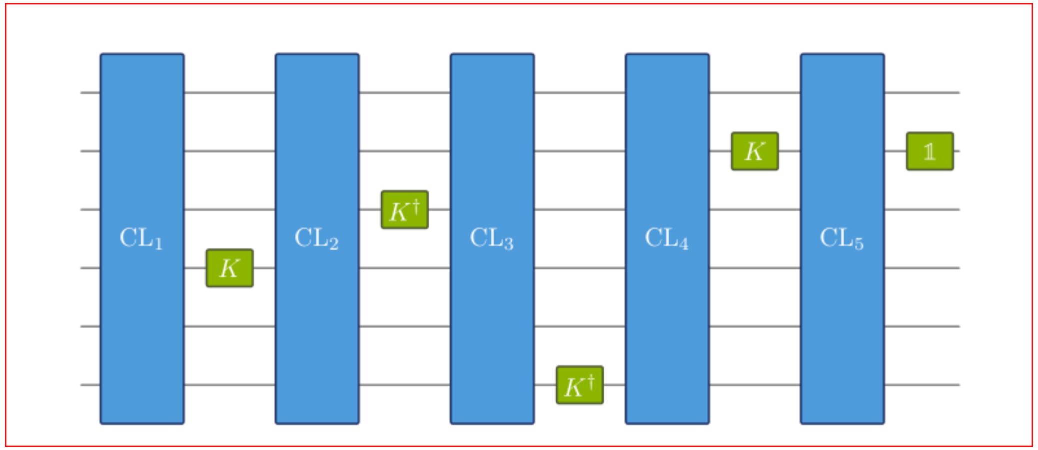
Quantum shuffling: Distance measure

Definition

Distribution ν on $U(2^n)$ is *unitary t-design* if

$$\rho \mapsto \int \mathrm{d}\nu \; U^{\otimes t} \rho U^{-\otimes t} = \int \mathrm{d}\nu_{\mathrm{Haar}} \; U^{\otimes t} \rho U^{-\otimes t}$$

It is ϵ-*approximate* t-design if the maps are ϵ-close in cb norm.



Prior results

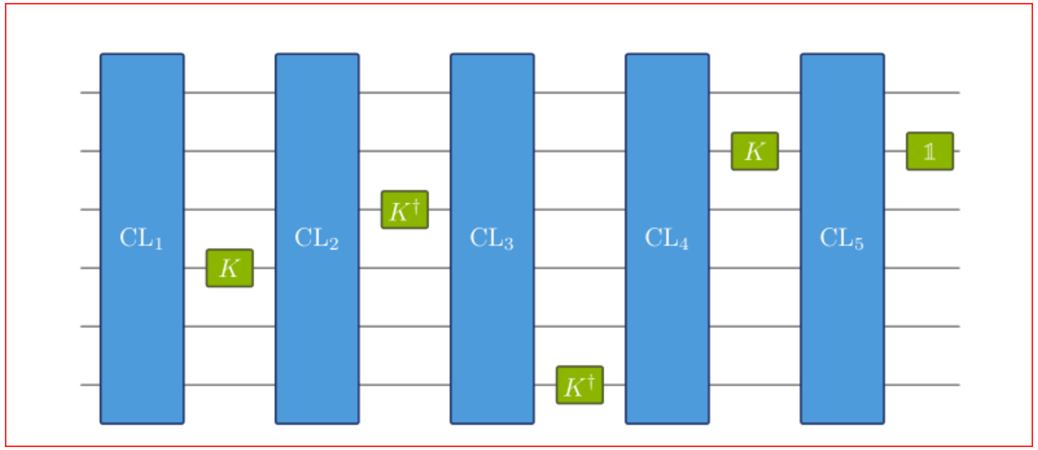
- $\mathrm{m} = O(n^2 t^{10} \log 1/\epsilon)$ is sufficient
- If one restricts to Clifford gates, can only reach t=3 ☹

Quantum shuffling: Result

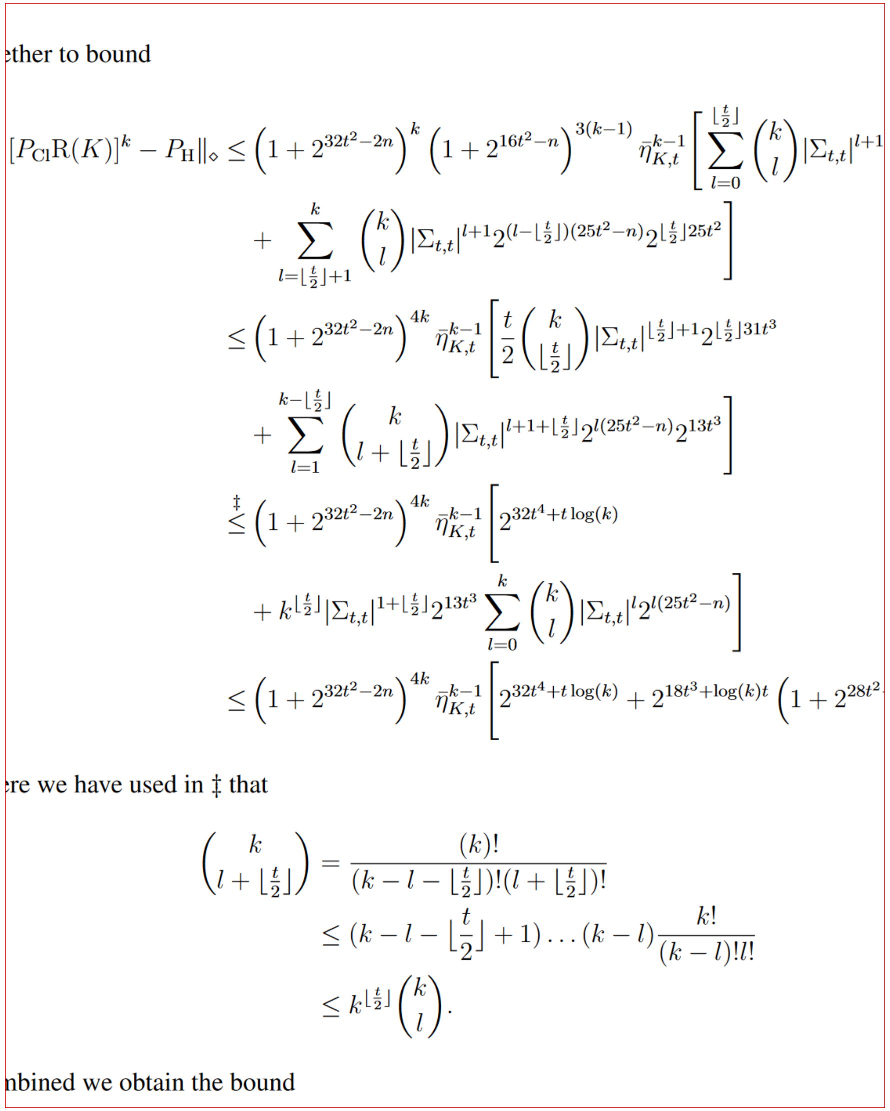
Thm [many authors, 2020]
$O(t^4 \log 1/\epsilon)$ non-Clifford gates are sufficient.

Yepp, density of non-Cliffords → 0 as *n* grows.

You see, a *homeopathic dose* is enough.



Quantum shuffling: Proof ingredients



ther to bound

$$[P_{\mathrm{Cl}}\mathrm{R}(K)]^k - P_{\mathrm{H}}\|_\diamond \leq \left(1+2^{32t^2-2n}\right)^k \left(1+2^{16t^2-n}\right)^{3(k-1)} \bar{\eta}_{K,t}^{k-1} \Bigg[\sum_{l=0}^{\lfloor \frac{t}{2}\rfloor} \binom{k}{l} |\Sigma_{t,t}|^{l+1}$$

$$+ \sum_{l=\lfloor \frac{t}{2}\rfloor+1}^{k} \binom{k}{l} |\Sigma_{t,t}|^{l+1} 2^{(l-\lfloor \frac{t}{2}\rfloor)(25t^2-n)} 2^{\lfloor \frac{t}{2}\rfloor 25t^2}\Bigg]$$

$$\leq \left(1+2^{32t^2-2n}\right)^{4k} \bar{\eta}_{K,t}^{k-1} \Bigg[\frac{t}{2}\binom{k}{\lfloor \frac{t}{2}\rfloor} |\Sigma_{t,t}|^{\lfloor \frac{t}{2}\rfloor+1} 2^{\lfloor \frac{t}{2}\rfloor 31t^3}$$

$$+ \sum_{l=1}^{k-\lfloor \frac{t}{2}\rfloor} \binom{k}{l+\lfloor \frac{t}{2}\rfloor} |\Sigma_{t,t}|^{l+1+\lfloor \frac{t}{2}\rfloor} 2^{l(25t^2-n)} 2^{13t^3}\Bigg]$$

$$\overset{\ddagger}{\leq} \left(1+2^{32t^2-2n}\right)^{4k} \bar{\eta}_{K,t}^{k-1} \Bigg[2^{32t^4+t\log(k)}$$

$$+ k^{\lfloor \frac{t}{2}\rfloor} |\Sigma_{t,t}|^{1+\lfloor \frac{t}{2}\rfloor} 2^{13t^3} \sum_{l=0}^{k} \binom{k}{l} |\Sigma_{t,t}|^{l} 2^{l(25t^2-n)}\Bigg]$$

$$\leq \left(1+2^{32t^2-2n}\right)^{4k} \bar{\eta}_{K,t}^{k-1} \Bigg[2^{32t^4+t\log(k)} + 2^{18t^3+\log(k)t} \Big(1+2^{28t^2}$$

re we have used in ‡ that

$$\binom{k}{l+\lfloor \frac{t}{2}\rfloor} = \frac{(k)!}{(k-l-\lfloor \frac{t}{2}\rfloor)!(l+\lfloor \frac{t}{2}\rfloor)!}$$

$$\leq (k-l-\left\lfloor \frac{t}{2}\right\rfloor+1)\dots(k-l)\frac{k!}{(k-l)!l!}$$

$$\leq k^{\lfloor \frac{t}{2}\rfloor}\binom{k}{l}.$$

nbined we obtain the bound

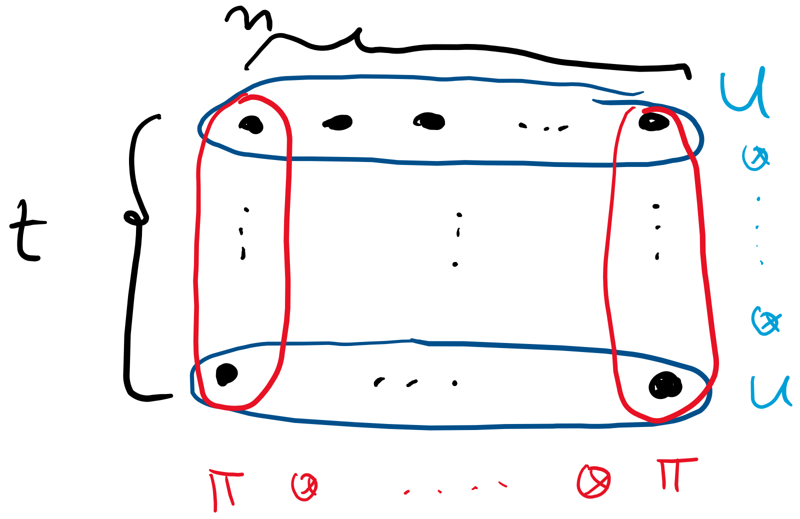
Proof by blood, toil, tears, and sweat…

(Using uniform description of commutant)

Quantum shuffling: How it *should* look like

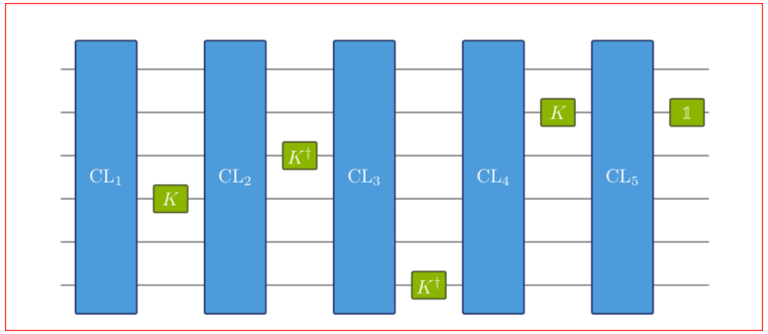
Schur-Weyl case [Collins, Sniady]:

$$\sum_{\pi} \pi \operatorname{tr}(\pi^{-1}\rho) \to \int \mathrm{d}\nu_{\mathrm{Haar}}\, U^{\otimes t} \rho U^{-\otimes t}$$



Want good understanding of:

$$\sum_{E \in O(t),\mathrm{CSS}} E \operatorname{tr}(E^{-1}\rho) \to \int \mathrm{d}\nu_{\mathrm{Clifford}}\, U^{\otimes t} \rho U^{-\otimes t}$$



Summary of Quantum Information results

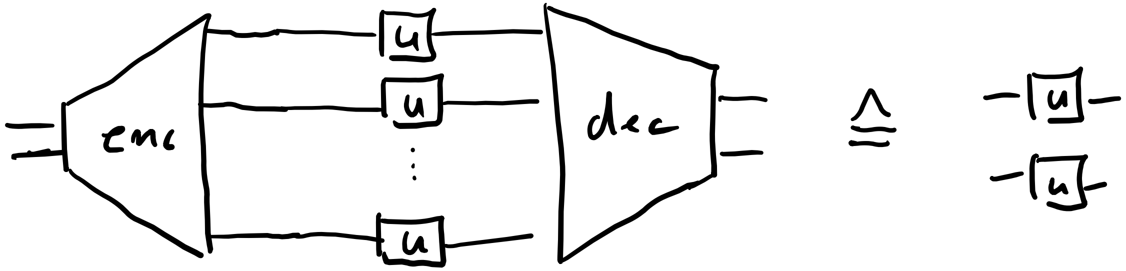
We have found the commutant of tensor powers of the Clifford group.

- Generated by stochastic orthogonal transformations and CSS code projections
- Many applications: Stabilizer rank, stabilizer testing, unitary designs, …

$$\overline{\mathbb{1}}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

More representation theory

Howe-Kashiwara-Vergne duality



Howe Duality – Continuous Variables

- Consider *metaplectic* representation

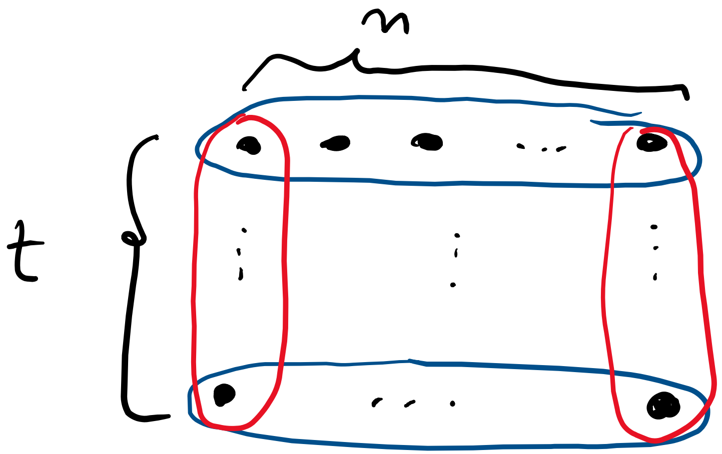
$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n), \quad \mu\colon \mathrm{Sp}(\mathbb{R}^{2n}) \to U(\mathcal{H})$$

- Tensor power $\mu^{\otimes t}$…
- …commutes with O(t) $\supset S_t$.

- Under $O(t) \times \mathrm{Sp}(\mathbb{R}^{2n})$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ irrep of $\mathrm{Sp}(\mathbb{R}^{2n})$.



$\circ = L^2(\mathbb{R})$

$\Downarrow$

$|F\rangle \qquad F \in \mathbb{R}^{t\times n}$

$O: |F\rangle \mapsto |OF\rangle$

Howe Duality – finite (and odd) dimensions

- Consider *metaplectic* representation

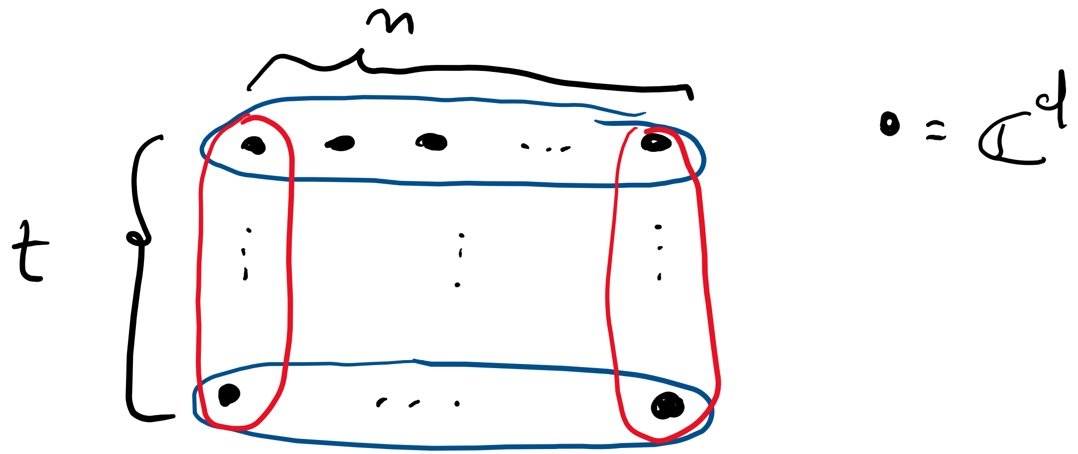
$$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}, \quad \mu\colon \mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right) \to U(\mathcal{H})$$

- Tensor power $\mu^{\otimes t}$…
- …commutes with O(t) ⊃ S_t.

- Under $O(t) \times \mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right)$:

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ reducible rep of $\mathrm{Sp}\left(\mathbb{Z}_d^{2n}\right)$.



ψ

$$|F\rangle \qquad F \in \mathbb{Z}_d^{t\times n}$$

$$O : |F\rangle \mapsto |OF\rangle$$

Howe Duality – finite (and odd) dimensions

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$

- τ irrep of $O(t)$, $\Theta(\tau)$ reducible.
- Failure of Howe duality over finite fields known since 70s…
- …building on Nezami-Walter-DG, Gurevich-Howe 2016…
- we can reduce out this space ☺

Theorem [Montealegre, DG 2019]

$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{r} \bigoplus_{\tau} \eta(\tau) \otimes \mathrm{Ind}(\eta(\tau))$$

Rank of Sp(V)-representations

- Sp(V) contains a large Abelian subgroup

$$\begin{bmatrix} \mathbb{1} & A_1 \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & A_2 \\ 0 & \mathbb{1} \end{bmatrix} = \begin{bmatrix} \mathbb{1} & A_1 + A_2 \\ 0 & \mathbb{1} \end{bmatrix}$$

- ⇒ Restriction of any rep π to Abelian group decomposes Hilbert space into 1D irreps:

$$\pi \begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} |\Phi_B\rangle = \exp(i \operatorname{tr} AB) |\Phi_B\rangle$$

Def.: $\operatorname{rank} \pi = \max_{B} \operatorname{rank} B$

[Gurevich-Howe 2017]

The rank of Sp(V)-representations

Def.: $\operatorname{rank} \pi = \max_B \operatorname{rank} B$

Fact.: The rank of $\mu^{\otimes t}$ is t.

<u>t = 1:</u>

$$\mu\begin{pmatrix} \mathbb{1} & A \\ 0 & \mathbb{1} \end{pmatrix} |\overset{\overset{\mathbb{Z}_d^m}{\downarrow}}{\underline{x}}\rangle = \omega^{(\underline{x}, A\underline{x})} |\underline{x}\rangle = \omega^{\operatorname{tr} A \underbrace{\underline{x}\,\underline{x}^T}_{B}} |\underline{x}\rangle .$$

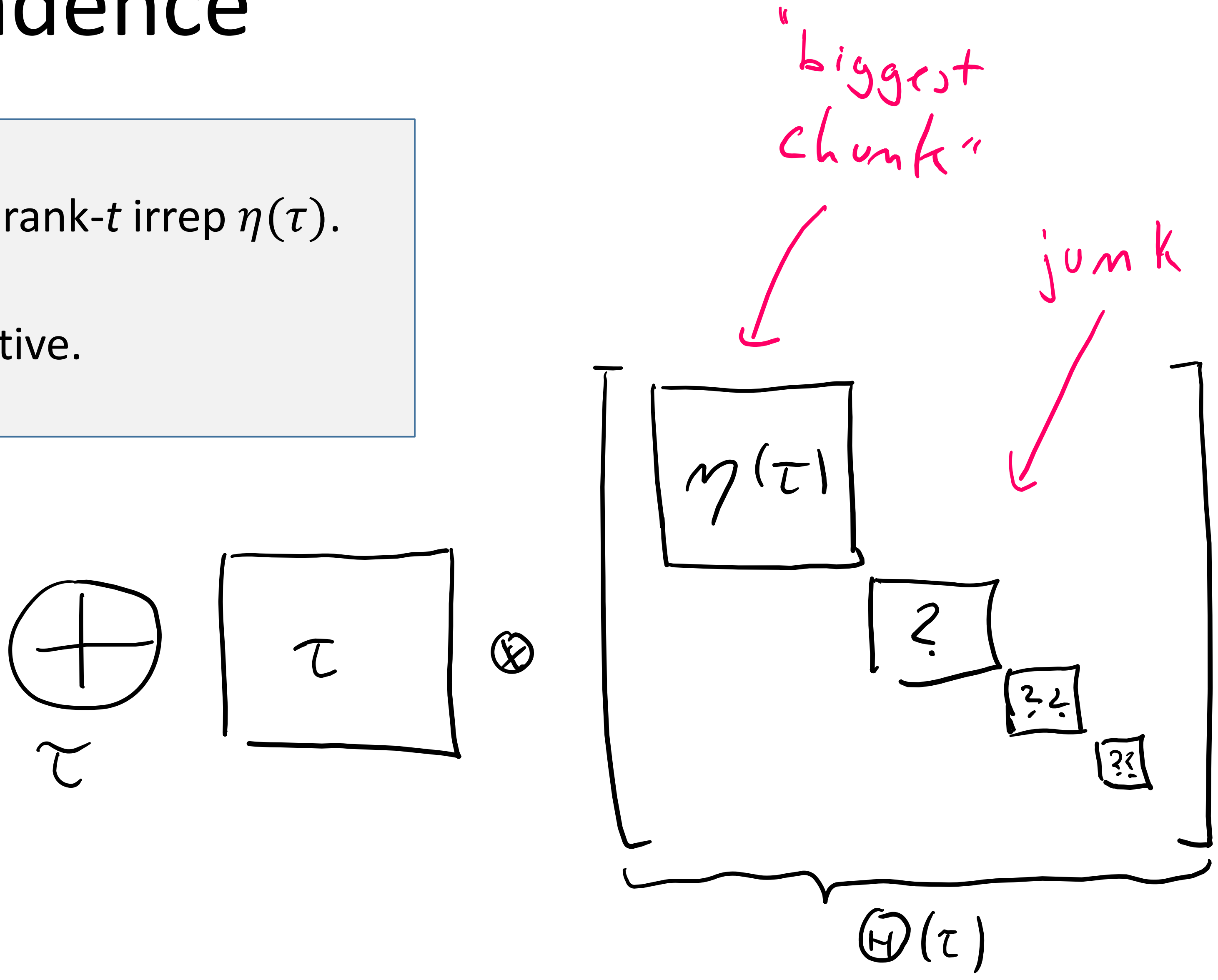
[Gurevich-Howe 2017]

The η-correspondence

Thm [Gurevich-Howe `17]

- $\Theta(\tau)$ contains exactly one rank-t irrep $\eta(\tau)$.
- The map $\tau \mapsto \eta(\tau)$ is injective.

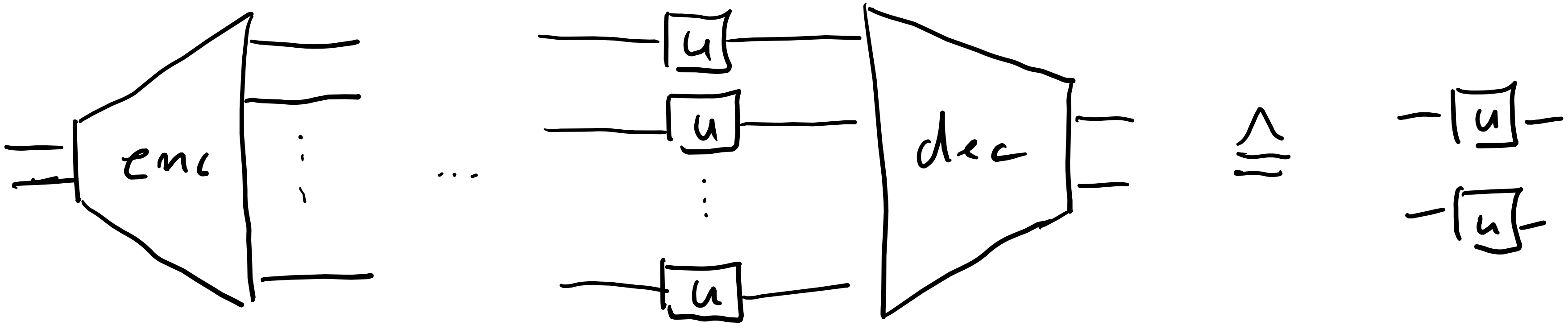
$$\mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \Theta(\tau)$$



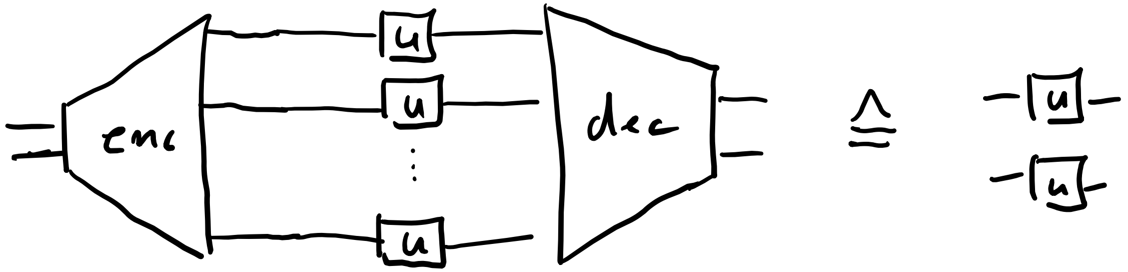
Where do the rank-deficient reps come from?

Idea: Can one “imbed lower tensor powers into *t*-th tensor power”?

…that’s what transversal gates on quantum codes do!



…from CSS codes!



Thm [Montealegre-Mora, DG]

Let $N \subset \mathbb{Z}_d^t$ be isotropic, let C_N be the associated CSS code.

- Then $C_N^{\otimes t}$ is isomorphic to $\mu^{\otimes s}$, $s = t - 2\dim N$.
- All rank-deficient subreps arise this way!

$$\mathcal{H}^{\otimes t} = \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_t} \tau \otimes \eta(\tau)\right] \oplus \left[\bigoplus_{\tau \in \mathrm{Irr}\, O_{t-2}} \tau \otimes \eta(\tau)\right] \otimes \mathbb{C}^{\# N's} \oplus \cdots$$

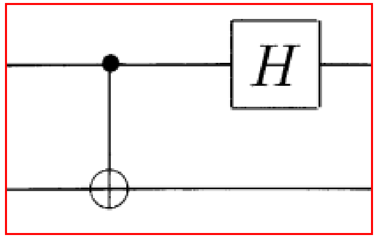
irred. and inequ.

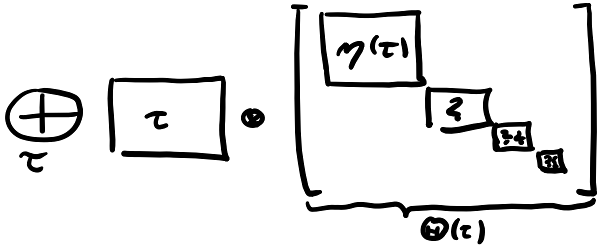
Summary: Howe-Kashiwara-Vergne duality

We have looked at the representation theory of tensor powers of the Clifford group.

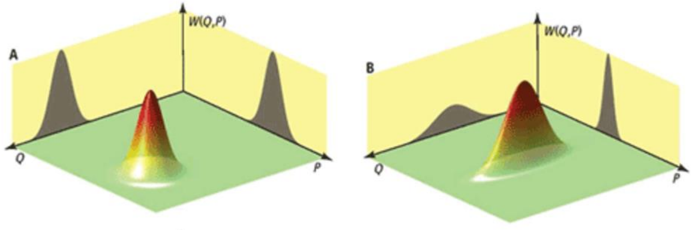
- Clifford group results motivates to consider action on CSS codes…
- …transversal action of the Clifford group on these codes explains failure of Howe duality over finite fields. ☺

Thank you!





$$\mathbb{1}_{6\times 6} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$



David Gross, University of Cologne

With:

Sepehr Nezami, Michael Walter, Felipe Montealegre, Huangjun Zhu, Markus Heinrich, Jonas Haferkamp, Ingo Roth, Jens Eisert (that’s it, though)

Application: Robust Hudson

Thm. [Nezami, Walter, DG 18]
Pure ψ on n qudits, d odd.

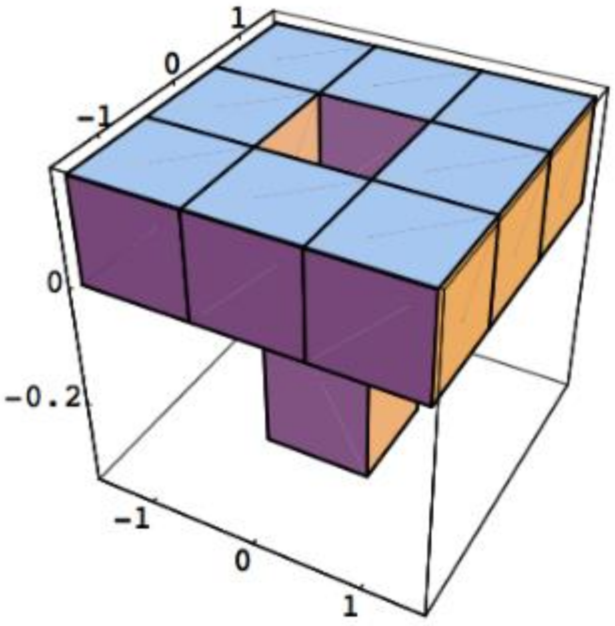
Wigner sum negativity for pure state:

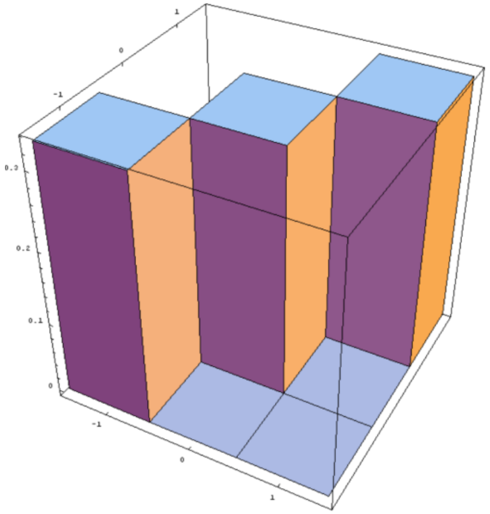
$$\operatorname{sn}(\psi) = \sum_{v, W_\psi(v) \leq 0} |W_\psi(v)|.$$

Then

$$\max_S |\langle \psi | S \rangle|^2 \leq 1 - d^2 \operatorname{sn}(\psi),$$

independent of n.





Application: Exponential de Finetti

Thm.

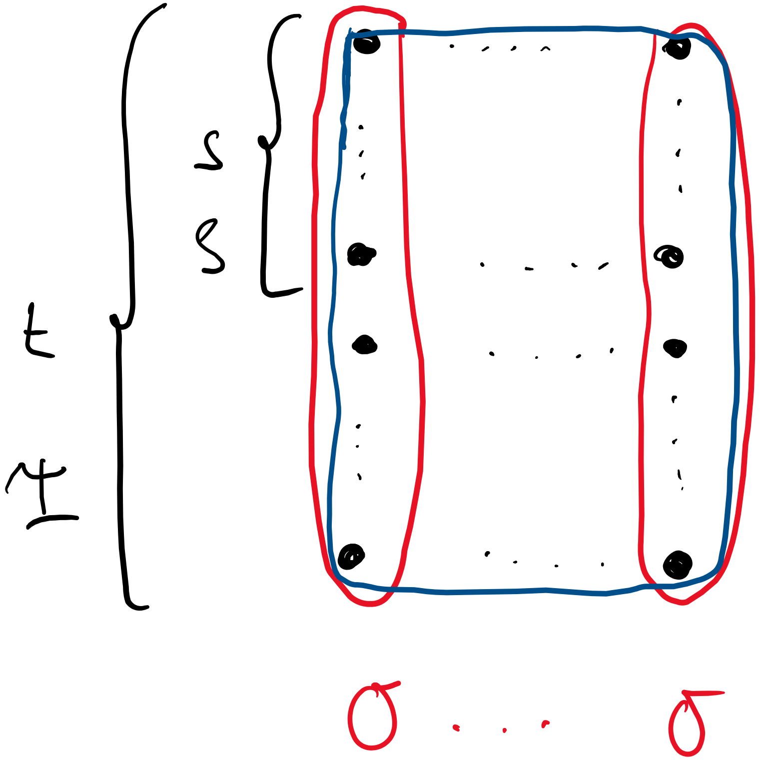
Let $\psi \in \left(\mathbb{C}^{2^n}\right)^{\otimes t}$ be invariant under stochastic orthogonal group.

Let ρ be the reduction to the first s copies.

There is a distribution over stabs s.t.:

$$\left\| \rho - \sum_S |S\rangle\langle S|^{\otimes s} p(S) \right\|_{tr}$$

$$\leq \exp\left(n^2 - (t-s) \right)$$



Finite analogue of [Leverrier 2017]