A non-nuclear C^*-algebra with Weak Expectation Property (WEP) and Local Lifting Property (LLP)

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Mathematical Picture Language Project Seminar, August 4, 2020

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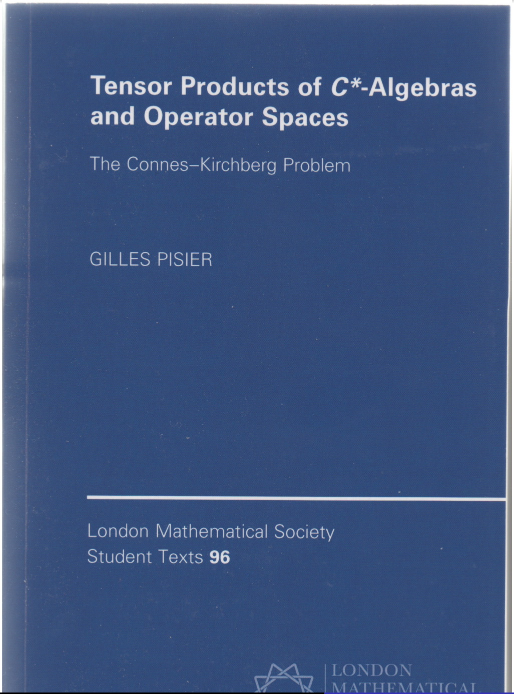
Tensor Products of C*-Algebras and Operator Spaces
The Connes–Kirchberg Problem
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Summary

We describe the construction of the first example of a non nuclear C^*-algebra A with WEP and LLP. This gives a new example of non-nuclear A for which there is a unique C^*-norm on

$$A^{op} \otimes A.$$

This example is of particular interest in connection with **the Connes-Kirchberg problem**, which is equivalent to the question[1] whether $C^*(\mathbb{F}_\infty)$ (or $C^*(\mathbb{F}_2)$), which is known to have the LLP, also has the WEP.
The C^*-algebras A and $C^*(\mathbb{F}_\infty)$ are "locally equivalent" in the sense that they have the same collection of finite dimensional operator subspaces, or phrased differently each embeds in an ultraproduct of the other.

[1] **according to a recent paper entitled MIP* = RE posted on arxiv in Jan. 2020 by Ji, Natarajan, Vidick, Wright, and Yuen the answer is negative**

Plan

Introduction: Back to Grothendieck
Nuclear pairs
WEP
LLP
The construction as a sort of inductive limit

Historical introduction: Grothendieck

The key word today is

NUCLEAR

We will use it for C^*-algebras but it was originally introduced by Grothendieck for locally convex spaces.

Let X, Y be two locally convex spaces. Grothendieck proved in his famous PhD thesis (Mem. AMS 1953) that there was
a minimal tensor product

$$X \overset{\vee}{\otimes} Y$$

and a maximal tensor product

$$X \overset{\wedge}{\otimes} Y$$

Grothendieck observed that there were locally convex spaces X such that for any other such space Y there was only one possible reasonable tensor product on the algebraic $X \otimes Y$ so that

$$\forall Y \quad X \overset{\vee}{\otimes} Y = X \overset{\wedge}{\otimes} Y$$

He called such spaces "nuclear"
He also observed that for Banach spaces

nuclear= finite dimensional

Grothendieck then asked whether there could exist a pair of non-nuclear spaces X, Y such that

$$X \overset{\vee}{\otimes} Y = X \overset{\wedge}{\otimes} Y$$

("**a nuclear pair**")
In particular he asked whether there is an infinite dimensional Banach space X such that

$$X \overset{\vee}{\otimes} X = X \overset{\wedge}{\otimes} X$$

In Acta Math. 1981 I constructed the first example of an infinite dimensional Banach space X such that

$$X \overset{\wedge}{\otimes} X = X \overset{\vee}{\otimes} X$$

(sometimes denoted $X \otimes_{\pi} X = X \otimes_{\varepsilon} X$)

This means that

projective norm $\| \ \|_{\wedge} \simeq \| \ \|_{\vee}$ injective norm

on the algebraic tensor product $X \otimes X$.
Nevertheless (Ann. ENS 1979) for finite dimensional Banach spaces X_n, Y_m

$$X_n \overset{\wedge}{\otimes} Y_m \simeq X_n \overset{\vee}{\otimes} Y_m \text{ uniformly over } n, m$$

implies

$$\sup_{n,m} \min\{\dim(X_n), \dim(Y_m)\} < \infty$$

In particular
X as above fails Grothendieck's approximation property

I would like to quote here a recent paper
Universal gaps for XOR games from estimates on tensor norm ratios
by G. Aubrun, L. Lami, C. Palazuelos, S. J. Szarek and A. Winter
that just appeared
Commun. Math. Phys. 375, 679-724 (2020)

where the preceding phenomenon for finite dimensional spaces is used

In the 1950's and 1960's similar (maximal and minimal) tensor products of C^*-algebras were introduced
and in 1973 Lance asked the natural analogue of Grothendieck's question:
If a C^*-algebra A is such that $A^{op} \otimes_{\min} A = A^{op} \otimes_{\max} A$ does it follow that $B \otimes_{\min} A = B \otimes_{\max} A$ for **ANY** C^*-algebra B ?

In his remarkable 1993 Inventiones paper Kirchberg gave a counterexample

The case of C^*-algebras

In Japan in the late 1950's tensor products of two C^*-algebras A, B were introduced (notably by Turumaru) and it was discovered (Takesaki, Guichardet) that there again was a minimal and a maximal tensor product denoted respectively by $A \otimes_{\min} B$ and $A \otimes_{\max} B$.

These are obtained by completing the algebraic $A \otimes B$ with respect to the minimal and maximal C^*-norms $\| \ \|_{\min}$ or $\| \ \|_{\max}$

When $A \subset B(H)$ and $B \subset B(K)$ then

$$\forall t \in A \otimes B \quad \|t\|_{\min} = \|t\|_{B(H \otimes_2 K)} \text{“spatial norm”}$$

and

$$\forall t \in A \otimes B \quad \|t\|_{\max} = \sup\{\|\pi \cdot \sigma(t)\|_{B(\mathcal{H})} \mid \pi, \sigma \text{ with commuting ranges}\}$$

where sup runs over all $\mathcal{H}$'s and all pairs (π, σ) of $*$-homomorphisms

$$A \xrightarrow{\pi} B(\mathcal{H}) \xleftarrow{\sigma} B$$

Basic facts on Tensorization of linear maps

$$\|u : A \to B\|_{cb} = \sup_n \|Id_{M_n} \otimes u : M_n(A) \to M_n(B)\|$$

Let

$$u_j : A_j \to B_j \text{ linear maps}$$

$$\|u_1 \otimes u_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2\| \leq \|u_1\|_{cb}\|u_2\|_{cb}$$

$$\|u_1 \otimes u_2 : A_1 \otimes_{\max} A_2 \to B_1 \otimes_{\max} B_2\| \leq \|u_1\|_{dec}\|u_2\|_{dec}$$

If u is completely positive $\|u\|_{dec} = \|u\|$.

Exactness

$$\forall A/I \quad B \otimes_{\max} (A/I) = [B \otimes_{\max} A]/[B \otimes_{\max} I] \qquad \text{for all } B$$

but this **fails for** min:

$$\forall A/I \quad B \otimes_{\min} (A/I) = [B \otimes_{\min} A]/[B \otimes_{\min} I] \;\Leftrightarrow B \text{ exact}$$

Exact=subnuclear (Kirchberg)

Nuclear pairs

Definition

A pair of C^* algebras (A, B) will be called a nuclear pair if

$$A \otimes_{\min} B = A \otimes_{\max} B,$$

or equivalently if the min-norm is equal to the max-norm on the algebraic tensor product $A \otimes B$.

Definition

A is called nuclear if (A, B) is nuclear for all B.

Main early references: Takesaki (1964) Lance (1973), Effros-Lance, Choi-Effros+Connes, $\cdots$

Examples: commutative case, $K(H)$, $C^*(G)$ for amenable G

The fundamental pair $(\mathcal{C}, \mathcal{B})$

We denote

$$\mathcal{B} = B(\ell_2) \text{ and } \mathcal{C} = C^*(\mathbb{F}_\infty)$$

Definition

A is called WEP if $(A, \mathcal{C})$ is nuclear

Definition

A is called LLP if $(A, \mathcal{B})$ is nuclear

Theorem (Kirchberg)

The fundamental pair $(\mathcal{C}, \mathcal{B})$ is nuclear
More generally,

$$A \text{ WEP and } B \text{ LLP } \Rightarrow (A, B) \text{ is a nuclear pair}$$

Prototype of A WEP: $A = \mathcal{B}$ **Prototype of A LLP:** $A = \mathcal{C}$

Kirchberg proved :

$$A \text{ WEP and } B \text{ LLP } \Rightarrow A \otimes_{\min} B = A \otimes_{\max} B.$$

Thus

$$A \text{ both WEP and LLP } \Rightarrow A \otimes_{\min} A = A \otimes_{\max} A.$$

and in fact (since LLP and WEP each remain valid for A^{op})

$$A \text{ both WEP and LLP } \Rightarrow A^{op} \otimes_{\min} A = A^{op} \otimes_{\max} A.$$

$$* * *$$

Constructing such an A is our goal.

At the end of his landmark 1993 paper Kirchberg formulated two series of equivalent conjectures

Conjecture A: $(\mathcal{B}, \mathcal{B})$ (or $(\mathcal{B}^{op}, \mathcal{B})$) is a nuclear pair

Conjecture B: $(\mathcal{C}, \mathcal{C})$ (or $(\mathcal{C}^{op}, \mathcal{C})$) is a nuclear pair

Moreover: Conjecture A is equivalent to WEP $\Rightarrow$ LLP

while: Conjecture B is equivalent to LLP $\Rightarrow$ WEP

However, Conjecture A was disproved by Junge and myself (1995).

Kirchberg (1993) proved that Conjecture B is equivalent to the Connes Embedding Problem.

Thus the Connes Embedding Problem is equivalent to the assertion that $\mathcal{C}$ is WEP

or that $\mathcal{C}$ is both WEP and LLP.

Another tensor product characterization of WEP

Kirchberg (1993) observed that

$$A^{op} \otimes_{\min} A = A^{op} \otimes_{\max} A \Rightarrow A \text{ WEP (i.e. } (A, \mathcal{C}) \text{ nuclear)}$$

but the converse is not true (Junge-P. 1995 converse fails for $A = B(H)$).
However:

Theorem (Haagerup, unpublished)

A C^*-algebra A has the WEP IFF

$$\forall n \forall a_j \in A \quad \|\sum a_j^* \otimes a_j\|_{A^{op} \otimes_{\min} A} = \|\sum a_j^* \otimes a_j\|_{A^{op} \otimes_{\max} A}.$$

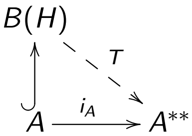
Equivalently iff the min and max norms coincide on the set of "positive definite tensors" in $A^{op} \otimes A$

Equivalent definitions of the WEP (Lance)

Theorem

Consider a C^*-algebra $A \subset B(H)$ TFAE
(i) A has the WEP
(ii) There is a contractive linear map $T : B(H) \to A^{**}$ such that $T(a) = a \ \forall a \in A$, or in other words $T_{|A} = i_A : A \to A^{**}$.

Note: Also true with c.p. and completely contractive T (cf. Tomiyama)



Such a T is called a **"weak expectation"**

$$\{\text{nuclear } C^*\text{-alg.}\} \cup \{\text{injective vN alg.}\} \subset \{WEP\}$$

Local Liftings: Kirchberg's LLP

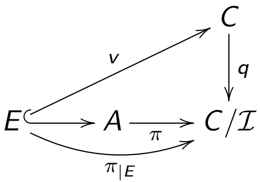
Theorem

TFAE

(i) A has LLP

(ii) For any $*$*-homomorphism* $\pi : A \to C/\mathcal{I}$ *into any quotient* C^**-algebra, for any f.d. subspace* $E \subset A$ *and any* $\varepsilon > 0$ *the restriction* $\pi_{|E}$ *admits a lifting* $v : E \to C$ *with* $\|v\|_{cb} \leq (1 + \varepsilon)$.

(iii) Same with $\varepsilon = 0$.



Digression: The Lifting property LP

A has the lifting property (LP) if any unital $*$-homomorphism $\pi : A \to C/\mathcal{I}$ admits a **global** unital c.p. lifting
(for A not unital, consider unitization)
Kirchberg (1993) proved that $\mathcal{C}$ has the LP.

Important remark: if his conjecture that $\mathcal{C}$ has the WEP is correct, then the LLP implies the LP in the separable case.

LLP and asymptotic $*$-morphisms

One consequence of the LLP that we will use is

Corollary

Assume C has LLP. D another C^*-algebra.
Let $\psi_n : C \to D$ be maps with $\sup \|\psi_n\| < \infty$ that form an asymptotic $*$-morphism in the sense that
$\|\psi_n(x)^* - \psi_n(x^*)\| \to 0$ *and* $\|\psi_n(xy) - \psi_n(x)\psi_n(y)\| \to 0 \; \forall x, y$
then for any f.d. subspace $E \subset C$

$$\limsup \|\psi_{n|E}\|_{cb} \leq 1.$$

Proof.

One just applies Local lifting to the $*$-homomorphism

$$(\psi_n) : C \to \ell_\infty(D)/c_0(D)$$

to get $v_n : E \to D$ with $\|v_n\|_{cb} \leq 1$ such that $(\psi_n - v_n)_{|E} \to 0$ pointwise

where

$$\ell_\infty(D) = \{(x_n) \in D^{\mathbb{N}} \mid \sup \|x_n\| < \infty\}$$

$$c_0(D) = \{(x_n) \mid \|x_n\| \to 0\} \subset \ell_\infty(D)$$

Notation
Let Z, Z' be operator spaces
We say that Z' is completely $(1+\varepsilon)$-isomorphic to Z
if there is a linear isomorphism

$$v : Z' \to Z$$

such that

$$\|v\|_{cb}\|v^{-1}\|_{cb} \leq 1+\varepsilon.$$

We will abbreviate this by

$$Z' \overset{1+\varepsilon}{\simeq} Z$$

LLP implies local embeddability in $\mathcal{C}$

Definition

A, C C^*-algebras (or operator spaces). We say that A "locally embeds" in C if $\forall\varepsilon > 0$ $\forall Z \subset A$ f.d. $\exists Z' \subset C$ f.d. such that

$$Z' \overset{1+\varepsilon}{\simeq} Z$$

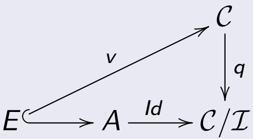
A and C are "locally equivalent" if each locally embeds in the other

Corollary

If A has LLP then A locally embeds in $\mathcal{C}$

Proof.

Assume A separable, so $A = \mathcal{C}/\mathcal{I}$ for simplicity



$$E \simeq v(E) \subset \mathcal{C}$$

With Junge (1995) we showed that WEP $\not\Rightarrow$ LLP,
by showing that
there are A's with WEP that do not locally embed in $\mathcal{C}$.

$* * *$

However if we remove this obstruction then the implication is true :

Proposition

Let A be a C^*-algebra. Assume that A locally embeds in $\mathcal{C}$.
Then A WEP $\Rightarrow$ A has LLP.

Main result

Theorem

There is a non-nuclear C^*-algebra A with both WEP and LLP. Moreover we can ensure that A is locally equivalent to $\mathcal{C}$

Plan of construction

This variant of the proof is based on the following criterion for WEP:

here

$\ell_1^n = \mathrm{span}[1, U_1, U_2, \cdots U_{n-1}] \subset \mathcal{C}$ free unitary generators of $\mathcal{C}$

Theorem

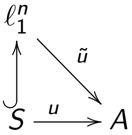
A has the WEP

$\Leftrightarrow$

$\forall n \geq 1\ \forall S \subset \ell_1^n\ \forall u : S \to A\ \forall \varepsilon > 0\ \exists \tilde{u} : \ell_1^n \to A$

such that $\tilde{u}_{|S} = u$ *with*

$$\|\tilde{u}\|_{cb} \leq (1+\varepsilon)\|u\|_{cb}.$$



Plan of construction

We will find f.d. subspaces A_n and set $A = \overline{\cup A_n}$

$$\cdots \subset A_n \subset A_{n+1} \subset \cdots \subset \mathcal{L}$$

$\varepsilon_n > 0$ $\varepsilon_n \to 0$
$\forall n, \forall S\ \forall u \exists \tilde{u}$ with $\|\tilde{u}\|_{cb} \leq (1 + \varepsilon_n)\|u\|_{cb}$.

$$\begin{array}{ccc} \ell_1^n & \xrightarrow{\tilde{u}} & A_{n+1} \\ \uparrow & & \uparrow \\ S & \xrightarrow{u} & A_n \end{array}$$

Clearly the resulting A will have WEP
If we can locally embed the A_n's (and hence A) in $\mathcal{C}$ the LLP will be automatic

What are the obstacles ?

1:

$$\begin{array}{ccc} \ell_1^n & \xrightarrow{\tilde{u}} & ? \\ \uparrow & & \uparrow \\ S & \xrightarrow{u} & A_n \end{array}$$

The easy solution :
Use embed $A_n \subset B(H)$ and replace ? by $A_{n+1} = B(H)$
But (Junge-P. 1995) $B(H)$ does *NOT* locally embed in $\mathcal{C}$!
So this does not solve the problem one has to start with A_n well embedded in $\mathcal{C}$ and produce the extension into A_{n+1} that *still* well embeds in $\mathcal{C}$

Nevertheless this is easy to solve by reproducing the same basic idea as for my 1981 Banach space counterexample, and this leads to an example of WEP operator space with the OLLP (op. space analogue of LLP)

BUT unfortunately, NO inf. dim. C^*-algebra has the OLLP !

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BUT unfortunately, NO inf. dim. C^*-algebra has the OLLP !

2:

In the preceding screen $A_n \to A_{n+1}$ is implicitly a completely isometric embedding

to produce a C^*-algebra we need to find maps $A_n \to A_{n+1}$ that in addition are close to multiplicative

Definition

Let B, C be C^*-algebras Let $\mathcal{E} \subset B$ (self-adjoint), $\varepsilon > 0$. A linear map $\psi : \mathcal{E} \to C$ is an ε-morphism if

(i) $\|\psi\| \leq 1 + \varepsilon$,

(ii) for any $x, y \in \mathcal{E}$ with $xy \in \mathcal{E}$ we have $\|\psi(xy) - \psi(x)\psi(y)\| \leq \varepsilon \|x\| \|y\|$,

(iii) ψ is self-adjoint i.e. for any $x \in \mathcal{E}$ we have $\psi(x^*) = \psi(x)^*$.

The cone algebra

For this the following ingredient is crucial:
Let $C_0 = C((0,1])$ and $C = C([0,1])$. For any C^*-algebra A, let

$$C_0(A) = C_0 \otimes_{\min} A \text{ (cone algebra of } A)$$

Notation: $\forall u : A \to B$, let $u_0 = Id_{C_0} \otimes u : C_0(A) \to C_0(B)$.

$$\ell_\infty(A) = \{(x_n) \in A^{\mathbb{N}} \mid \sup \|x_n\| < \infty\}$$

$$c_0(A) = \{(x_n) \mid \|x_n\| \to 0\} \subset \ell_\infty(A)$$

Lemma

Let $q : C \to B$ (and hence $q_0 : C_0 \to B_0$) be a surjective $*$-homomorphism
Let $q_0 = Id_{C_0} \otimes q : C_0(C) \to C_0(B)$
Then q_0 "almost" lifts :
$\forall \varepsilon > 0\ \forall \mathcal{E} \subset C_0(B)$ f.d.s.a. there is an ε-morphism $\psi : \mathcal{E} \to C_0(C)$ such that $\|q_0\psi(x) - x\| \leq \varepsilon \|x\|$ for any $x \in \mathcal{E}$.

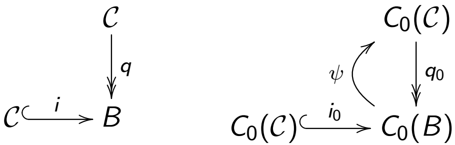
Use of the cone algebra appears in Connes-Higson's 1990 work on E-theory (but apparently unrelated)

Basic setup

By the universality of $\mathcal{B}$, $\exists$ embedding $\mathcal{C} \subset \mathcal{B}$
Known fact:
$\exists B$ separable C^*-algebra with WEP such that

$$\mathcal{C} \subset B \subset \mathcal{B}$$

We will work with B (separable) in place of $\mathcal{B} = B(\ell_2)$
Let $q : \mathcal{C} \to B$ be a surjective $*$-homomorphism.
The relevant diagrams are as follows:



KEY STEP:

$$\begin{array}{ccccc}
\ell_1^n & \xrightarrow{\tilde{u}} & E_{n+1} & \hookrightarrow & C_0(\mathcal{C}) \\
\big\uparrow & & \big\uparrow T_n & & \\
S & \xrightarrow{u} & E_n & \hookrightarrow & C_0(\mathcal{C})
\end{array}$$

T_n will be

• completely $(1+\varepsilon_n)$-isometric,

$$(\ \|T_n\|_{cb} \leq 1+\varepsilon_n, \quad \|T_n^{-1}{}_{T_n(E_n)}\|_{cb} \leq 1+\varepsilon_n\ )$$

• an ε_n-morphism such that

$$\bullet T_n(E_n)T_n(E_n) \subset E_{n+1}$$

and we will need

$$\bullet \sum \varepsilon_n < \infty$$

The inductive limit

$$A = \mathrm{Inductive}\lim(\{E_n\}, \{T_n : E_n \to E_{n+1}\})$$

Let

$$\mathcal{L} = \ell_\infty(C_0(\mathcal{C}))/c_0(C_0(\mathcal{C}))$$

with quotient map

$$Q : \ell_\infty(C_0(\mathcal{C})) \to \ell_\infty(C_0(\mathcal{C}))/c_0(C_0(\mathcal{C}))$$

$$A_n = \{Q(0, \cdots, 0, x, T_n(x), T_{n+1}T_n(x), \cdots) \mid x \in E_n\} \subset \mathcal{L}$$

$$A = \overline{\cup A_n}$$

Proof of KEY STEP:

We start with

$$\begin{array}{ccccc} \ell_1^n & & & & \\ \uparrow & & & & \\ S & \xrightarrow{u} & E_n & \hookrightarrow & C_0(\mathcal{C}) \end{array}$$

We need to produce E_{n+1} and T_n

$$\begin{array}{ccccccc} \ell_1^n & & & & & & \\ \uparrow & & & & & & \\ S & \xrightarrow{u} & E_n & \hookrightarrow & C_0(\mathcal{C}) & \overset{i_0}{\hookrightarrow} & C_0(B) \end{array}$$

$$\begin{array}{ccccccc} \ell_1^n & & & & & & \\ \uparrow & & & & & & \\ S & \xrightarrow{u} & E_n & \hookrightarrow & C_0(\mathcal{C}) & \underset{i_0}{\hookrightarrow} & C_0(B) \end{array}$$

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$$\begin{array}{ccccc} \ell_1^n & & & & \\ \uparrow & & \searrow & & \\ S & \xrightarrow{u} & E_n & \hookrightarrow & \mathcal{E}_n & \hookrightarrow & C_0(B) \end{array}$$

$$\begin{array}{ccccccc}
\ell_1^n & & & & C_0(\mathcal{C}) & & \\
\uparrow & & & \searrow & & \searrow & \\
S & \xrightarrow{u} & E_n & \hookrightarrow & \mathcal{E}_n & \hookrightarrow & C_0(B)
\end{array}$$

$$\begin{array}{ccccccc}
\ell_1^n & & & & C_0(\mathcal{C}) & & \\
\uparrow & & & \searrow & \uparrow{\scriptstyle \psi} & \searrow & \\
S & \xrightarrow{u} & E_n & \hookrightarrow & \mathcal{E}_n & \hookrightarrow & C_0(B)
\end{array}$$

$$\begin{array}{ccccccc}
\ell_1^n & \longrightarrow & E_{n+1} & \hookrightarrow & C_0(\mathcal{C}) & & \\
\uparrow & & \uparrow{\scriptstyle T_n} & & \uparrow{\scriptstyle \psi} & \searrow & \\
S & \xrightarrow{u} & E_n & \hookrightarrow & \mathcal{E}_n & \hookrightarrow & C_0(B)
\end{array}$$

Let $E_{n+1} = \psi(E_n)$ and $T_n = \psi_{|E_n} : E_n \to E_{n+1}$

QED !!

Main Lemma (recapitulation)

Let (Z_n) be an arbitrary sequence of f.d.s.a. subspaces of $C_0(\mathcal{C})$.

Lemma

Let $\varepsilon_n > 0$ with $\sum \varepsilon_n < \infty$. There are $E_n \subset C_0(\mathcal{C})$ f.d.s.a. and $T_n : E_n \to E_{n+1}$ s.a. such that $\forall n \geq 1$

- (i) $\forall S \subset \ell_1^n, \forall u : S \to E_n \; \exists \tilde{u} : \ell_1^n \to E_{n+1}$ *such that*

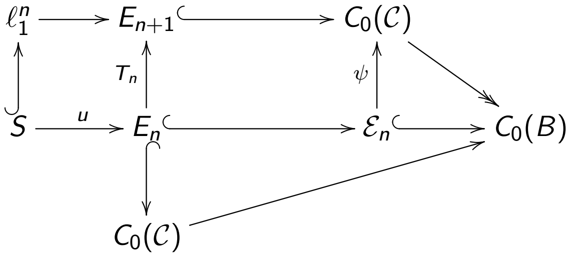
$$\tilde{u}_{|S} = T_n u \text{ and } \|\tilde{u}\| \leq (1+\varepsilon_n)\|u\|_{cb}.$$

- (ii) $\|T_n\|_{cb} \leq 1 + \varepsilon_n$ *and* $\|{T_n^{-1}}_{|T_n(E_n)}\|_{cb} \leq 1 + \varepsilon_n$.
- (iii) *For any* $n \geq 2$ *we have* $T_{n-1}(E_{n-1})T_{n-1}(E_{n-1}) \subset E_n$ *and*

$$\forall x, y \in T_{n-1}(E_{n-1}) \quad \|T_n(x)T_n(y) - T_n(xy)\| \leq \varepsilon_n \|x\|\|y\|.$$

- (iv) *For any* $n \geq 2$ *we have* $Z_n \subset E_n$.

Missing ingredient: how to show: $\|T_n\|_{cb} \leq 1 + \varepsilon_n$?
We use (as previously seen)



Then $T_n = \psi_{|E_n} : E_n \to E_{n+1}$ with $E_{n+1} = \psi(\mathcal{E}_n)$
If $C_0(C)$ LLP then $\lim_{(\mathcal{E},\varepsilon)\to\infty} \|\psi_{|E_n}\|_{cb} \leq 1$
by LLP for asymptotic $*$-morphisms mentioned earlier

Remark

Let $C_0(\mathcal{C}) = \overline{\cup Z_n}$ with $\cdots Z_n \subset Z_{n+1} \subset \cdots$ f.d. subspaces
We can additionally obtain easily :

$$\forall n \quad Z_n \subset E_n$$

so that $C_0(\mathcal{C})$ (and hence $\mathcal{C}$) locally embeds in A
It follows that A and $\mathcal{C}$ are locally equivalent, so that each embeds in an ultrapower of the other.

Indeed, given $T_n : E_n \to E_{n+1}$ we can always enlarge E_{n+1} to ensure that $E_{n+1} \supset Z_{n+1}$

Thank you !