Integrability, rationality and convolutions

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Abstract

The Eisenstein-Kronecker function is a useful object in number theory and physics. It figures in a proof of rationality for period integrals of cusp forms. It generates integration kernels of elliptic polylogarithms which express Feynman diagrams on the torus. It is the central object in the construction of elliptic R-matrices. Vector bundles on elliptic curves constitute a mathematical context for all of these structures. Convolutions of their sections provide a new tool for applications. Here we describe the simplest case, the iterated convolutions of the Kronecker zeta function.

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1 Motivation (survey)

1.1 Vector bundles over an elliptic curve

Indecomposable vector bundles over an elliptic curve $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, $(\tau \in \mathfrak{h})$ have been classified by Atiyah in 1957.

Their isomorphism classes are classified by the pair of (rank, degree), and a point on the curve (the latter plays the role of $\hbar$ for elliptic R-matrices).

Atiyah did not determine their global sections, which have become important in recent years.

As an example, consider vector bundles $\mathcal{V}$ of rank 2,

$$0 \to \mathcal{E} \to \mathcal{V} \to \mathcal{L} \to 0$$

where $\mathcal{E}$ is the trivial line bundle.

This leads to upper triangular matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ as transition functions, where a defines the line bundle $\mathcal{L}$. For shifts by τ, it is of the form $a = \exp(2\pi i m(z - c))$, where $m = -\deg \mathcal{L}$ and $c \in X$.

For $m = 1$, the canonical solution is given by the Appell-Lerch sum (and if Atiyah had followed up on this case he would have found Mock-Jacobi forms which Zwegers discovered in 2002).

Explicitly, let $q = e^{2\pi i\tau}$ and

$$\mu(z, z_0) := \frac{e^{\pi iz}}{\vartheta(z_0)} \sum_{n\in\mathbb{Z}} \frac{(-e^{2\pi iz_0})^n q^{\frac{n(n+1)}{2}}}{1 - e^{2\pi iz}q^n}$$

For $a(z) = -e^{2\pi i(z-z_0)}q^{1/2}$, $\mathcal{L}$ is defined by the divisor $D = -[z_0-\frac{\tau}{2}]$ and the pair $(e^{\pi iz}\mu, 1)$ defines a section in $\mathcal{V}$.

Here and in the following, ϑ denotes the classical Jacobi theta function.

What I will talk about can be formulated as iterated extensions of the trivial line bundle by trivial line bundles. That means that my transition functions will be upper triangular matrices with diagonal $1, 1, 1, \ldots$. Such functions are quasi-elliptic.

1.2 Feynman graphs on elliptic curves

The calculation of Feynman integrals on $\mathbb{P}^1_{\mathbb{C}}$ can be reduced to that of integrals of rational functions along paths on the Riemann sphere (multiple polylogarithms).

The generalisation to elliptic curves (Levin & Brown) uses a family of functions $g^{(n)}(x)$ for $n \geq 0$ which are linearly independent and have at most simple poles (Broedel, Duhr, Dulat, Tancredi 2018).

Fix an elliptic curve X and $x_1, \ldots, x_k \in X$. A class of iterated integrals is defined recursively by

$$\Gamma\left(\begin{smallmatrix} n_1,\ldots,n_k \\ x_1,\ldots,x_k \end{smallmatrix}; y\right) = \int_0^y dx\, g^{(n_1)}(x - x_1)\, \Gamma\left(\begin{smallmatrix} n_2,\ldots,n_k \\ x_2,\ldots,x_k \end{smallmatrix}; x\right)$$

where $\Gamma(\,;y) = 1$. Here $n_1, \ldots, n_k \in \mathbb{N}$. (The integral may require regularisation at $y = 0$.)

The $g^{(n)}$ for $n \geq 2$ are regular on $\mathbb{R}$, while for $n = 1$,

$$g^{(1)}(x) = \zeta(x) - e_2 x$$

for Weierstrass' ζ-function ($\zeta' = -\wp$), and $e_2 = \zeta(x+1) - \zeta(x)$ is the quasi-modular Eisenstein series of weight two.

These integration kernels are generated by the Eisenstein-Kronecker function

$$\boxed{F_\tau(x, y) = \frac{\vartheta'(0)\vartheta(x+y)}{\vartheta(x)\vartheta(y)}}$$

We will consider these functions in more detail.

1.3 Elliptic integrable models

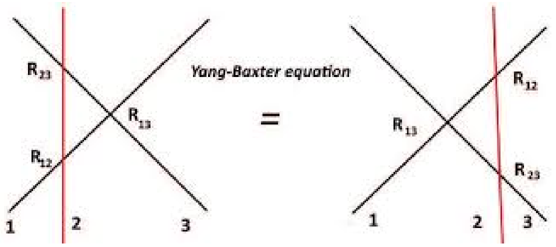


Figure 2: star triangle identity $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

Two-particle scattering matrices of 2D integrable models have the same structure as Baxter's R-matrices for the 8-vertex model when the rapidity is taken as the spectral parameter (Zamolodchikov 1979).

Solvability of the 8-vertex model relies crucially on the fact that the corresponding Boltzmann weights define an elliptic curve. Their ratio can be written in terms of $\mathrm{sn}(u), \mathrm{sn}(u - w), \mathrm{sn}(w)$ and their product, where $\mathrm{sn}(u)$ equals a shift of $F_\tau(z, 0)$ by half-integer periods (here $z = u/\text{const.}$).

Simpler eq. which depends on a second parameter (Polishchuk 2002), later called $\hbar$:

Associative Yang-Baxter (AYB):

$$\begin{aligned} r_{12}(-w,v)r_{13}(u+w,v+v') - r_{23}(u+w,v')r_{12}(u,v) \\ + r_{13}(u,v+v')r_{23}(w,v') = 0 \end{aligned}$$

(Note that this formulation uses the additive structure on the curve.)

Here $r(u,v)$ is a meromorphic function of two complex variables u,v in the neighborhood of $(u,v) = (0,0)$ taking values in $A \otimes A$ where A is an associative algebra with 1. The pair of indices specifies an embedding $A \otimes A \hookrightarrow A \otimes A \otimes A$, e.g. when A is a matrix algebra, then $r_{13}(u,v) = \sum r_{ijkl}(u,v)e_{ij} \otimes 1 \otimes e_{kl}$.

Theorem 1 (Polishchuk 2009)**.** *Let $r(u, v)$ be a solution of (AYB) with Laurent expansion at $v = 0$ given by*

$$r(u, v) = \frac{1 \otimes 1}{v} + r_0(u) + vr_1(u) + O(v^2)$$

Equivalent:

1. *For fixed v, $r(u, v)$ satisfies (w.r.t. u)*

   Quantum Yang-Baxter (QYB), $\hbar \in X$ fixed:

   $$R^{\hbar}_{12}(u - w)R^{\hbar}_{13}(u)R^{\hbar}_{23}(w) = R^{\hbar}_{23}(w)R^{\hbar}_{13}(u)R^{\hbar}_{12}(u - w)$$

2. *$r(u, v)$ is unitary:* $\boxed{r(u, v)r(u, -v) \propto 1 \otimes 1}$.

Burban & Kreussler worked out Polishchuk's tensor-valued solutions of (AYB) explicitely (2012).

For the individual matrix elements, the solution reduces to

("Fay's identity" - really a Jacobi ϑ function id.)

$$r(-w,v)r(u+w,v+v') - r(u+w,v')r(u,v) \\ + r(u,v+v')r(w,v') = 0 \, .$$

Theorem 2 (Polishchuk 2000)**.** *If* $r(u,v)$ *is a non-zero meromorphic function in the neighborhood of* $(0,0)$ *satisfying*

$$r(-u,-v) = -r(u,v)$$

and "Fay's identity", then $r(u,v) \sim F_\tau(u,v)$ *is the Eisenstein-Kronecker function (or its trigonometric resp. rational degeneration).*

Solutions with higher pole order have been obtained from $F_\tau(u,v)$ and its derivatives (Burban & Henrich). They do, however, not satisfy unitarity.

For the integrable chiral Potts model, the weights are parametrized by a higher genus curve and involve higher genus theta functions (Au-Yang, McCoy & Perk).

1.4 Number theory

For $\tau \in \mathfrak{h}$, $q = e^{2\pi i\tau}$, let

$$f(\tau) = \sum_{\ell=1}^{\infty} a_f(l) q^l$$

a cusp form (i.e. $a_f(0) = 0$) of weight k on $PSL(2, \mathbb{Z})$. (The space of such cusps forms has a basis of Hecke eigenforms.)

Period polynomial of degree $\leq k-2$ in X:

$$\begin{aligned} r_f(X) &= \int_0^{i\infty} f(\tau)(\tau - X)^{k-2} d\tau \\ &= \sum_{n=0}^{k-2} (-)^n \binom{k-2}{n} X^{k-2-n} \underbrace{\int_0^{i\infty} f(\tau)\tau^n d\tau}_{n^{\text{th}} \text{ period of } f} \end{aligned}$$

Let $(r_f(X) r_f(Y))^{\text{odd}}$ be the odd part of $r_f(X) r_f(Y)$ w.r.t. $(X, Y) \leftrightarrow (-X, -Y)$, normalised with the Petersson scalar product (f, f).

Zagier & Kohnen consider (for $k \geq 2$ even, $l \geq 0$),

$$c_k(X, Y; \tau) := \frac{1}{(2i)^{k-3}} \sum_{f \in M_k} \frac{(r_f(X) r_f(Y))^{\mathrm{odd}}}{(f, f)} f(\tau)$$

Here M_k is a basis of the vector space of Hecke eigenforms of weight k (including the Eisenstein series e_k, where one needs to adapt the definition of r_f). We consider the generating series

$$C(X, Y; \tau, T) = \frac{(XY - 1)(X + Y)}{X^2 Y^2} T^{-2} + \sum_{k=2}^{\infty} c_k(X, Y; \tau) T^{k-2}$$

Theorem 1.1: (Zagier 1991)

The function $C(X, Y; \tau, T) \in (XYT)^{-2} \mathbb{Q}[X, Y][[q, T]]$ is given by

$$C(X, Y; \tau, T) = F_\tau(T, -XYT) F_\tau(XT, YT)\,.$$

Remark 1. *1. Arguments of F_τ are added, as in statistical mechanics and QFT, but also multiplied. The four arguments of the theta factors involved satisfy an equation of the form $ad - bc = 0$.*

2. *The above formula the* $c_k(X, Y; \tau)$ *for* $k \geq 2$ *are modular forms (*e_2 *drops out). For* $X = 1$*, the degenerate Fay identity yields*

   $$F_\tau(T, -YT)F_\tau(T, YT) = \wp(T) - \wp(YT)\ .$$

   ($Y = 1$ *similar.)*

3. *Unitarity condition for R-matrices of Baxter-Belavin type for* $N \times N$ *matrices*

   $$R^{\hbar}_{12}(z)R^{\hbar}_{21}(-z) = N^2(\wp(N\hbar) - \wp(z))\, 1 \otimes 1$$

   which reduces to the above equation for $N = 1$.

1.5 Virasoro N-point functions in CFT

Fields are operators and N-point functions of the holomorphic Virasoro field are given by the functional derivative w.r.t. the metric (Weinberg 1972/ Einstein)

$$\boxed{\langle T(x_1)\dots T(x_N)\rangle = \frac{\delta}{\delta g(x_1)}\dots\frac{\delta}{\delta g(x_N)}\langle 1\rangle}$$

Though in this talk, only $g = 1$ is addressed, we really aim for higher genus (hyperelliptic curves).

Theorem 1.2: (L 2018) Graphical representation for $g = 1$

For $N \geq 1$,

$$\langle 1\rangle^{-1}\langle T(z_1)\dots T(z_N)\rangle = \sum_{\Gamma} F(\Gamma)\,,$$

Γ oriented graph with vertices $z_1, \dots, z_N$, where $\forall i$,
z_i has at most one ingoing and at most one outgoing line (no tadpole).

For $n \geq 1$, $\exists\, M_{2n} : \mathfrak{h} \;\to\; \mathbb{C}$ such that

$$F(\Gamma) = \left(\frac{c}{2}\right)^{\sharp\text{loops}} \prod_{(z_i,z_j)\in\Gamma} \wp(z_i - z_j|\tau)\, M_{2\cdot(N-\sharp\text{edges})}$$

M_{2n} is a modular form of weight $2n$.

Proof. $\boxed{T(z)\otimes T(0) \mapsto \frac{c/2}{z^4}.1 + \frac{2}{z^2}T(0) + \frac{1}{z}T'(0) + O(1)}$ □

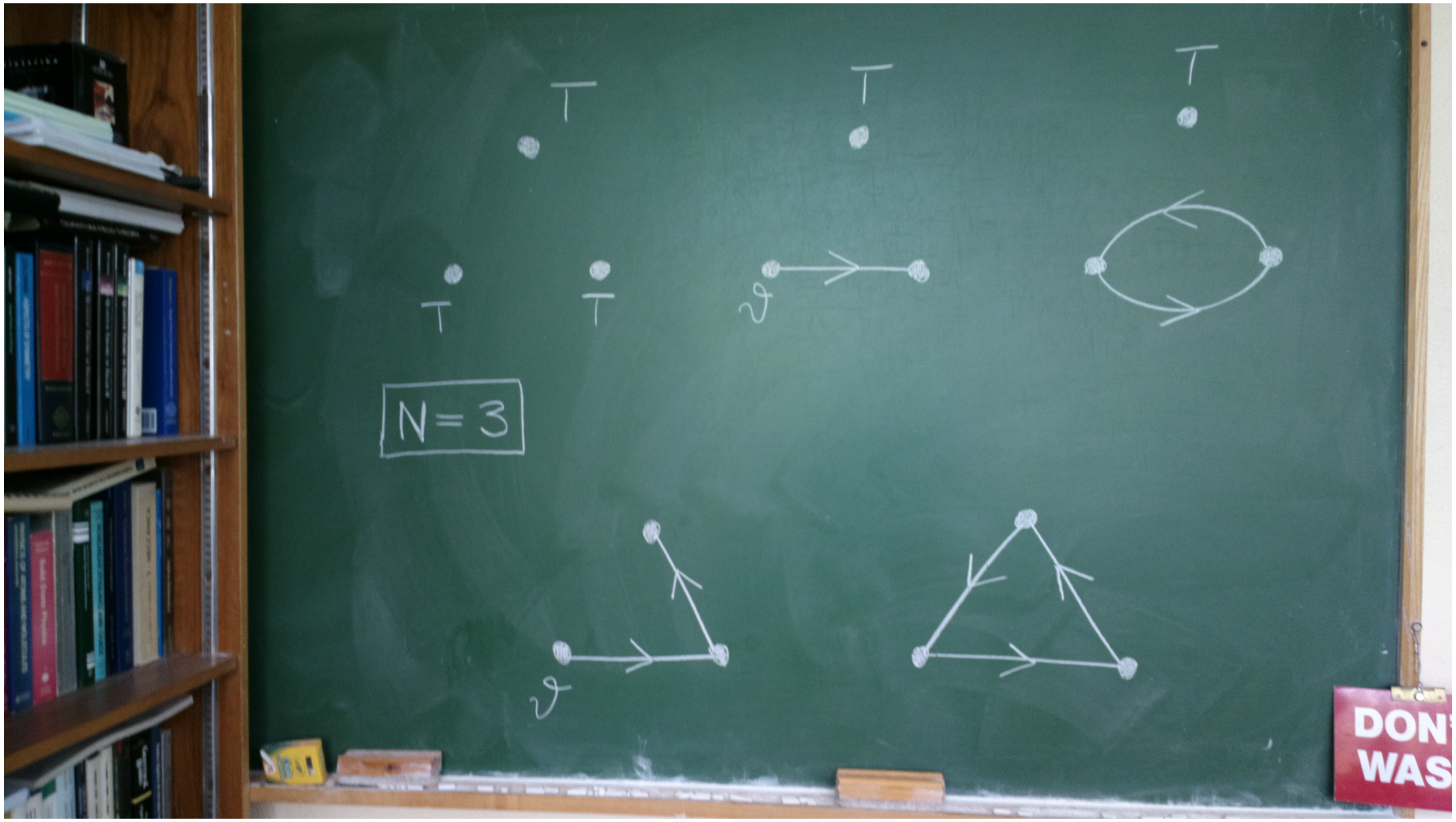


Figure 3: Virasoro 3-pt function. The graphs can be interpreted as baby Feynman graphs.

$N = 0$: $\langle 1 \rangle$ constant in position but function of the metric g.

In genus $= 1$, every metric is conformally equivalent to a flat metric, so g is a function of $\tau \in \mathfrak{h}$ (Teichmüller space).

$\langle T \rangle$ is constant in position, and for $N \geq 1$,

$$\frac{d^N}{d\tau^N}\langle 1 \rangle_{\text{flat}} = \underbrace{\oint_{\text{period}} \dots \oint_{\text{period}}}_{N-1} \langle T(z_1) \dots T(z_N) \rangle \frac{dz_2 \dots dz_N}{(2\pi i)^N}$$

where the contour is taken along the real period.

This leads to expressions like (τ fixed)

$$\oint_{\text{period}} \wp(x_3 - x_1) \underbrace{\left(\oint_{\text{period}} \wp(x_1 - x_2)\wp(x_2 - x_3) dx_2 \right)}_{\text{function of } (x_1 - x_3)} dx_3$$

$$= (\wp * \wp * \wp)(0) \quad \text{Only loops contribute!}$$

Integration by parts: $(\wp * \wp * \wp)(0) = 35e_6 + \dots e_2$. (The e_2 terms can be read off easily by modular invariance).

In order to actually compute such iterated convolutions, introduce Kronecker's modified ζ function (cf. Weil 1976):

$$\mathcal{Z}(x) := \zeta(x) - e_2 x$$

$\mathcal{Z}$ is not an elliptic function (though it almost is):

$$\mathcal{Z}(x+1) - \mathcal{Z}(x) = 0$$
$$\Delta\mathcal{Z}(x) = \mathcal{Z}(x+\tau) - \mathcal{Z}(x) = -2\pi i$$

So if $\mathcal{V}$ is spanned locally by 1 and $\mathcal{Z}$, then $\mathcal{V}/\mathcal{E} \cong \mathcal{L}$ is trivial.

This leads to the study of quasi-elliptic functions.

2 Convolution algebra of *SK* functions

Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be meromorphic with period 1.

$$\Delta f(x) := f(x+\tau) - f(x)\,, \quad \forall x \in \mathbb{C}\,.$$

Set $\Delta^k = \Delta \circ \ldots \circ \Delta$ (k factors, where $\Delta^0 = \mathrm{id}$). For $k \geq 0$, let

$$K_k := \ker \Delta^k$$

f is **quasi-elliptic** if $f \in K_k$ for some k. $\Delta : K_{k+1} \to K_k$ defines a K_1-linear map and $K = \cup_{k \geq 0} K_k$ is a filtered algebra over K_1 w.r.t. pointwise multiplication. $f \in K$ is **special** (written $f \in SK$) if all poles of f are located on points of $\Lambda = \tau\mathbb{Z} + \mathbb{Z}$. Let $SK_k := SK \cap K_k$.

Examples

- $K_0 = \{0\}$
- $K_1 = \mathbb{C}(\wp, \wp')$ field of elliptic functions.
- $SK_1 = \mathbb{C}[\wp, \wp']$ ring of special elliptic functions.

- $K_2 \ni \mathcal{Z}$ since $\Delta\mathcal{Z} \in \mathbb{C}$ (in the following, $\Delta\mathcal{Z} = 1$).
- As a function in x_1, $(\mathcal{Z}(x_1 + x_2) - \mathcal{Z}(x_1)) \in K_1$ but not special for generic x_2.
- $(\mathcal{Z}(x_1 + x_2) - \mathcal{Z}(x_1) - \mathcal{Z}(x_2))\,(\wp(x_1) - \wp(x_2)) \in SK_1$ (namely $= \frac{1}{2}\,(\wp'(x_1) - \wp'(x_2))$ by the addition theorem)

Theorem 2.1: Structure theorem for *SK* functions (L 2019)

a) $SK = SK_1[\mathcal{Z}]$ and is a free module over SK_1 with basis $\mathcal{Z}^k$ for $k \geq 0$.

b) For $f \in SK$ and $x = m\tau$, $(m \in \mathbb{Z})$, $\exists\, f_\ell \in \mathrm{Map}(\mathbb{Z}, \mathbb{C})$ such that

$$f(x) = \sum_{\ell \in \mathbb{N}} \frac{f_\ell(m)}{(x - m\tau)^\ell} + \text{reg.}$$

For every $\ell \in \mathbb{N}$, f_ℓ is a polynomial, $f_\ell \in \mathcal{P}^{[\ell]} \cong \mathbb{C}[m]$.

c) The map

$$\sigma : \quad SK/\mathbb{C} \rightarrow \oplus_{\ell=1}^{\infty} \mathcal{P}^{[\ell]}$$

defined by

$$\widehat{f} \mapsto \sigma(\widehat{f}) = (f_1, f_2, f_3, \ldots)$$

is an isomorphism.

Let $f, g \in SK$. For $0 < \Im(x) < 2\Im(\tau)$,

$$(f \circledast g)(x) := \oint_{\Im(z)=\Im(x)/2} f(x-z)g(z)dz\,.$$

$\circledast : SK \times SK \;\to\; SK$ with

1. $\circledast$ is **symmetric**: Setting $x - z = x_1 + iy_1$ and $z = x_2 + iy_2$, we have
$$(f{\circledast}g)(x) = \int_0^1 \int_0^1 f(x_1{+}iy_1)g(x_2{+}iy_2)\delta(x_1{+}x_2{-}\Re(x))dx_1dx_2$$
provided $y_1 = y_2 = \frac{\Im(x)}{2} \in (0, \Im(\tau))$. Product is manifestly symmetric.

When $x \in \mathbb{R}$, then $y_1 = -y_2$ so $\nexists$ symmetric formulation.

2. $\circledast$ is also **associative** (similar argument).
3. If f is regular on $0 < \Im(x) < n\Im(\tau)$ and g is regular on $0 < \Im(x) < m\Im(\tau)$ then $f \circledast g$ is regular on $0 < \Im(x) < (n+m)\Im(\tau)$.

SK becomes a filtered algebra in two ways

1. $(SK, \circledast)$ is a filtered algebra over $\mathbb{C}$ (w.r.t. pole order):

For $f, g \in SK$ with maximal pole order o_f resp. o_g, we have

$$o_{f \circledast g} \le o_f + o_g - 1$$

2. Δ descends to classes of functions mod $\mathbb{C}$ (We have $\widehat{f} \cdot \widehat{g} \neq \widehat{f \cdot g}$ but $\widehat{f} \circledast \widehat{g} = \widehat{f \circledast g}$).

For $\widehat{f} \in SK_r/\mathbb{C}$ and $\widehat{g} \in SK_s/\mathbb{C}$,

$$\widehat{f} \circledast \widehat{g} \in SK_{r+s}/\mathbb{C}$$

So $(SK/\mathbb{C}, \circledast)$ is a filtered algebra over $SK_1/\mathbb{C}$ (w.r.t. Δ):

Theorem 2.2: (L 2019)

Let $V = \{f \in SK | \, o_f \le 1\}$.

1. $V/\mathbb{C} = \mathrm{span}_{\mathbb{C}}\{\widehat{\mathcal{Z}^{\circledast n}} | n \ge 1\}$
2. $\mathrm{Res}_{x=m\tau}\,[\mathcal{Z}^{\circledast n}] = \binom{m-1}{n-1}$

3 Applications

Let $g^{(n)}$ be a generating sequence for $F_{tau}(x, y)$:

$$F_{tau}(x, y) =: \frac{1}{y} \sum_{n \geq 0} g^{(n)}(x)\, y^n \,.$$

E.g. $g^{(0)}(x) = 1$ and

$$\begin{aligned} g^{(1)}(x) &= \mathcal{Z}(x) \\ g^{(2)}(x) &= \frac{1}{2}\mathcal{Z}(x)^2 - \frac{1}{2}\wp(x) \\ g^{(3)}(x) &= \frac{1}{6}\mathcal{Z}(x)^3 - \frac{1}{2}\wp(x)\mathcal{Z}(x) - \frac{1}{6}\frac{d}{dx}\wp(x) \,. \end{aligned}$$

Theorem 3.1: (Zagier 1991)

For $n \geq 0$, $g^{(n)} \in V$, and we have for $m \in \mathbb{Z}$,

$$\mathrm{Res}_{x=m\tau}\left[g^{(n)}(x)\right] = \frac{m^{n-1}}{(n-1)!} \,.$$

In particular, $g^{(n)}$ for $n \geq 2$ is regular on $\mathbb{R}$.

Theorem 3.2: (L 2019)

For $n \geq 0$, $g^{(n)}$ define elements in $V \cap SK_{n+1}$ (or sections in a vector bundle of rank $n+1$ over X).

1. For $n \geq 0$, we have

$$\mathcal{Z}^{\circledast n}(x) = \sum_{k=1}^{n} (-1)^{n-1} \frac{(k-1)!}{(n-1)!} s(n,k) g^{(k)}(x) + \text{const.} \quad (1)$$

   where for $1 \leq k \leq n$, $s(n,k)$ are the signed Stirling numbers of the 1st kind (and the constant is known, e.g. equal to 1 for $n = 0$).

2. For $n \geq 0$, the inverse transformation

$$g^{(n)}(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{(n-1)!} S(n,k) \mathcal{Z}^{\circledast k}(x) + \frac{2B_n}{n!}(1-2^{n-1}) \quad (2)$$

   where for $1 \leq k \leq n$, $S(n,k)$ are the Stirling numbers of the 2nd kind. (Here $B_0 = 1$ and for $n \geq 2$, B_n is the nth Bernoulli number.)

Proof. Comparing residues.

The additive constant is computed using that

$$\int_0^1 g^{(n)}(x)dx = \frac{B_n}{n!}\,, \qquad n \geq 2\,,$$

The $\mathcal{Z}^{\circledast k}$ integral equals $(1/2)^k$ (half the full contour integral for $k = 1$).

$\square$

$g^{(1)} = \mathcal{Z}$ and we have

$$g^{(2)} = -\mathcal{Z}^{\circledast 2} + \mathcal{Z} - \frac{1}{6}$$

$$g^{(3)} = \mathcal{Z}^{\circledast 3} - \frac{3}{2}\mathcal{Z}^{\circledast 2} + \frac{1}{2}\mathcal{Z}$$

$$g^{(4)} = -\mathcal{Z}^{\circledast 4} + 2\mathcal{Z}^{\circledast 3} - \frac{7}{6}\mathcal{Z}^{\circledast 2} + \frac{1}{6}\mathcal{Z} + \frac{7}{360}$$

$$g^{(5)} = \mathcal{Z}^{\circledast 5} - \frac{5}{2}\mathcal{Z}^{\circledast 4} + \frac{25}{12}\mathcal{Z}^{\circledast 3} - \frac{5}{8}\mathcal{Z}^{\circledast 2} + \frac{1}{24}\mathcal{Z}$$

Conversely,

$$\mathcal{Z}^{\circledast 2} = -g^{(2)} + g^{(1)} - \frac{1}{6}$$

$$\mathcal{Z}^{\circledast 3} = g^{(3)} - \frac{3}{2}g^{(2)} + g^{(1)} - \frac{1}{4}$$

$$\mathcal{Z}^{\circledast 4} = -g^{(4)} + 2g^{(3)} - \frac{11}{6}g^{(2)} + g^{(1)} - \frac{103}{360}$$

$$\mathcal{Z}^{\circledast 5} = g^{(5)} - \frac{5}{2}g^{(4)} + \frac{35}{12}g^{(3)} - \frac{25}{12}g^{(2)} + g^{(1)} - \frac{43}{144}$$

where $g^{(k)} \in SK_1[\mathcal{Z}]$ is computable from $F_{tau}(x, y)$.

3.1 Addition theorems for iterated convolutions of $\mathcal{Z}$

Using the addition theorems for $\mathcal{Z}$ and $\wp$, respectively, yields

$$\mathcal{Z}^{\circledast 2}(x_1 + x_2) = -\frac{1}{2}A^2 - \frac{1}{2}\frac{\wp'(x_1) - \wp'(x_2)}{\wp(x_1) - \wp(x_2)}A - \frac{1}{2}\left(\wp(x_1) - \wp(x_2)\right) + \frac{2}{3}$$

where $A = \mathcal{Z}(x_1) + \mathcal{Z}(x_2) - 1$.

and the corresponding addition formulae for higher n.

(For $n = 2$, we may alternatively use "Fay's identity" though for higher n formulae don't seem to come out nicely.)

3.2 Computation of iterated convolutions of $\wp$

$g^{(k)}$ for $k \geq 2$ are regular at $x = 0$, so we can compute

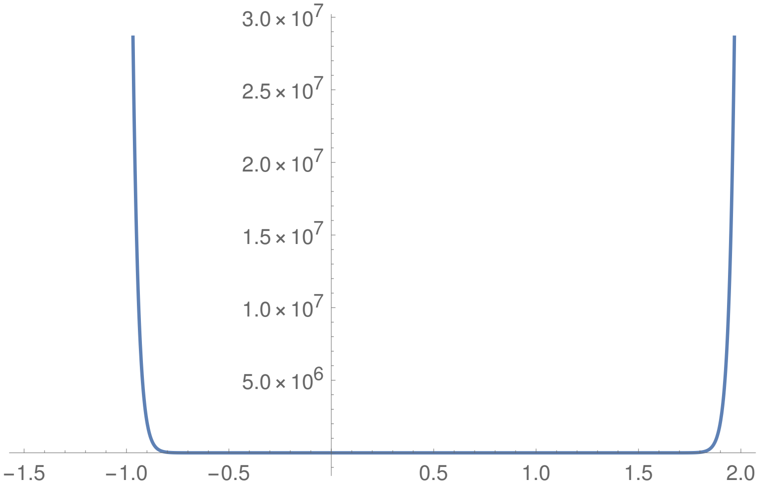
$$\wp^{\circledast n}(x) = (-1)^n \frac{d^n}{dx^n}\mathcal{Z}^{\circledast n}(x) + (-1)^n e_2^n$$

3.3 Convolution polynomials

For $n \geq 1$, let

$$p_n(x) := \sum_{k=1}^{n} (-1)^{k-1} \frac{(k-1)!}{(n-1)!} S(n,k)\, x^k$$

$n = 50$:



Theorem 3.3: (L 2020)

The sequence of polynomials $p_n(x)$ for $n \geq 1$ is generated by

$$G(x, y) := \frac{x \cdot \exp(y)}{1 - x + x \cdot \exp(y)} .$$

1. We have $p_1(x) = x$, $p_2(x) = x(1 - x)$ and
$$p_n(1 - x) = (-1)^n p_n(x) , \quad n \geq 2 . \tag{3}$$
2. For $n \geq 1$, $p_n(x)$ has n simple zeroes $x_j^{[n]}$, $j = 1, \ldots, n$, in the interval $[0, 1]$ so that $x_1^{[n]} = 0$, $x_n^{[n]} = 1$ and $x_j^{[n]} < x_{j+1}^{[n+1]} < x_{j+1}^{[n]}$ for $j = 1, \ldots, n - 1$. Thus the zeroes of p_n and of p_{n+1} are interlaced.

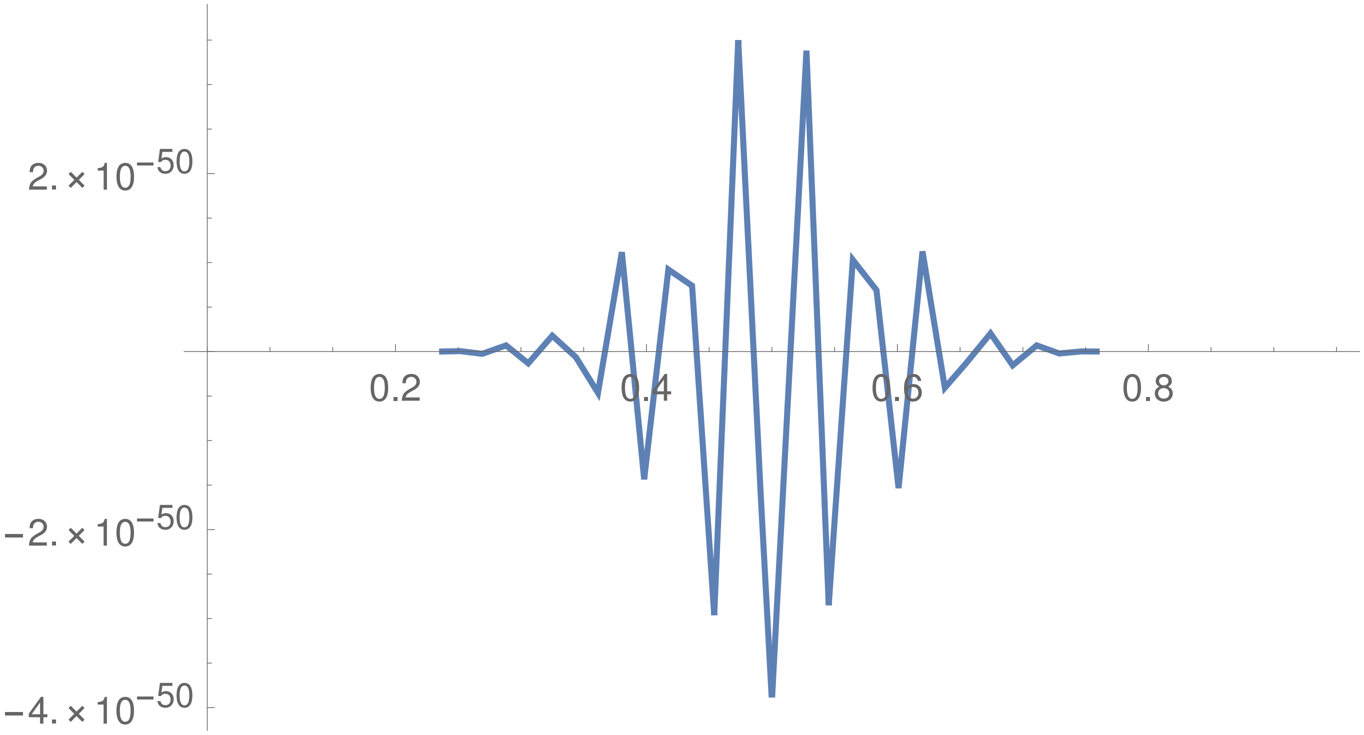
Key observation. $G(x, y)$ satisfies the PDE $\frac{\partial G}{\partial y} = x(1 - x) \frac{\partial G}{\partial x}$ so for $n \geq 1$, p_n satisfy the recursion relation

$$np_{n+1}(x) = x(1 - x)p_n'(x) .$$

□

Note: There is no inner product w.r.t. which the $p_n(x)$ are orthogonal.

Figure 4: Gaussian enveloping curve ($n = 100$)



4 Conclusion & outlook

- The vector space of quasi-elliptic functions is under complete control.
- The $g^{(n)}$ and the iterated convolutions of $\mathcal{Z}$ both yield natural bases for a subspace $V \subset SK$ with many applications. Each set has its own merits.
- The methods used here should work for arbitrary vector bundles on elliptic curves, in particular for convolutions of Mock-Jacobi forms.
- The perspective developed here should help with the understanding of higher genus.