

Discriminating between unitary quantum processes

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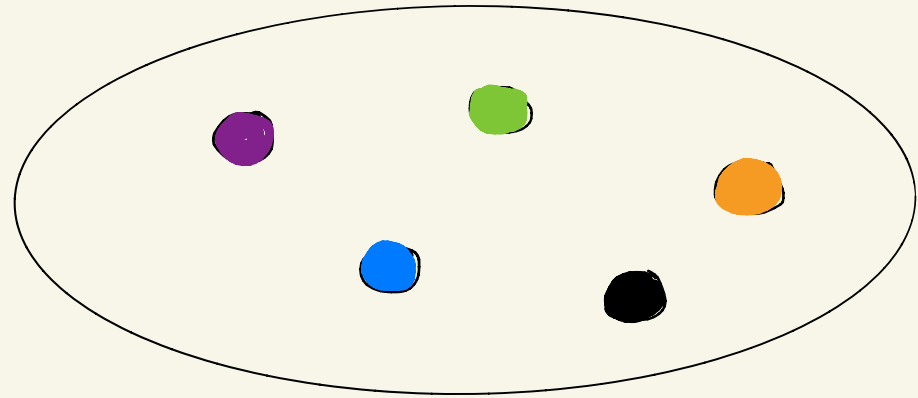
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Harvard


Discriminating between unknown objects
in a finite set :

S



is a fundamental task in any experimental
science

Discriminating between quantum states

- You are given:  A : a quantum system in an **unknown** quantum state ρ

- You are told: ρ  with equal probability

i.e. $\rho \in S = \{\rho_1, \rho_2\}$

e.g. A : an electron; $\mathcal{H} \simeq \mathbb{C}^2$
 $\rho_1 = |0\rangle\langle 0|$, $\rho_2 = |+\rangle\langle +|$
 $|0\rangle = |\uparrow\rangle$; $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$

Discriminating between quantum states

• You are given: $\textcircled{\text{A}}$: a quantum system in an **unknown** quantum state ρ

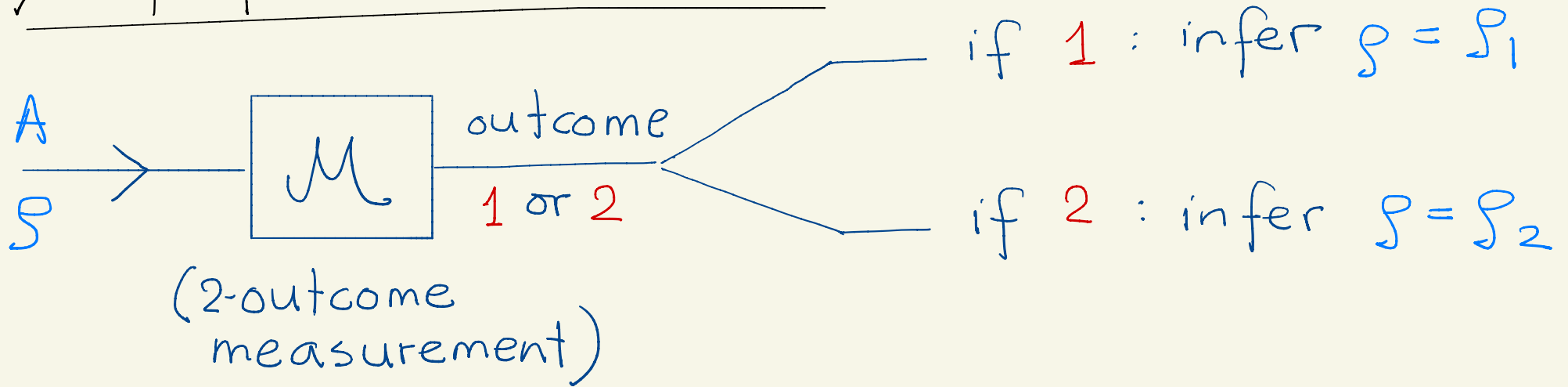
• You are told: ρ $\begin{cases} \rho_1 \\ \rho_2 \end{cases}$ with equal probability

i.e. $\rho \in \mathcal{S} = \{\rho_1, \rho_2\}$

• Your task: to determine whether the state is ρ_1 or ρ_2

Discriminating between quantum states

- You perform a measurement:



- Average probability of error in discrimination:

$$P_{\text{err}}(\rho_1, \rho_2; \mathcal{M}) = \frac{1}{2} P(\text{infer } \rho_1 | \rho_2) + \frac{1}{2} P(\text{infer } \rho_2 | \rho_1)$$

- Optimal error probability

$$P_{\text{err}}(\rho_1, \rho_2) = \inf_{\mathcal{M}} P_{\text{err}}(\rho_1, \rho_2; \mathcal{M})$$

Discriminating between quantum states

Holevo-Helström Theorem

$$\|A\|_1 = \text{Tr} \sqrt{A^\dagger A}$$

$$P_{\text{err}}(\rho_1, \rho_2) = \frac{1}{2} \left(1 - \frac{1}{2} \|\rho_1 - \rho_2\|_1 \right)$$

- $0 \leq \|\rho_1 - \rho_2\|_1 \leq 2$
- $\|\rho_1 - \rho_2\|_1 = 2$ iff $\rho_1 \perp \rho_2$ ($\rho_1 \rho_2 = 0$)
(mutually orthogonal!)

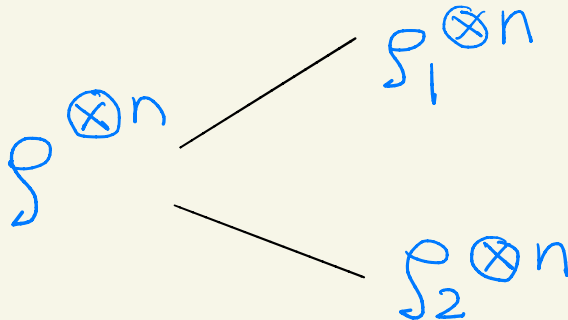
⇒

$$P_{\text{err}}(\rho_1, \rho_2) = 0 \text{ iff } \rho_1 \perp \rho_2$$

[In Quantum Mechanics only mutually orthogonal states can be perfectly discriminated.]

Discriminating between quantum states

- What if you are given multiple (n) identical copies of the unknown state?

- You get: $\rho^{\otimes n}$  with equal probability

- Optimal error probability (by Holevo-Helström Thm)

$$P_{\text{err}}^{(n)}(\rho_1, \rho_2) = P_{\text{err}}(\rho_1^{\otimes n}, \rho_2^{\otimes n}) = \frac{1}{2} \left[1 - \frac{1}{2} \|\rho_1^{\otimes n} - \rho_2^{\otimes n}\|_1 \right]$$

- [Nussbaum-Szkola & Audenaert et al].

$$P_{\text{err}}^{(n)}(\rho_1, \rho_2) \sim \exp(-n \xi_Q); \quad \xi_Q = \begin{array}{l} \text{quantum} \\ \text{Chernoff bound} \end{array}$$

Discriminating between quantum states

$$P_{\text{err}}^{(n)}(\rho_1, \rho_2) \sim \exp(-n \xi_Q) ; \xi_Q = -\log \left(\inf_{0 \leq s \leq 1} \text{Tr}(\rho_1^s \rho_2^{1-s}) \right)$$

Quantum Chernoff bound

$$0 \leq \xi_Q \leq \infty$$

- $\xi_Q = 0$ when $\rho_1 = \rho_2$
- $\xi_Q = \infty$ when $\rho_1 \perp \rho_2$; $P_{\text{err}}^{(n)}(\rho_1, \rho_2) = 0$

Discriminating between quantum states

$$P_{\text{err}}^{(n)}(\rho_1, \rho_2) \sim \exp(-n \xi_Q) ; \xi_Q = -\log \left(\inf_{0 \leq s \leq 1} \text{Tr}(\rho_1^s \rho_2^{1-s}) \right)$$

Quantum Chernoff bound

• For $0 < \xi_Q < \infty$; $\lim_{n \rightarrow \infty} P_{\text{err}}^{(n)}(\rho_1, \rho_2) = 0$

• Perfect discrimination of two states ρ_1 & ρ_2 such that $\rho_1 \not\perp \rho_2$ is only possible in the asymptotic limit $n \rightarrow \infty$

Discriminating between quantum processes

- **Focus**: the task of discriminating between **unitary** quantum processes / quantum channels (involve **unitary operators** / **superoperators**)

- Motivation: **unitary operators** are the building blocks for quantum computation:

In the **quantum circuit model** for quantum computation

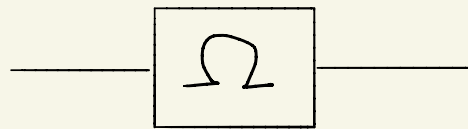
quantum logic gates \equiv **unitary operators**

- **Unitary channel** (superoperator): $\mathcal{U}(\rho) = U\rho U^\dagger$
(U)

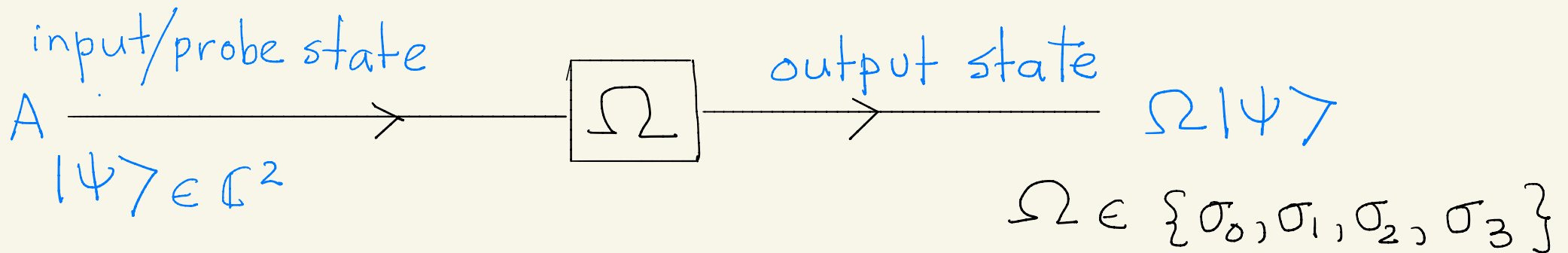
ρ : state of a quantum system ; U : unitary operator

Discriminating between unitaries: an example

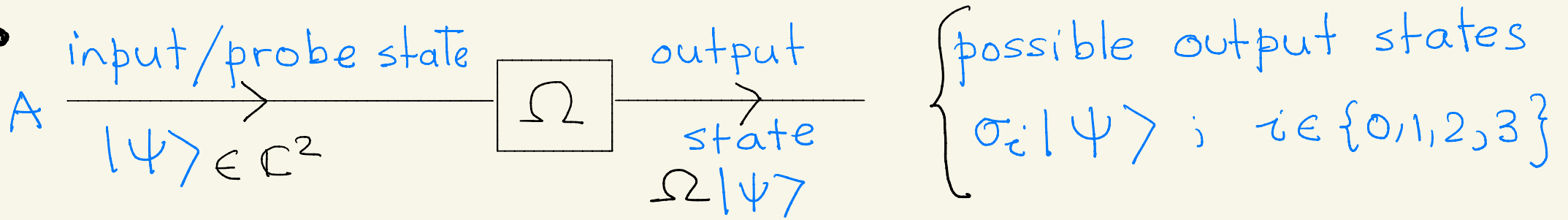
- $S = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$; $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- unitary operators on \mathbb{C}^2
- form a group
- You are given the unitary operator as a black box:



- Your task: determine whether Ω is $\sigma_0, \sigma_1, \sigma_2$ or σ_3



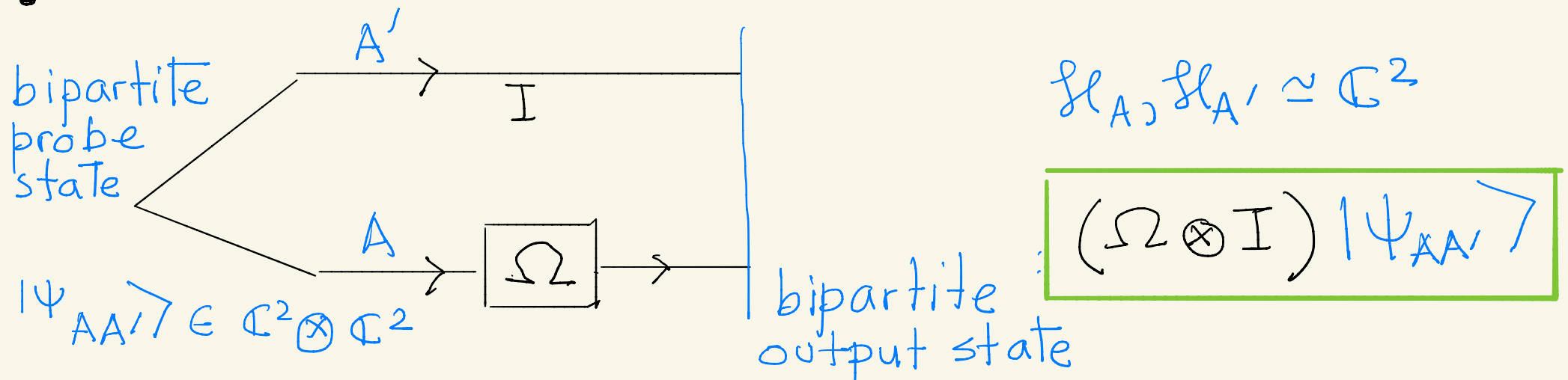
Discriminating between unitaries: an example



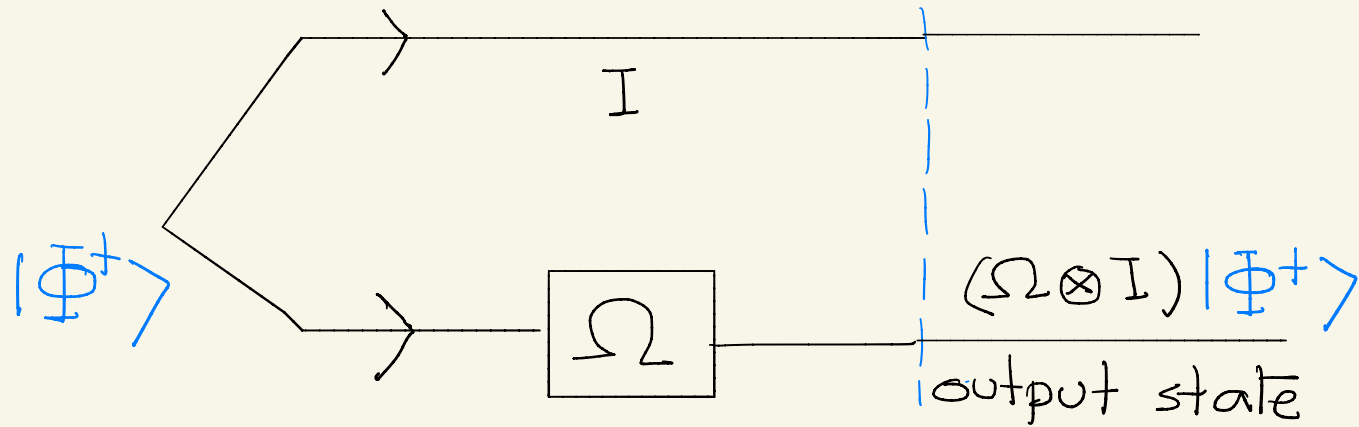
BUT: • $\left\{ \sigma_i |\psi\rangle ; i = 0,1,2,3 \right\}$ a linearly dependent set of states

• \therefore cannot be perfectly discriminated.

• Instead use an ancilla A' & a bipartite probe state



Discriminating between unitaries: an example



- Choose $|\Psi_{AA'}\rangle \equiv |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (entangled state)

- Possible output states: ($\Omega \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$)

$$(\sigma_0 \otimes I) |\Phi^+\rangle = |\Phi^+\rangle$$

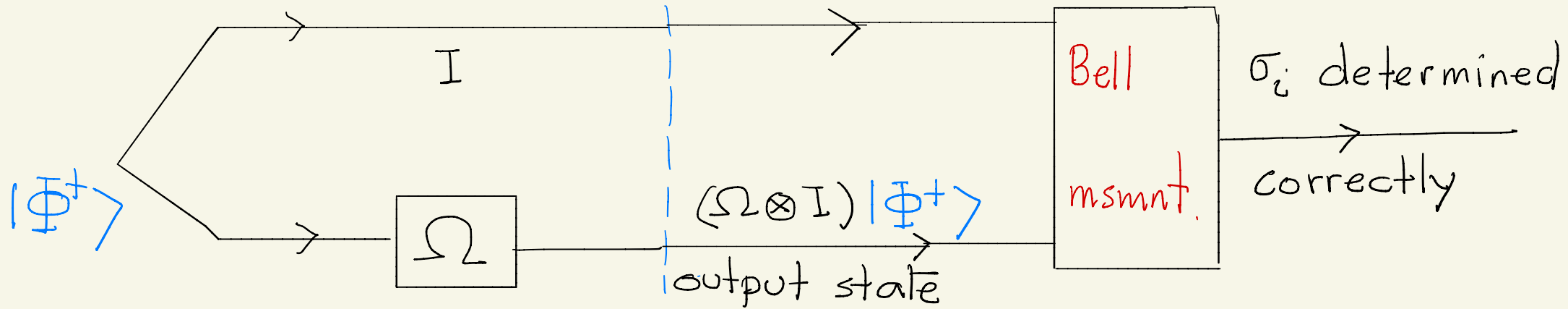
$$(\sigma_1 \otimes I) |\Phi^+\rangle = |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$(\sigma_2 \otimes I) |\Phi^+\rangle = |\Psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

$$(\sigma_3 \otimes I) |\Phi^+\rangle = |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

- Bell states
- mutually \perp

Discriminating between unitaries: an example

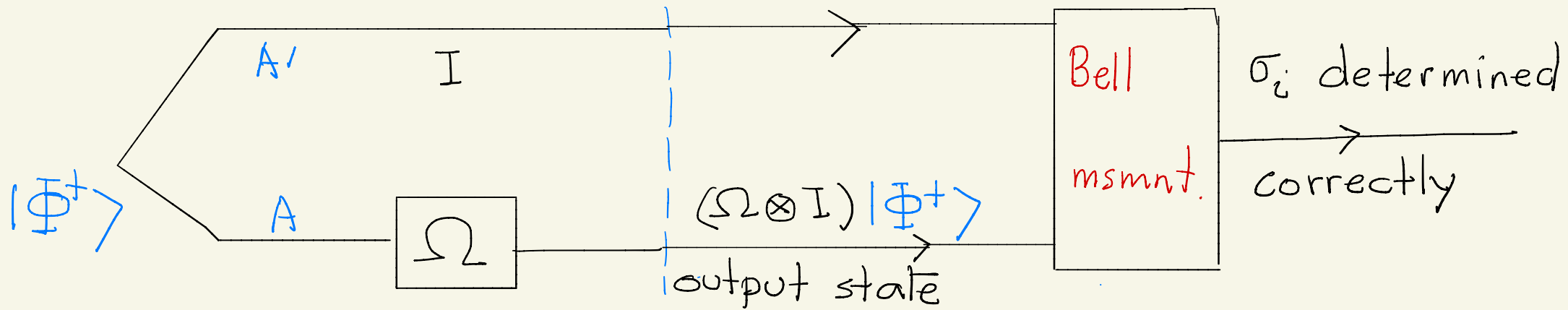


- Choose $|\Psi_{AA'}\rangle \equiv |\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (entangled state)
- Possible output states: ($\Omega \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$)

$$|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

-
- Bell states
 - mutually \perp
 - can be perfectly discriminated
-

Discriminating between unitaries: an example



- Use of an ancilla (A') & a bipartite probe state
 $|\Psi_{AA'}\rangle = |\Phi^+\rangle$ (Bell state)

allows perfect discrimination of the unitaries in the set

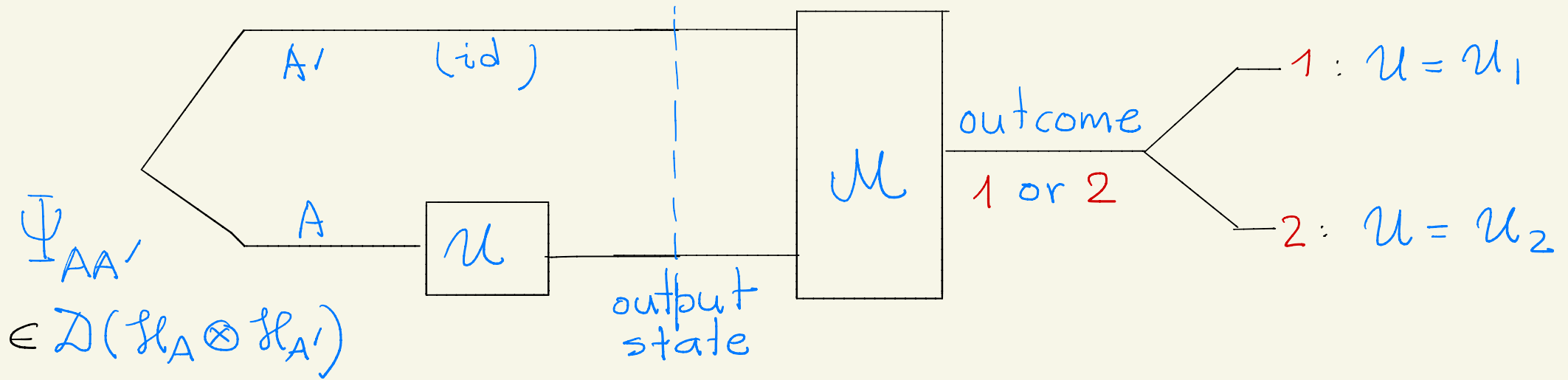
$$S = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$$

Next: consider the task of discriminating
between unitary quantum channels

Discriminating between unitary channels

- You are given a unitary channel (\mathcal{U}) as a black box
- You are told $\mathcal{U} \begin{cases} \mathcal{U}_1 \\ \mathcal{U}_2 \end{cases}$ with equal probability
- A : quantum system; \mathcal{H}_A
 $\mathcal{D}(\mathcal{H}_A) :=$ set of quantum states $\left[\rho \geq 0, \text{Tr} \rho = 1 \right]$ [density operators]
- $\mathcal{U}_1, \mathcal{U}_2: \mathcal{D}(\mathcal{H}_A) \rightarrow \mathcal{D}(\mathcal{H}_A)$
 $\mathcal{U}_1(\rho) = U_1 \rho U_1^\dagger$; $\mathcal{U}_2(\rho) = U_2 \rho U_2^\dagger$; U_1, U_2 unitary operators
- Your task: determine whether \mathcal{U} is \mathcal{U}_1 or \mathcal{U}_2

Discriminating between unitary channels



$$\Psi_{AA'} = |\Psi_{AA'}\rangle\langle\Psi_{AA'}|, \quad U \in \{U_1, U_2\}$$

- Need to discriminate between 2 possible output states:

$$\omega_1 := (U_1 \otimes \text{id}) \Psi_{AA'}, \quad \& \quad \omega_2 := (U_2 \otimes \text{id}) \Psi_{AA'}$$

- By Holevo-Helstrom Theorem: optimal error probability

$$P_{\text{err}}(U_1, U_2) = \inf_{\Psi_{AA'}} \frac{1}{2} \left(1 - \frac{1}{2} \|\omega_1 - \omega_2\|_1 \right)$$

Discriminating between unitary channels

- Optimal error probability in discrimination

$$p_{\text{err}}(U_1, U_2) = \inf_{\Psi_{AA'}} \frac{1}{2} \left(1 - \frac{1}{2} \left\| (U_1 \otimes \text{id}) \Psi_{AA'} - (U_2 \otimes \text{id}) \Psi_{AA'} \right\|_1 \right)$$

$$\bar{\Psi}_{AA'} = |\Psi_{AA'}\rangle\langle\Psi_{AA'}|$$

$$= \frac{1}{2} \left(1 - \frac{1}{2} \sup_{\Psi_{AA'}} \left\| (U_1 \otimes \text{id}) \bar{\Psi}_{AA'} - (U_2 \otimes \text{id}) \bar{\Psi}_{AA'} \right\|_1 \right)$$

a measure of discrimination for channels

- Diamond norm distance:

$$\|U_1 - U_2\|_{\diamond} = \sup_{\Psi_{AA'}} \left\| (U_1 \otimes \text{id}) \bar{\Psi}_{AA'} - (U_2 \otimes \text{id}) \bar{\Psi}_{AA'} \right\|_1$$

($\mathcal{H}_A \cong \mathcal{H}_{A'}$)

$$p_{\text{err}}(U_1, U_2) = \frac{1}{2} \left(1 - \frac{1}{2} \|U_1 - U_2\|_{\diamond} \right)$$

Discriminating between unitary channels U_1 & U_2

- Optimal error probability in discrimination

$$P_{\text{err}}(U_1, U_2) = \frac{1}{2} \left(1 - \frac{1}{2} \|U_1 - U_2\|_{\diamond} \right)$$

$$\|U_1 - U_2\|_{\diamond} = \sup_{\Psi_{AA'}} \|((U_1 - U_2) \otimes \text{id}) \Psi_{AA'}\|_1$$

$$0 \leq \|U_1 - U_2\|_{\diamond} \leq 2$$

$$P_{\text{err}}(U_1, U_2) = 0 \quad \text{iff} \quad \|U_1 - U_2\|_{\diamond} = 2$$

Discriminating between unitary channels \mathcal{U}_1 & \mathcal{U}_2

- Optimal error probability in discrimination

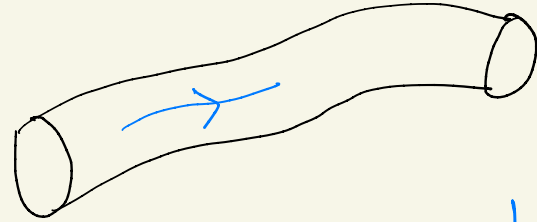
$$P_{\text{err}}(\mathcal{U}_1, \mathcal{U}_2) = \frac{1}{2} \left(1 - \frac{1}{2} \|\mathcal{U}_1 - \mathcal{U}_2\|_{\diamond} \right)$$

- $P_{\text{err}}(\mathcal{U}_1, \mathcal{U}_2) = 0$ iff $\|\mathcal{U}_1 - \mathcal{U}_2\|_{\diamond} = 2$

In this case \mathcal{U}_1 & \mathcal{U}_2 can be perfectly discriminated

Henceforth : we focus on discriminating between a pair of unitary channels acting on an infinite-dimensional quantum system

- e.g. a collection of electromagnetic modes travelling along an optical fibre



- each mode is modelled by a quantum harmonic oscillator
- a continuous variable (cv) quantum system [e.g. in quantum optics]
- relevance : such systems are candidates for many protocols of quantum communication & quantum computation

Continuous variable (CV) quantum systems

- We consider an m mode CV quantum system
- Hilbert space $\mathcal{H}_m = L^2(\mathbb{R}^m)$
- a_j^\dagger, a_j : creation & annihilation operators corrs. to the j th mode; $j = 1, 2, \dots, m$
- Canonical commutation relations (CCR): $[a_j, a_k^\dagger] = \delta_{jk} I$
- Quantum states: $\rho \in \mathcal{T}_1(\mathcal{H}_m)$ trace class operator
 $\rho \geq 0$; $\text{Tr} \rho = 1$
- Hamiltonian, H : densely defined self-adjoint operator
assume $\left\{ \begin{array}{l} \text{its spectrum } \sigma(H) \text{ is bounded from below} \\ \text{min } \sigma(H) = 0 \quad \text{"grounded"} \end{array} \right.$

Continuous variable (CV) quantum systems

- Experimentally, for a CV quantum system governed by a Hamiltonian H , one only has access to states ρ with bounded mean energy : $\text{Tr}(\rho H) \leq E$ for some $E < \infty$

- a natural Hamiltonian energy constraint

$$\text{Number operator } N = \sum_{k=1}^m a_k^\dagger a_k$$

- This energy constraint leads to a modified measure of discrimination for channels acting on CV quantum systems

Energy constrained diamond norm distance

[Shirokov;
Winter;
Pirandola]

Energy-constrained diamond (ECD) norm distance

$$\|u_1 - u_2\|_{\diamond}^{H, E} := \sup \left\| (u_1 - u_2) \otimes \text{id} \right\|_{\Psi_{AA'}}$$

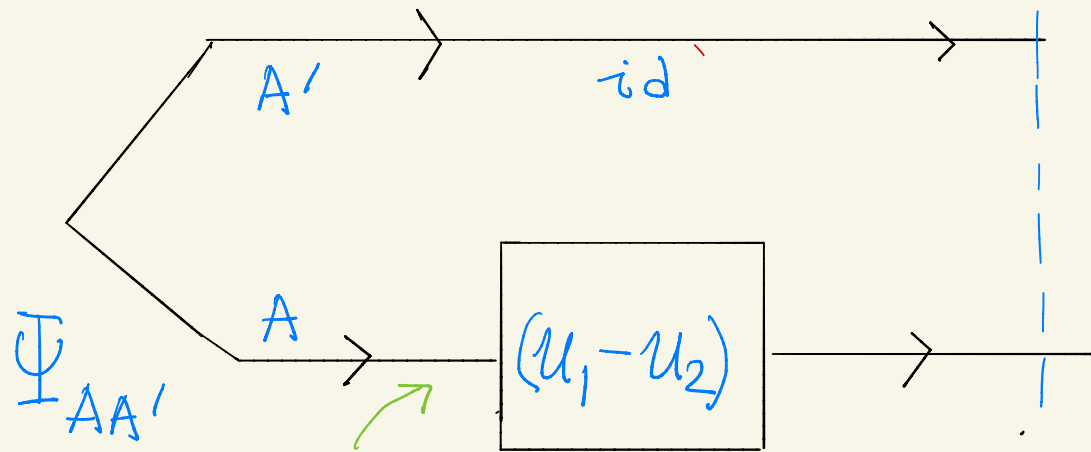
$$\Psi_{AA'} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{A'})$$

$$\text{Tr}(\rho_A H) \leq E$$

(energy constraint)

$$\rho_A = \text{Tr}_{A'} \Psi_{AA'}$$

(reduced state of A)



$$\sup_{E \geq 0} \|u_1 - u_2\|_{\diamond}^{H, E} = \|u_1 - u_2\|_{\diamond}$$

$$\rho_A = \text{Tr}_{A'} \Psi_{AA'}$$

$$\left((u_1 - u_2) \otimes \text{id} \right) \Psi_{AA'}$$

- We study: discrimination of pairs of unitary channels acting on CV quantum systems

Our Results

Results

- Result 1: no entanglement is needed to discriminate between 2 unitary channels acting on a CV quantum system.

- Theorem 1: Let \mathcal{U}, \mathcal{V} be 2 unitary channels acting on a CV quantum system A ; $\forall \rho \in \mathcal{D}(\mathcal{H}_A)$:
$$\mathcal{U}(\rho) = U \rho U^\dagger, \quad \mathcal{V}(\rho) = V \rho V^\dagger \quad ; \quad H: \text{s.a. operator acting on } A$$

(Hamiltonian)

$$\| \mathcal{U} - \mathcal{V} \|_{\diamond}^{H, E} = 2 \sqrt{1 - \inf_{\substack{|\psi_A\rangle \in \mathcal{H}_A \\ \langle \psi_A | H | \psi_A \rangle \leq E}} |\langle \psi_A | U^\dagger V | \psi_A \rangle|^2}$$

$E < \infty$

Results

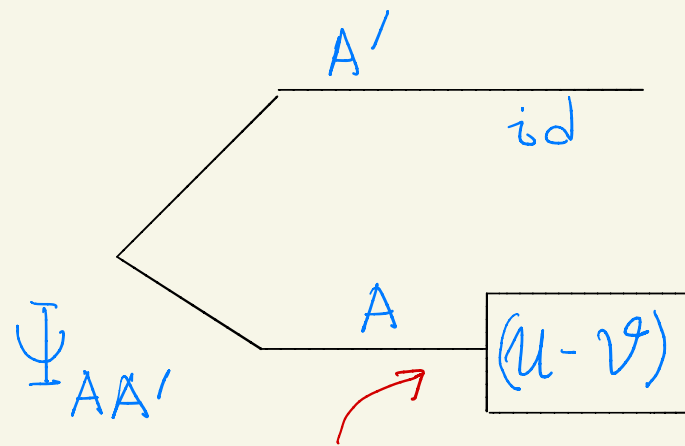
Theorem 1: $\|u-v\|_{\diamond}^{H,E} = 2 \sqrt{1 - \inf_{\substack{|\psi\rangle \in \mathcal{H}_A \\ \langle \psi | H | \psi \rangle \leq E}} |\langle \psi | U^\dagger V | \psi \rangle|^2}$

• How does this tell us that an **entangled probe state** is **not needed** for the discrimination task?

• $\|u-v\|_{\diamond}^{H,E} = \sup_{\substack{\Psi_{AA'} \\ \text{Tr } \rho_A H \leq E}} \|((u-v) \otimes \text{id}) \Psi_{AA'}\|_1$ — (a)

$\text{Tr } \rho_A H \leq E$

$(\Psi_{AA'} = |\Psi_{AA'}\rangle\langle\Psi_{AA'}|)$



(mixed state) $\rho_A = \text{Tr}_{A'} \Psi_{AA'}$

but in (a) only $|\psi\rangle \in \mathcal{H}_A$ appears
 $\Rightarrow \rho_A = |\psi\rangle\langle\psi|$ (pure state)
 $\Rightarrow |\Psi_{AA'}\rangle = |\psi\rangle \otimes |\psi\rangle$ product state
 $\Rightarrow \sup$ in (a) can be restricted to **unentangled states**.

Note: **Theorem 1** generalises the seminal
result by Aharonov, Kitaev & Nisan
to the infinite-dimensional setting

Results

- **Theorem 2**: in the setting of Theorem 1, there exists a positive integer n such that n parallel uses of u & v can be **perfectly discriminated** using inputs of finite energy, i.e.

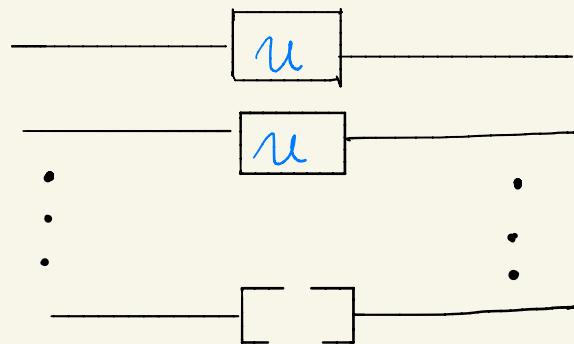
$$\|u^{\otimes n} - v^{\otimes n}\|_{H(n), E} = 2 \quad E < \infty$$

$$H(n) = \sum_{j=1}^n H_j \quad (\text{n-copy Hamiltonian})$$

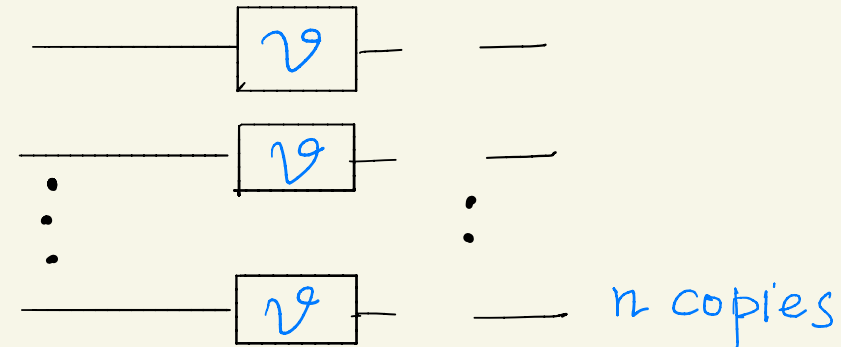
$$H_j = I^{\otimes \dots \otimes I^{\otimes} H^{\otimes} I^{\otimes \dots \otimes I}$$

↑
jth location

perfect
discrimination



vs.



Results

- In particular: let u, \hat{v} be such that

$$\|u - \hat{v}\|_{\diamond}^{H, E} \neq 2 \quad \text{cannot be perfectly discriminated}$$

However, there always $\exists n < \infty$ such that

$$\|u^{\otimes n} - \hat{v}^{\otimes n}\|_{\diamond}^{H_{(n)}, E} = 2 \quad \text{can be perfectly discriminated}$$

Results

- Note 1: This is in contrast to the case of state discrimination

If ρ_1, ρ_2 : 2 distinct states which are not mutually \perp , then $\rho_1^{\otimes n}$ & $\rho_2^{\otimes n}$ can be perfectly discriminated only in the asymptotic limit ($n \rightarrow \infty$)

- Note 2: Our Theorem 2 is an extension of a celebrated result by A. Acin (2001) (for unitary channels acting on finite-dimensional systems)

Results

- **Note**: the most familiar unitary channels are those arising from the **time evolution** of a closed quantum system evolving under the action of a **Hamiltonian**

• Let H & H' : Hamiltonians governing the evolution of 2 such quantum systems A, A'

- Corrs. **unitary channels**

$$U_t(\rho) = e^{-iHt} \rho e^{iHt} \quad ; \quad V_t(\rho) = e^{-iH't} \rho e^{iH't}$$

$$\rho \in \mathcal{D}(\mathcal{H})$$

$$\mathcal{H} \simeq \mathcal{H}_A \simeq \mathcal{H}_{A'}$$

Results

- Result 3: quantifies the **relative drift** caused by **2 different unitary dynamics** generated by Hamiltonians H and H' respectively

Theorem 3: If $(H - H')$ is relatively H -bounded

$$\|(H - H')|\psi\rangle\| \leq \alpha \|H|\psi\rangle\| + \beta, \text{ for some } \alpha, \beta > 0$$

and all states $|\psi\rangle$; then $\forall t \geq 0$ and $E \geq 0$:

$$\|u_t - v_t\|_{\square}^{H, E} \leq 2\sqrt{2}\sqrt{\alpha Et} + \beta$$

Results

Corollary

- a novel type of quantum speed limit:

What is meant by quantum speed limit?

Quantum Speed Limits

e.g.

- Consider a closed quantum system A .
- Let t_{\min} : the minimum time needed for the system A to evolve from a given initial state $|\psi(0)\rangle \equiv |\psi_{\text{in}}\rangle$ to a prescribed final state (or class of states)

Quantum Speed Limits

e.g.

- Consider a closed quantum system A.
- Let t_{\min} : the minimum time needed for the system A to evolve from a given initial state $|\psi(0)\rangle \equiv |\psi_{\text{in}}\rangle$ to a prescribed final state (or class of states)

e.g. $|\psi(t_{\min})\rangle \equiv |\psi_f\rangle$ such that $\langle \psi_f | \psi_{\text{in}} \rangle = 0$

i.e. $|\psi_f\rangle \perp |\psi_{\text{in}}\rangle$

Quantum Speed Limits

e.g.

- Consider a closed quantum system A.
- Let t_{\min} : the minimum time needed for the system A to evolve from a given initial state $|\psi(0)\rangle \equiv |\psi_{\text{in}}\rangle$ to a prescribed final state (or class of states)

e.g. $|\psi(t_{\min})\rangle \equiv |\psi_f\rangle$ such that $\langle \psi_f | \psi_{\text{in}} \rangle = 0$

i.e. $|\psi_f\rangle \perp |\psi_{\text{in}}\rangle$

- A quantum speed limit provides a lower bound on t_{\min}

Quantum Speed Limits

e.g. if $|\Psi(t_{\min})\rangle \perp |\Psi(0)\rangle \equiv |\Psi_{\text{in}}\rangle$

then $t_{\min} \geq \frac{\pi}{2\Delta E}$ [Mandelstamm & Tamm]

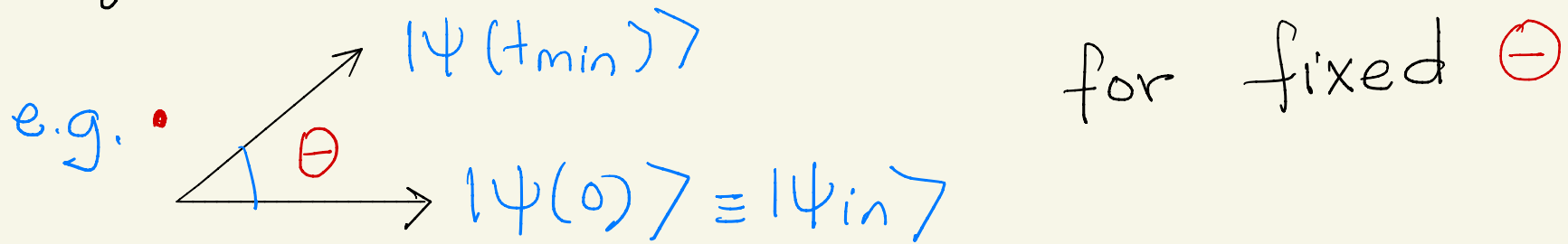
ΔE : variance of the energy of the initial state $|\Psi_{\text{in}}\rangle$

• if $|\Psi(t_{\min})\rangle \perp |\Psi(0)\rangle (\equiv |\Psi_{\text{in}}\rangle)$ & has expected energy \bar{E}

$t_{\min} \geq \max \left\{ \frac{\pi}{2\Delta E}, \frac{\pi}{2\bar{E}} \right\}$ [Mandelstamm & Tamm; Levitin & Toffoli]

Quantum Speed Limits

- There have been various generalizations of quantum speed limits:



- mixed states: $t_{\min} \geq ?$ such that $\rho(0) = \rho_{\text{in}}$
 $\text{dist}(\rho(t_{\min}), \rho(0)) = d$ for fixed d .

- open quantum systems (with bounded generators)

etc

Results

Corollary

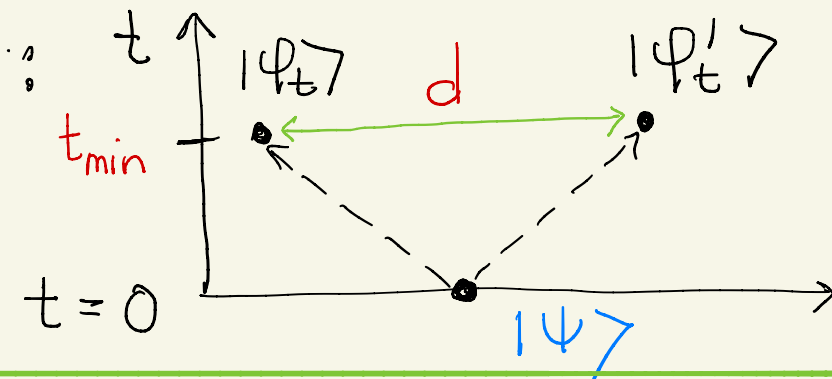
- a novel type of quantum speed limit:

a bound on the **minimum time needed** for the 2 different dynamics (corrs. to H & H') to evolve a given state $|\psi\rangle$ to a pair of states

$$|\varphi_t\rangle = e^{-iHt} |\psi\rangle \quad \& \quad |\varphi'_t\rangle = e^{-iH't} |\psi\rangle$$

such that $\| |\varphi_t\rangle - |\varphi'_t\rangle \| = d$ (fixed)

Schematically



$$t_{\min} = ?$$

$$t \geq ?$$

Results

Corollary

- a novel type of quantum speed limit:

a bound on the **minimum time needed** for the 2 different dynamics (corrs. to H & H') to evolve a given state $|\psi\rangle$ to a pair of states

$$|\varphi_t\rangle = e^{-iHt} |\psi\rangle \quad \& \quad |\varphi'_t\rangle = e^{-iH't} |\psi\rangle$$

such that $\| |\varphi_t\rangle - |\varphi'_t\rangle \| = d$ (fixed)

$$t \geq f(d, E, \alpha, \beta)$$

[explicitly determined]

Results

In particular, for $|\psi\rangle$ such that $\langle\psi|H|\psi\rangle = E$,
the minimum time t for which

$$\| |\psi_t\rangle - |\psi'_t\rangle \| = d \quad (\text{for some fixed } d)$$

satisfies

$$t \geq \frac{(\beta d + \alpha E - \sqrt{\alpha E})^2}{\beta},$$

(quantum
speed
limit)

where $\alpha, \beta > 0$ and



$$\| (H - H') |\psi\rangle \| \leq \alpha \| H |\psi\rangle \| + \beta$$

(relative boundedness)

Quantum Speed Limits

- Quantum speed limits in the literature: Bounds on the minimum time needed for a given initial state of a quantum system to evolve to a prescribed final state (or class of final states).
(closed/open) (pure/mixed)

- Our quantum speed limit pertains to 2 different quantum systems/unitary dynamics

-  $A; H$  $A'; H'$; both prepared in same initial state $|\psi\rangle$

• $\min t \geq ?$ s.t. $\| |\varphi_t\rangle - |\varphi'_t\rangle \| = d$

Results

Result 4 : extension of **Theorem 3** to quantify the relative drift caused by

① unitary dynamics (\mathcal{U}_t) of a **closed** quantum system ; Hamiltonian H

② dynamics of an **open** quantum system governed by a **QDS** (quantum dynamical semigroup)


$(\Lambda_t)_{t \geq 0}$ Λ_t : a quantum channel $\forall t \geq 0$

• $\Lambda_0 = \text{id}$ • $\Lambda_{t+s} = \Lambda_t \circ \Lambda_s$ (semigroup property)

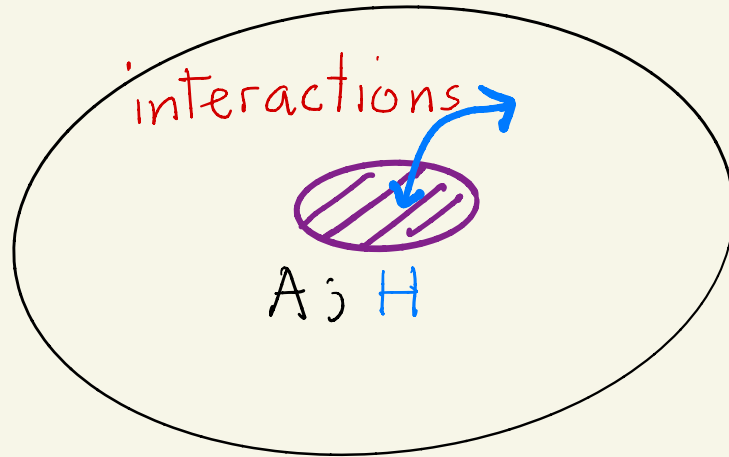
• $\lim_{t \rightarrow 0^+} \|\Lambda_t(\rho) - \rho\|_1 = 0$ (strong continuity)

Results

Scenario :


 $A; H$
 closed system

$$U_t(\cdot) = e^{-iHt} (\cdot) e^{iHt}$$



(interactions)

A: open system

B: bath

$$\Lambda_t = e^{t\mathcal{L}} ; \mathcal{L} : \text{unbounded generator}$$

Assume

$\mathcal{L} : \text{GKLS-type form}$

$$\mathcal{L}(X) = -i[H, X] + \underbrace{\frac{1}{2} \sum_{\ell} 2L_{\ell} X L_{\ell}^{\dagger} - L_{\ell}^{\dagger} L_{\ell} X - X L_{\ell}^{\dagger} L_{\ell}}_{\text{dissipative part}}$$

$\{L_{\ell}\}$: Lindblad operators

dissipative part

Results

- Theorem 4: If $\frac{1}{2} \|\sum L_\ell^\dagger L_\ell |\psi\rangle\| \leq \alpha \|H|\psi\rangle\| + \beta$

\forall states $|\psi\rangle$, for some constants $\beta \geq 0$, $0 \leq \alpha < 1$, then $\forall t \geq 0$
& $E \geq 0$:

$$\|U_t - \Lambda_t\|_{\diamond}^{H, E} \leq 4 \left(\sqrt{\sqrt{2} \alpha E t} + \beta t \right)$$

- Quantum speed limit: bound on minimum time t
needed such that $\|(U_t - \Lambda_t)\rho\|_1 = d$ (fixed)

$$t \geq g(E, d, \alpha, \beta) \quad (\text{explicitly determined})$$

Proof Sketches

Proof Sketch for Theorem 2

Theorem 2: U, V unitary channels acting on a CV-quantum system; (H : Hamiltonian).

$$\|U - V\|_{\diamond}^{H, E} \neq 2$$

but $\exists n < \infty$ such that

$$\|U^{\otimes n} - V^{\otimes n}\|_{\diamond}^{H, E} = 2$$

Q

Proof Sketch For Theorem 2

- For 2 unitary channels U, V (acting on a CV quantum system, under what condition can U & V be perfectly discriminated, i.e. $\|U - V\|_{\diamond}^{H, E} = 2$ iff $\textcircled{?}$

Theorem 1:

$$\|U - V\|_{\diamond}^{H, E} = 2 \sqrt{1 - \inf_{|\psi\rangle} |\langle \psi | W | \psi \rangle|^2}$$

$$W \equiv U^\dagger V$$

$$\therefore \|U - V\|_{\diamond}^{H, E} = 2 \text{ iff}$$

$$\underbrace{\langle \psi | W | \psi \rangle}_{\langle \psi | H | \psi \rangle \leq E} = 0$$

$$\mathcal{S}_E(W) := \{ \langle \psi | W | \psi \rangle ; \forall |\psi\rangle, \langle \psi | H | \psi \rangle \leq E \}$$

$$\therefore \|U - V\|_{\diamond}^{H, E} = 2 \text{ iff } 0 \in \mathcal{S}_E(W)$$

Proof Sketch For Theorem 2

- $\|u - v\|_{\diamond}^{H, E} = 2$ iff $0 \in \mathcal{S}_E(W)$, where

$$\mathcal{S}_E(W) := \{ \langle \psi | W | \psi \rangle; \langle \psi | H | \psi \rangle \leq E \}$$

- It can be shown that

$$\text{int}(\text{conv}(\sigma(W))) \subseteq \bigcup_{E > 0} \mathcal{S}_E(W) \quad \text{--- } \textcircled{a}$$

- If $0 \in \text{int}(\text{conv}(\sigma(W)))$ then $\textcircled{a} \Rightarrow 0 \in \mathcal{S}_E(W)$
for some $E < \infty$

- This would \Rightarrow $\|u - v\|_{\diamond}^{H, E} = 2$ for that E

Proof Sketch For Theorem 2

Theorem 2: $\exists n < \infty$ such that $\|u^{\otimes n} - v^{\otimes n}\|_{\diamond, H(n), E} = 2$

- It suffices to prove that $0 \in \text{int}(\text{conv}(\sigma(W^{\otimes n})))$
 $W = U^{\dagger}V$: unitary operator

- Let us see how : $0 \notin \text{int}(\text{conv}(\sigma(W)))$

$\|u - v\|_{\diamond, H, E} \neq 2$ (i.e. u, v cannot be perfectly discriminated)

but

$$0 \in \text{int}(\text{conv}(\sigma(W^{\otimes n})))$$

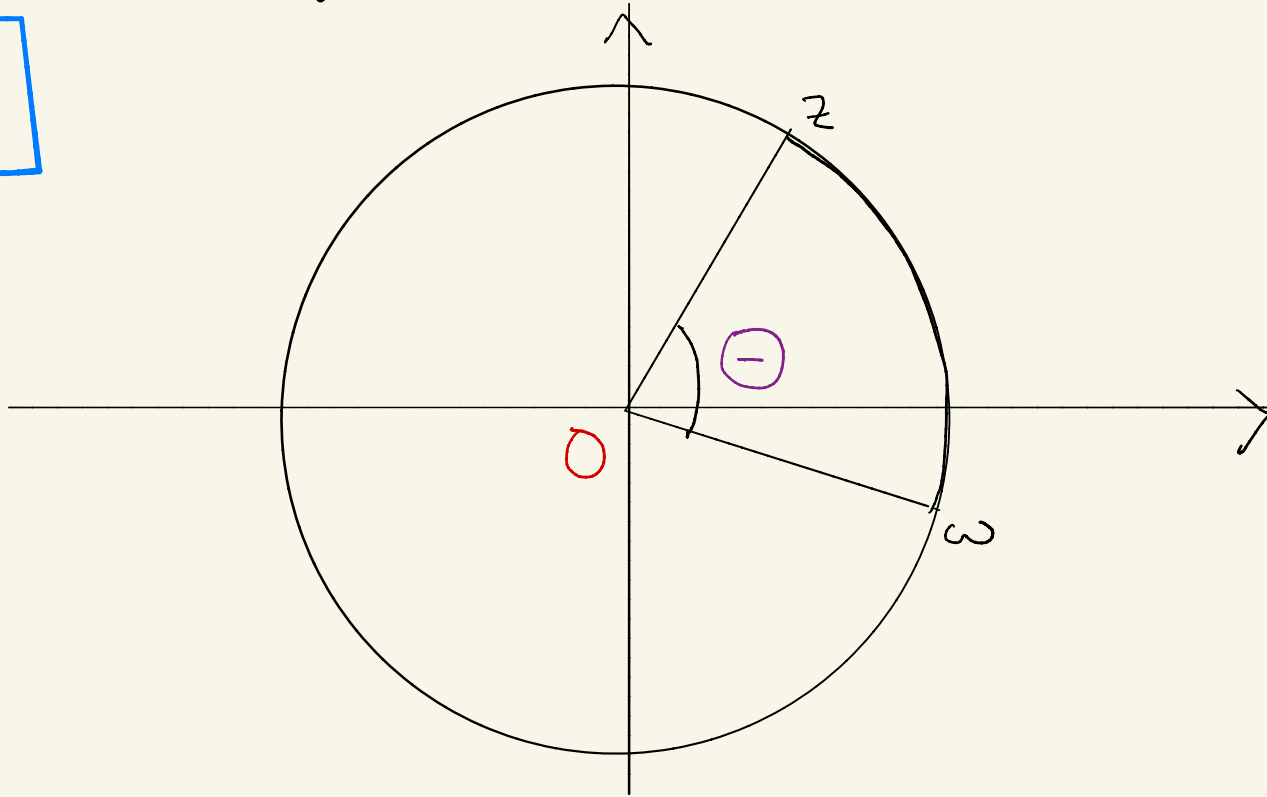
i.e. $\|u^{\otimes n} - v^{\otimes n}\|_{\diamond, H(n), E} = 2$ for some $E < \infty$

$\& \therefore u^{\otimes n}$ & $v^{\otimes n}$ can be perfectly discriminated

Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator
- Spectrum $\sigma(W)$: on the unit circle in the complex plane

$$W = U^\dagger V$$



Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator
- Spectrum $\sigma(W)$: on the unit circle in the complex plane

$$W = U^\dagger V$$

$$U \neq V \quad \therefore W \neq I$$

$$\ominus := \sup_{w, w' \in \sigma(W)} \{ \arg(w) - \arg(w') \} > 0$$

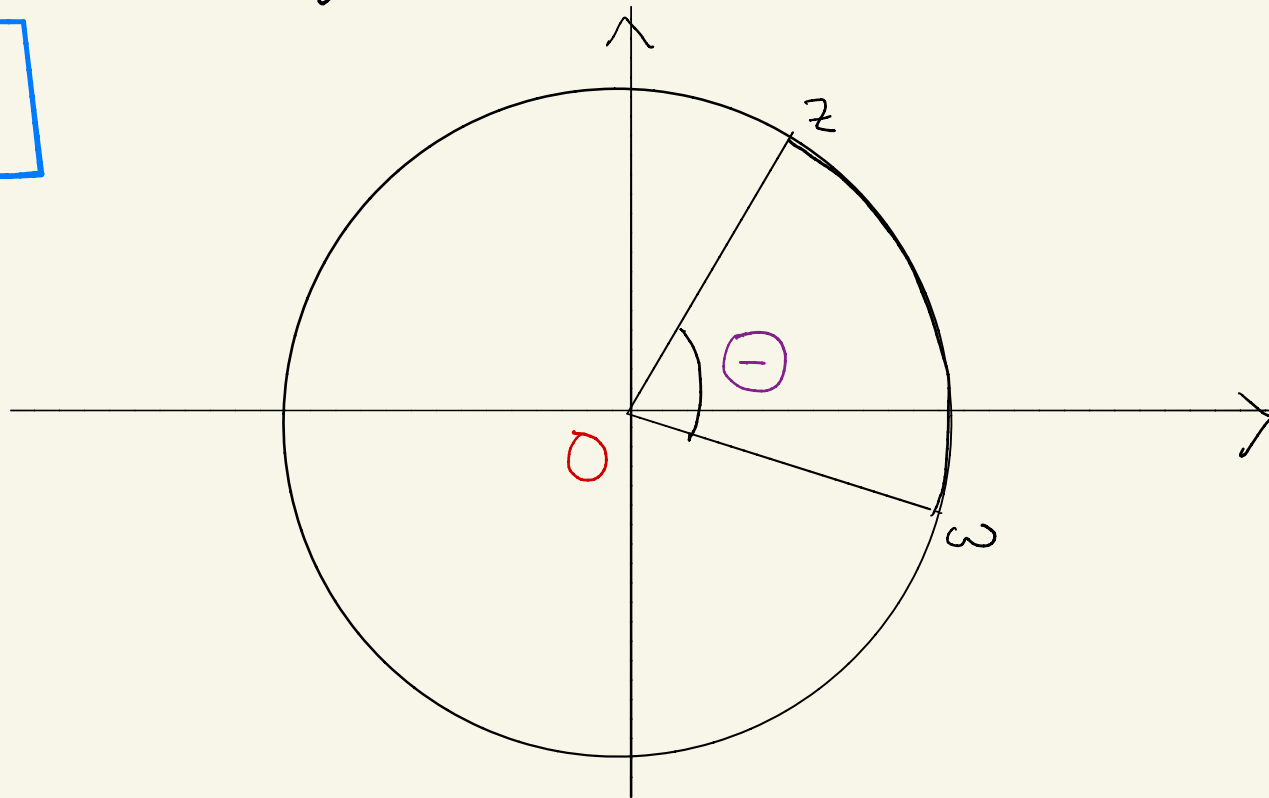
$$\arg(w) \in (-\pi, \pi)$$

Proof Sketch For Theorem 2

• Key idea (pictorially): W unitary operator

• Spectrum $\sigma(W)$:

$$W = U^{\dagger}V$$

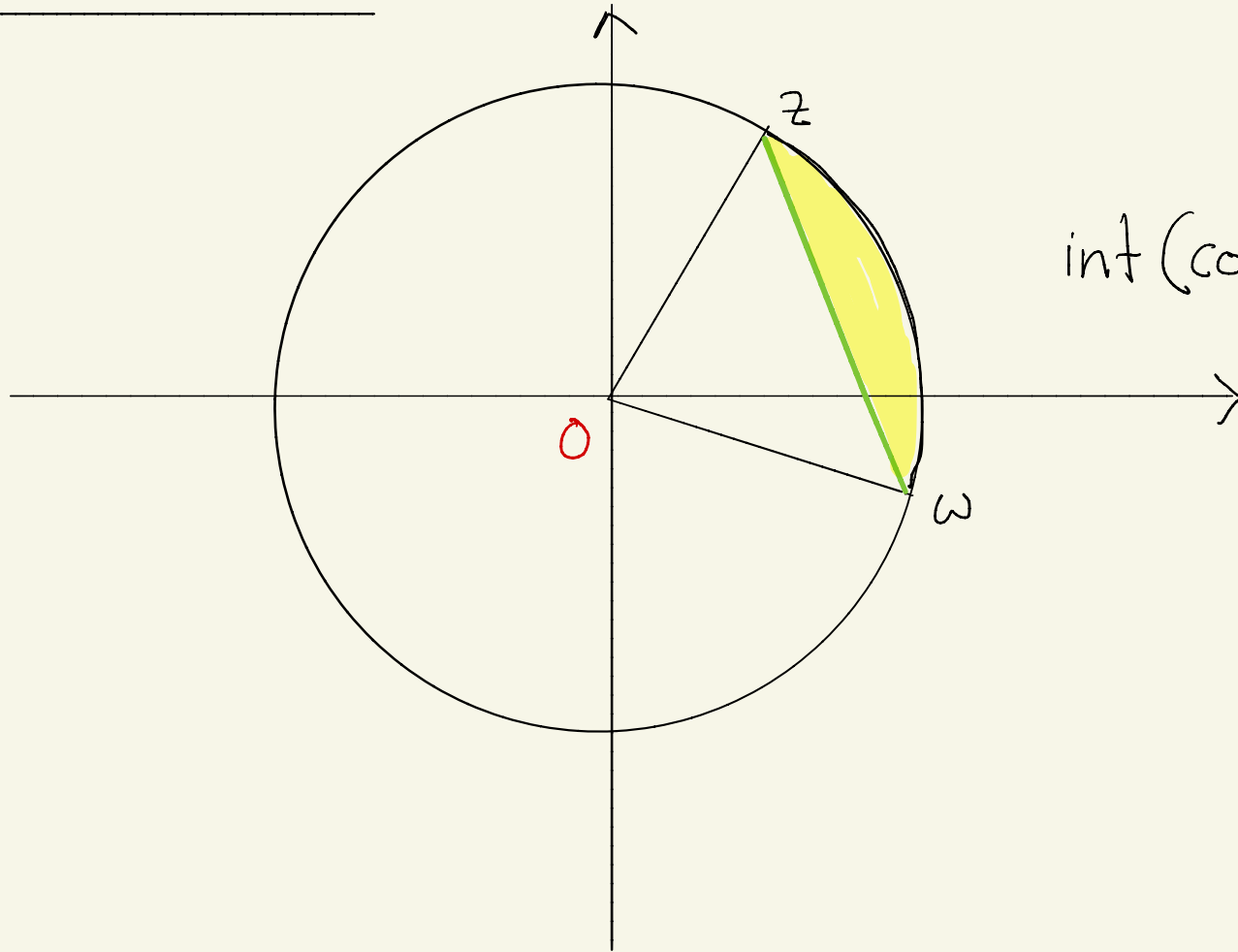


Ⓚ Is $0 \in \text{int}(\text{conv}(\sigma(W)))$?

Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

Spectrum $\sigma(W)$:

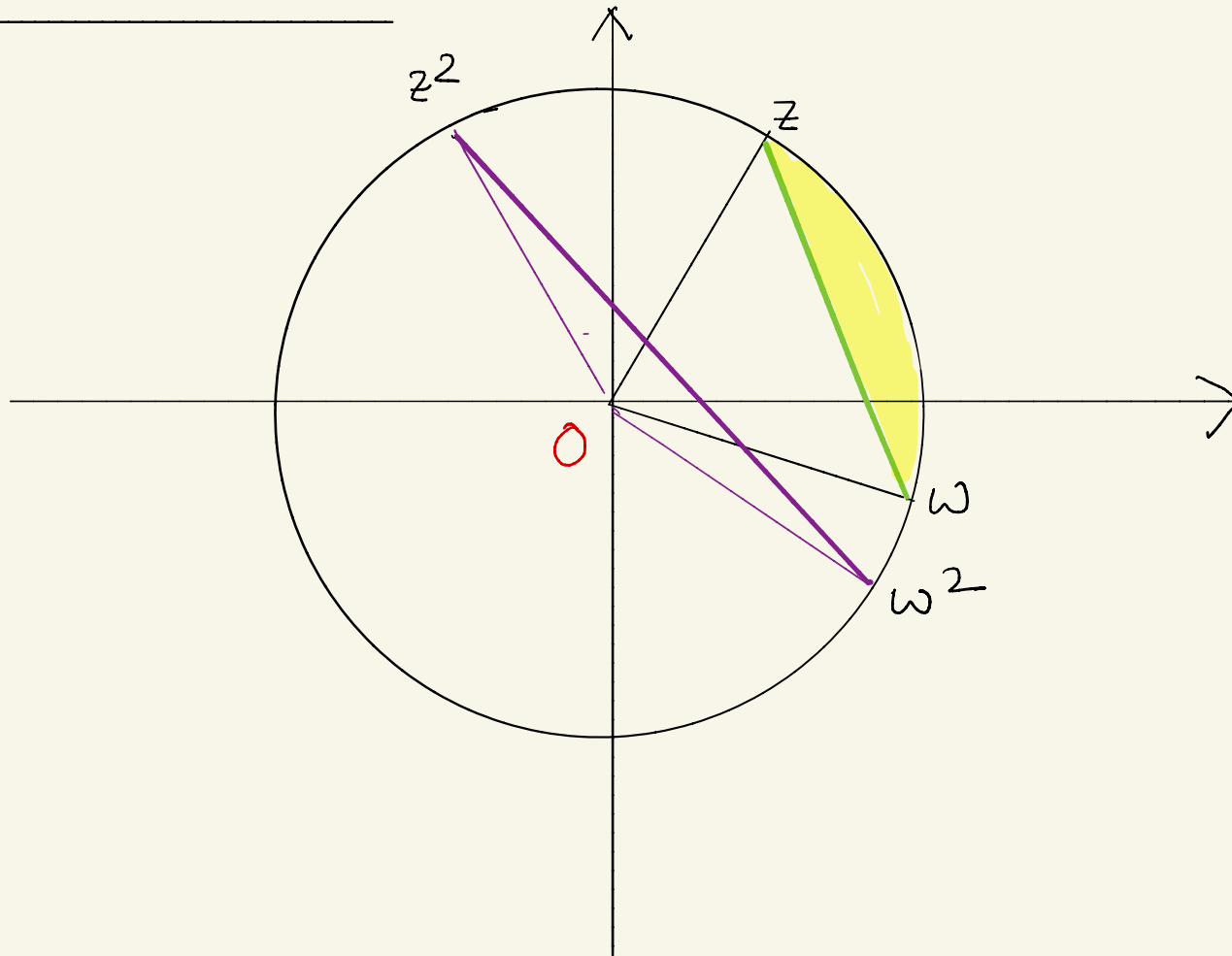


$\text{int}(\text{conv}(\sigma(W))) \neq \emptyset$

Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

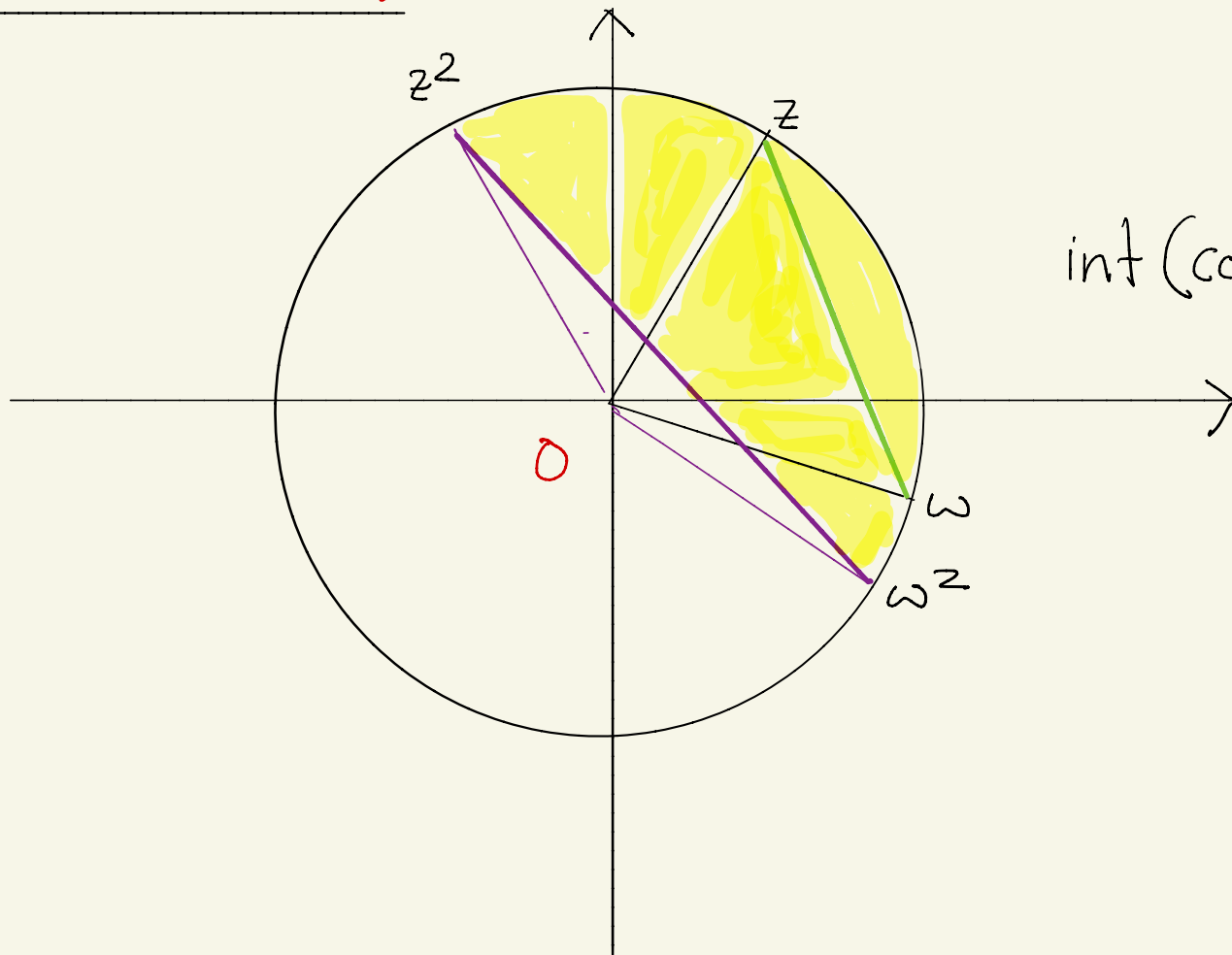
Spectrum $\sigma(W^{\otimes 2})$:



Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

Spectrum $\sigma(W^{\otimes 2})$:

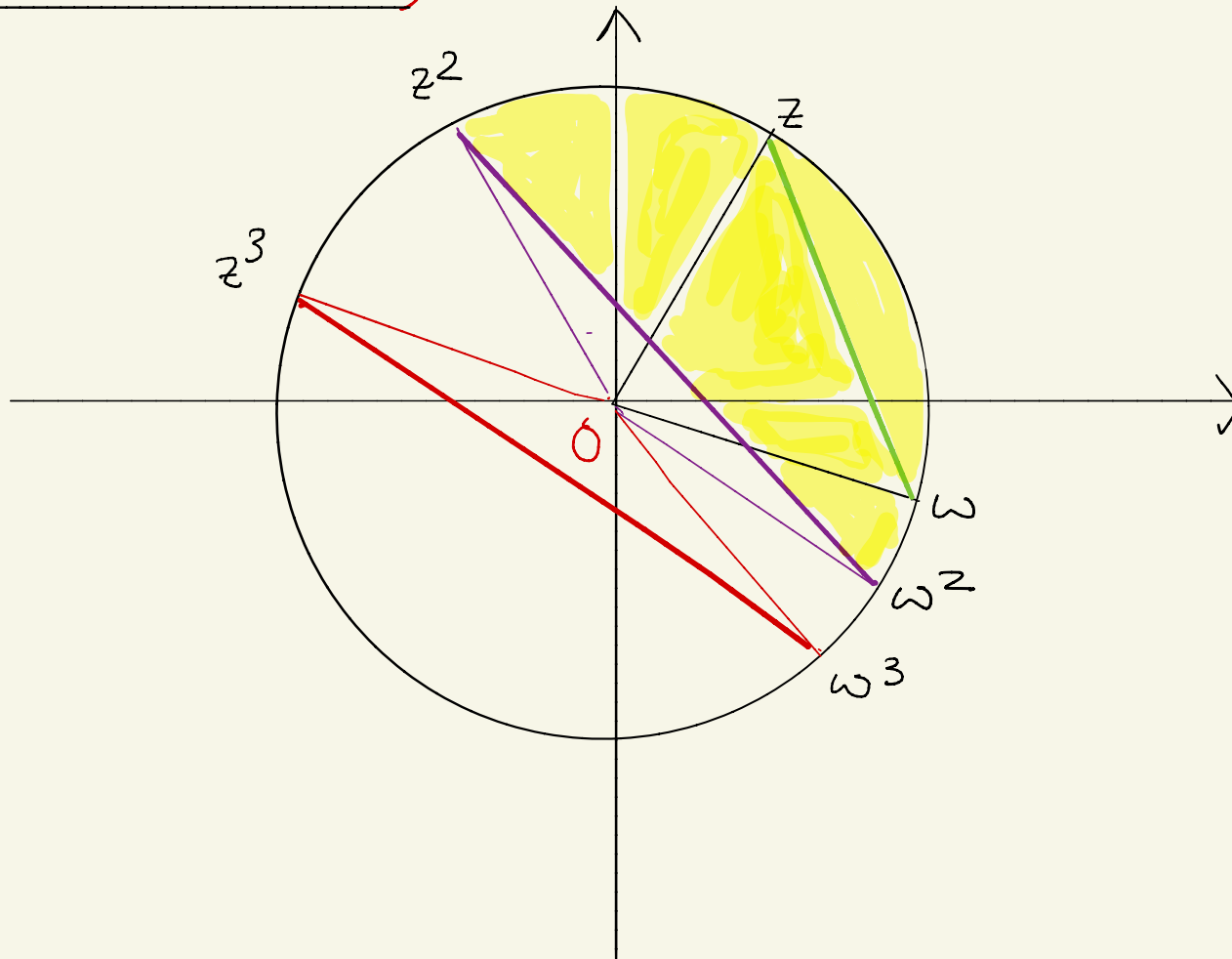


$$\text{int}(\text{conv}(\sigma(W^{\otimes 2}))) \neq 0$$

Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

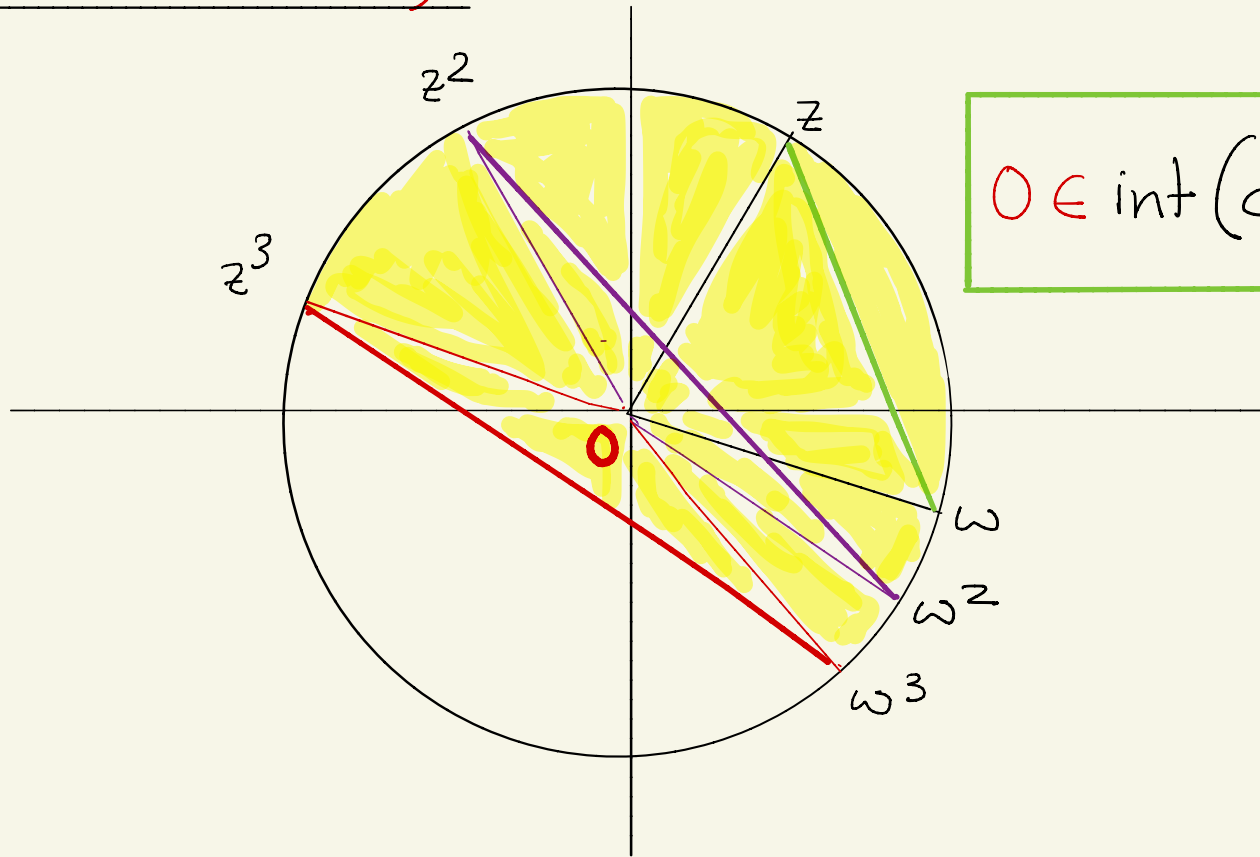
Spectrum $\sigma(W^{\otimes 3})$:



Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

Spectrum $\sigma(W^{\otimes 3})$:

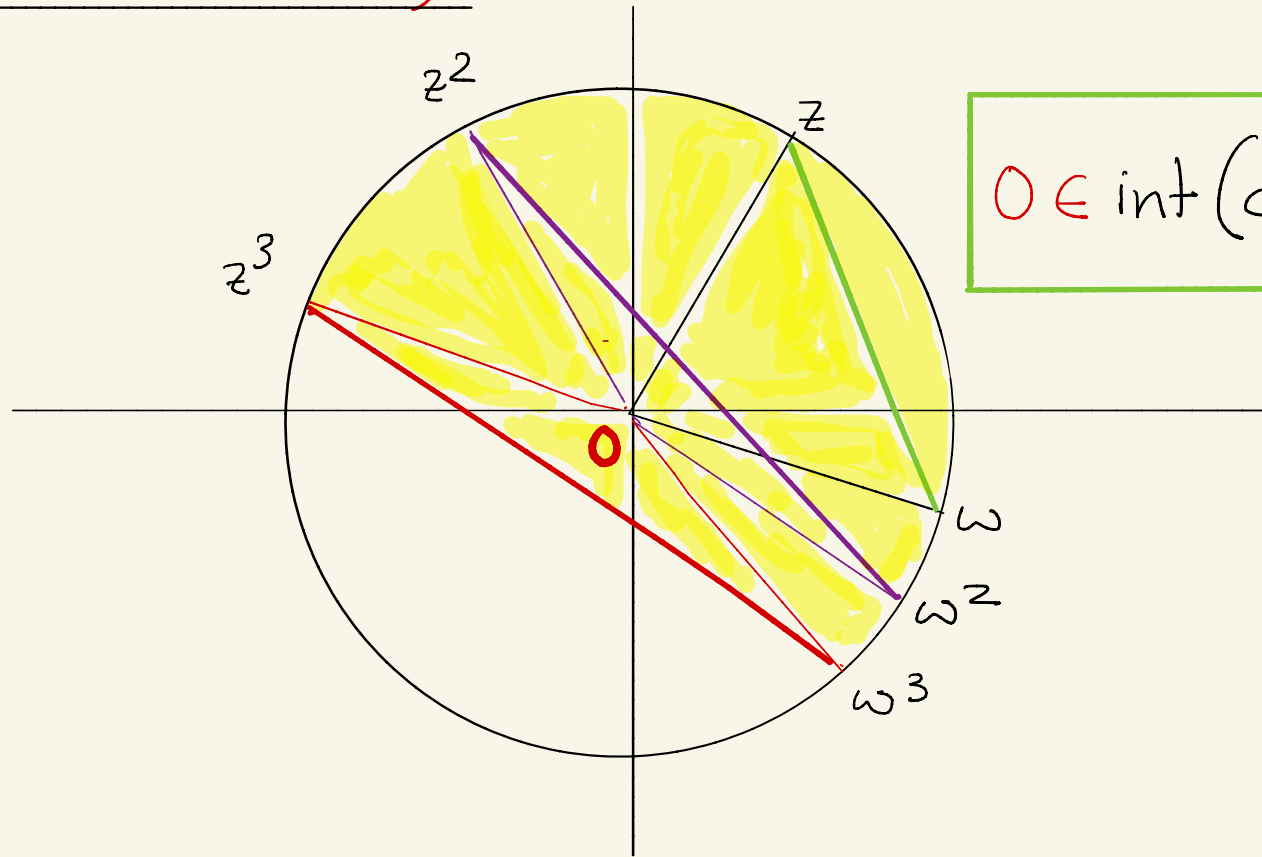


$$0 \in \text{int}(\text{conv}(\sigma(W^{\otimes 3})))$$

Proof Sketch For Theorem 2

- Key idea (pictorially): W unitary operator

Spectrum $\sigma(W^{\otimes 3})$:



$$0 \in \text{int}(\text{conv}(\sigma(W^{\otimes 3})))$$

$\Rightarrow u^{\otimes 3}$ & $v^{\otimes 3}$ can be perfectly discriminated.

Proof Sketch For Theorem 2

• More precisely:

• For $n = \left\lfloor \frac{\pi}{\Theta} \right\rfloor + 1$, $0 \in \text{int}(\text{conv}(\sigma(W^{\otimes n})))$

$\Rightarrow 0 \in \mathcal{S}_E(W^{\otimes n})$ for some $E < \infty$

where

$$\mathcal{S}_E(W^{\otimes n}) = \left\{ \langle \psi | W^{\otimes n} | \psi \rangle ; \langle \psi | H_{(n)} | \psi \rangle \leq E ; |\psi\rangle \in \mathcal{H}_A^{\otimes n} \right\}$$

$\Rightarrow \|u^{\otimes n} - v^{\otimes n}\|_{H_{(n)}, E} = 2$ for that E ■

Summary

- Discriminating between 2 unitary channels U, V acting on a CV quantum system (A)

[e.g. modes of an e.m. field travelling through an optical fibre]

- measure of discrimination: ECD norm

- entanglement is not needed for the discrimination task

- $\exists n < \infty$ such that $U^{\otimes n}$ & $V^{\otimes n}$ can be perfectly discriminated
(n parallel queries suffice)

- Novel speed limits: obtained by considering unitary channels U_t, V_t arising from the time evolution generated by Hamiltonians H & H' (or) U_t & $(V_t)_{t \neq 0}$

Thank you !

Thanks to my coauthors :

Simon Becker, Ludovico Lami, Cambyse Rouze'

arXiv ; 2006.06659

(all relevant citations
included therein)

Extra slides: Proof sketch for Theorem 1

Proof ingredients from Convex Analysis

Defn 1 [Field of values of a matrix]

For $Z \in M_n(\mathbb{C})$ ($n \times n$ complex matrix)

$$F(Z) = \{ \langle \psi | Z | \psi \rangle : |\psi\rangle \in \mathbb{C}^n \} \subset \mathbb{C}$$

Thm A [Toeplitz-Hausdorff]

$\forall Z \in M_n(\mathbb{C}), F(Z) \subset \mathbb{C}$ is convex

Proof ingredients from Convex Analysis

Defn 2 [k -dl field of values of k matrices]

For $Z_1, Z_2, \dots, Z_k \in M_n(\mathbb{C})$

$$F_k(Z_1, \dots, Z_k) := \left\{ \left(\langle \psi | Z_1 | \psi \rangle, \dots, \langle \psi | Z_k | \psi \rangle \right)^T : |\psi\rangle \in \mathbb{C}^k \right\}$$

$$\subset \mathbb{C}^k$$

If Z_1, \dots, Z_k are self-adjoint (s.a.) then

$$F_k(Z_1, \dots, Z_k) \subset \mathbb{R}^k$$

Proof ingredients from Convex Analysis

Theorem: [Au-Yeung-Poon]

For 3 self-adjoint matrices $X, Y, Z \in M_n(\mathbb{C})$.

$F_3(X, Y, Z) \subset \mathbb{R}^3$ is convex

Proof Sketch for Theorem 1

Theorem 1: for unitary channels U, V acting on a cv-quantum system, H : Hamiltonian, $E < \infty$

$$\|U - V\|_{\diamond}^{H, E} = 2 \sqrt{1 - \inf_{\substack{|\psi\rangle \in \mathcal{H}_A \\ \langle \psi | H | \psi \rangle \leq E}} |\langle \psi | W | \psi \rangle|^2}$$

$$W = U^\dagger V$$

• Using

① definition of the ECD-norm

② the identity $\| |\alpha\rangle\langle\alpha| - |\beta\rangle\langle\beta| \|_1 = 2 \sqrt{1 - |\langle \alpha | \beta \rangle|^2}$

③ Schmidt decomposition

$$|\Psi_{AA'}\rangle = \sum_{i=0}^{\infty} \sqrt{x_i} |e_i\rangle |f_i\rangle \quad \&$$

④ a standard truncation argument

we obtain

Proof Sketch for Theorem 1

$$\bullet \|u - v\|_{\diamond}^{H, E} = 2 \sqrt{1 - \nu_E(W)^2} \quad \text{--- (a)} \quad (W = U^\dagger v)$$

where

$$\nu_E(W) = \inf_{\substack{P_A : \text{rk } P_A < \infty \\ \text{Tr}(P_A H) \leq E}} |\text{Tr}(P_A W)|$$

• Theorem 1:

$$\|u - v\|_{\diamond}^{H, E} = 2 \sqrt{1 - \inf_{\substack{|\psi\rangle \in \mathcal{H}_A \\ \langle \psi | H | \psi \rangle \leq E}} |\langle \psi | W | \psi \rangle|^2} \quad \text{--- (b)}$$

∴ (a) & (b) \Rightarrow it suffices to prove

$$\nu_E(W) = \inf_{\substack{|\psi\rangle \in \mathcal{H}_A \\ \langle \psi | H | \psi \rangle \leq E}} |\langle \psi | W | \psi \rangle|$$

Proof Sketch for Theorem 1

$$\nu_E(W) = \inf_{\substack{\mathcal{S} \subset \text{Dom } H^{1/2} \\ 3 \leq \dim \mathcal{S} < \infty}} \inf_{\substack{P_A \in \mathcal{D}(\mathcal{S}) \\ \text{Tr}(P_A H) \leq E}} \underbrace{|\text{Tr}(P_A W)|}$$

• Let $\Pi_{\mathcal{S}}$: orthogonal projection onto \mathcal{S}

$$\& \forall X, \quad X^{\mathcal{S}} := \Pi_{\mathcal{S}} X \Pi_{\mathcal{S}}$$

$$\bullet W = W_R + iW_I$$

$$\bullet P_A \in \mathcal{D}(\mathcal{S}) \Rightarrow \text{Tr}(P_A W) = \text{Tr}(\Pi_{\mathcal{S}} P_A \Pi_{\mathcal{S}} W) = \text{Tr}(P_A W^{\mathcal{S}})$$

$$|\text{Tr}(P_A W)| = |\text{Tr}(P_A W_R^{\mathcal{S}}) + i \text{Tr}(P_A W_I^{\mathcal{S}})|$$

Proof Sketch for Theorem 1

$$V_E(W) = \inf_{\substack{\mathcal{S} \subset \text{Dom } H^{1/2} \\ 3 \leq \dim \mathcal{S} < \infty}} \inf_{\substack{P_A \in \mathcal{D}(\mathcal{S}) \\ \text{Tr}(P_A H) \leq E}} |\text{Tr}(P_A W)|$$

• For a pure state $P_A = |\psi\rangle\langle\psi|$ energy constraint

$$|\text{Tr}(P_A W)| = |\langle\psi| \underbrace{W_R^{\mathcal{S}} }_X + i \langle\psi| \underbrace{W_I^{\mathcal{S}} }_Y |\psi\rangle| \quad ; \quad \langle\psi| \underbrace{H^{\mathcal{S}} }_Z |\psi\rangle \leq E$$

s.a. matrices

$$R_3 \equiv F_3(x, y, z) = \left\{ \left(\langle\psi| X |\psi\rangle, \langle\psi| Y |\psi\rangle, \langle\psi| Z |\psi\rangle \right)^T ; |\psi\rangle \in \mathcal{S} \right\}$$

$(x, y, z)^T$

$$|\text{Tr}(P_A W)| = |x + iy| = \sqrt{x^2 + y^2}$$

$$; \text{ energy constraint } z \leq E$$

Proof Sketch for Theorem 1

$$\begin{aligned} \nu_E(W) &= \inf_{\mathcal{S} \subset \text{Dom } H^{1/2}} \inf_{\substack{P_A \in \mathcal{D}(\mathcal{S}) \\ \text{Tr}(P_A H) \leq E}} \underbrace{|\text{Tr}(P_A W)|} \end{aligned}$$

$$= \inf_{\mathcal{S} \subset \text{Dom } H^{1/2}}$$

$$\inf_{\substack{(x, y, z) \in \text{conv } \mathcal{R}_3 \\ z \leq E}} \sqrt{x^2 + y^2}$$

$$\begin{aligned} \nu_E(W) &= \inf_{\substack{|\psi\rangle \in \mathcal{H}_A \\ \langle \psi | H | \psi \rangle \leq E}} |\langle \psi | W | \psi \rangle| \end{aligned}$$

\mathcal{R}_3 is convex
[Thm: Au-Yeung Poon]

