

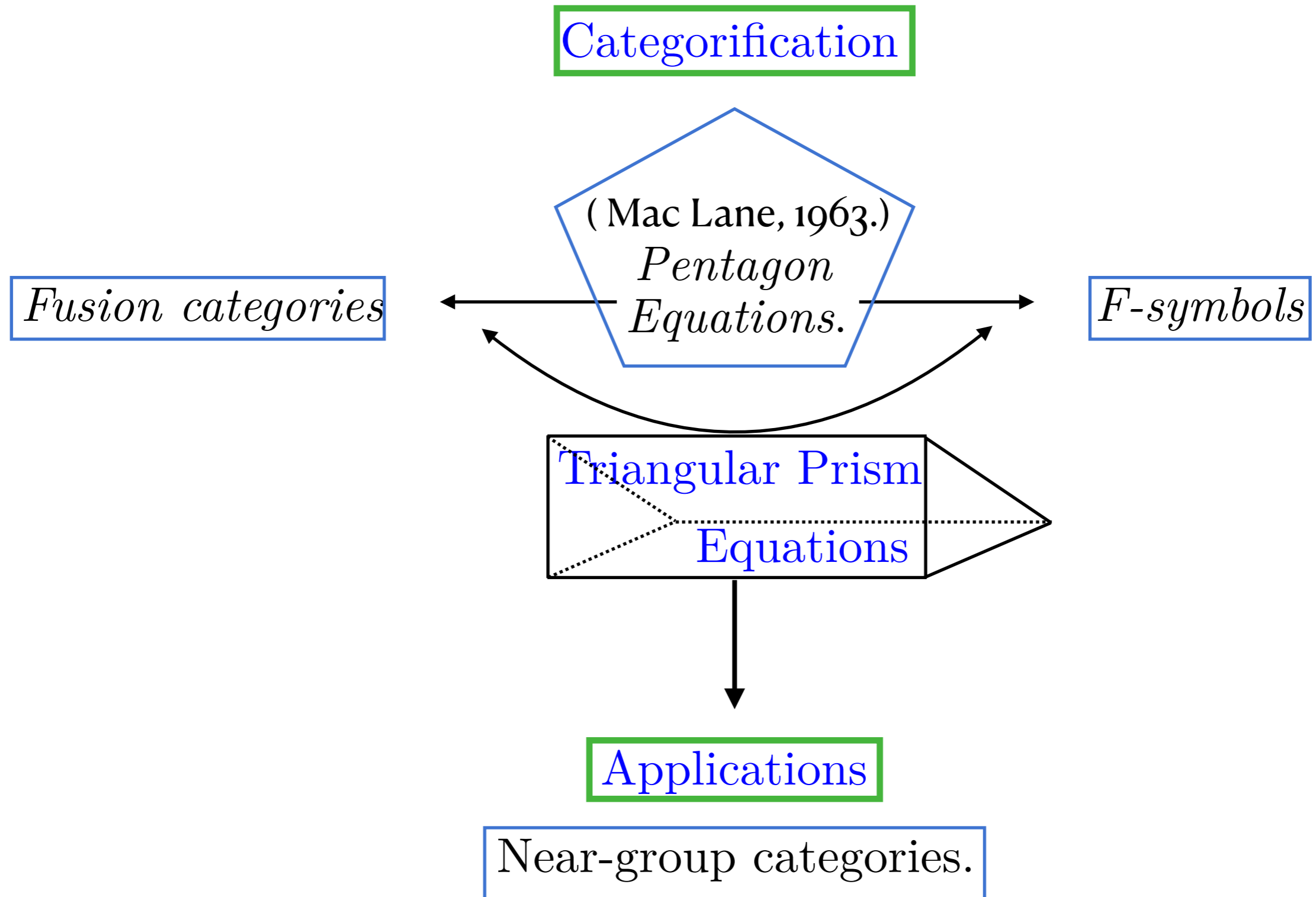
TRIANGULAR PRISM EQUATIONS AND CATEGORIFICATION

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Outline

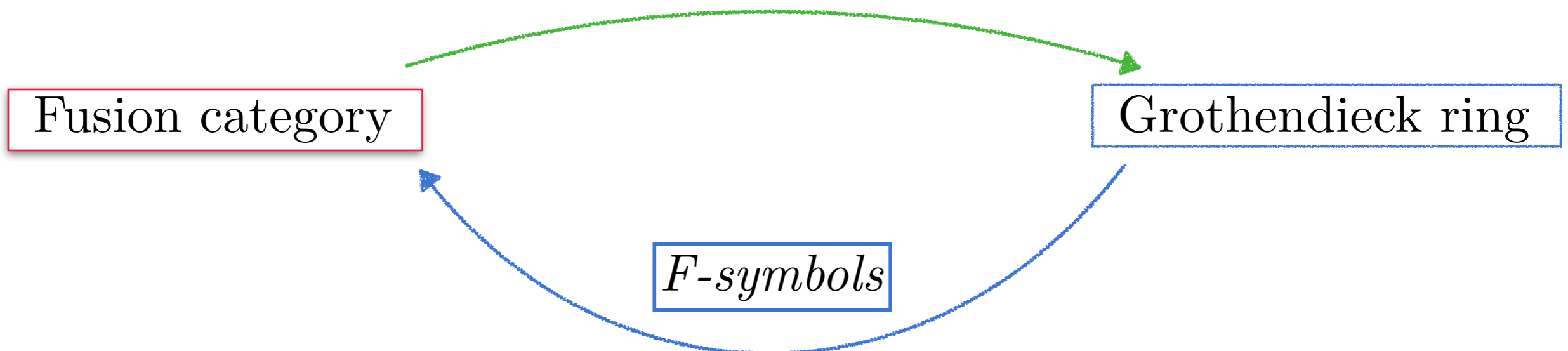


Fusion categories: a fundamental example

- **Fusion categories** have been a central subject in mathematics, and have tremendous important applications in physics: *anyon*, *topological quantum computation*, and *topological orders*.
- **Fusion category**: Let G be a finite group. The representation category $Rep(G)$ has the following data:
 - Simple objects: $\{V_i : V_i \text{ is a irreducible representation}\}$
 - Morphism: G -invariant map between representations.
 - More structures: Direct sum and decomposition of representation; Tensor product of representations and G -invariant maps

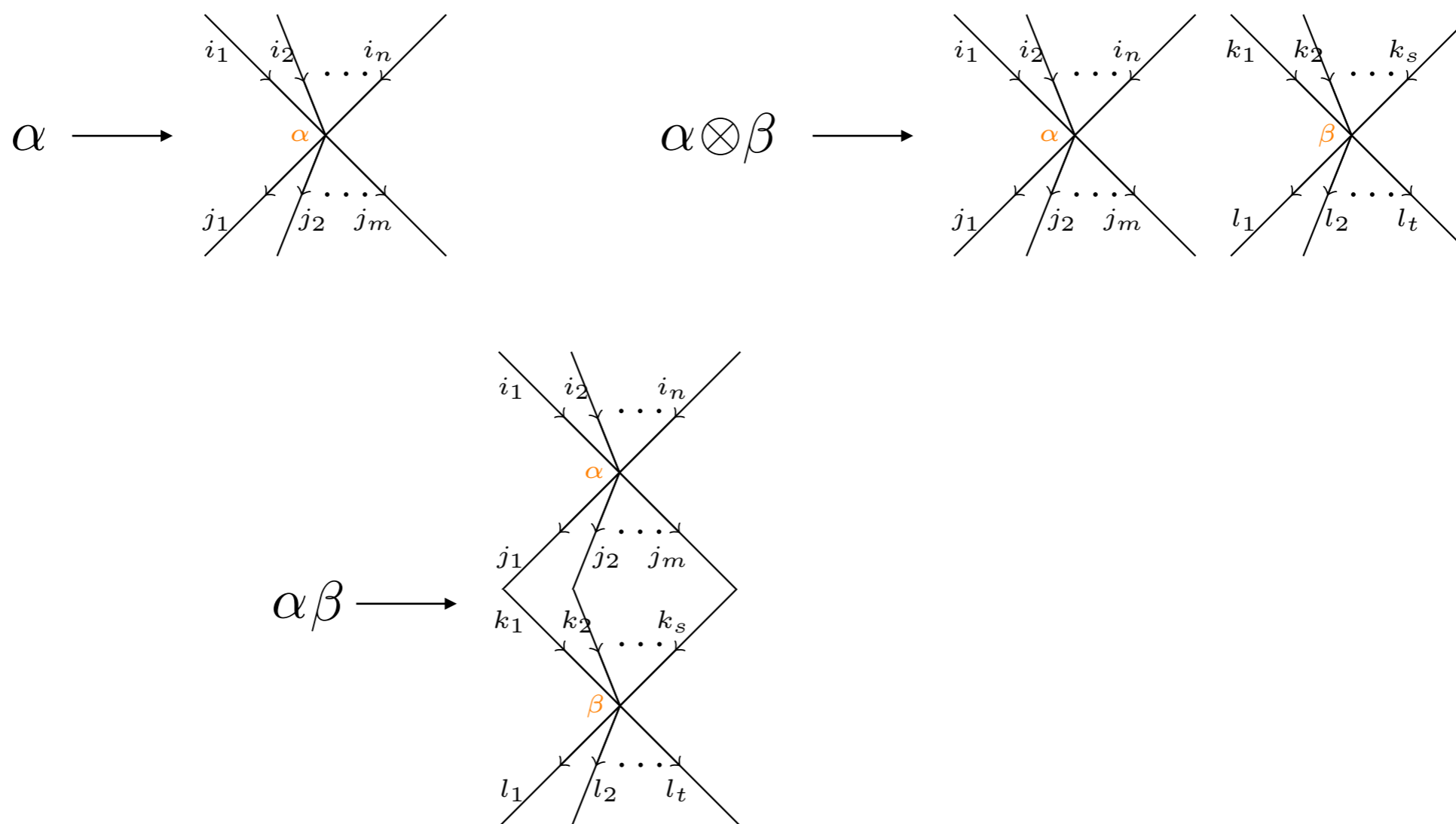
Fusion categories: a formal description

- Some basic data of a fusion category \mathcal{C} :
 - Simple objects: $I = \{x_0, x_1, x_2, \dots, x_n\}$;
 - Fusion rules: $x_i \otimes x_j = \bigoplus_k N_{i,j}^k x_k$. $N_{i,j}^k \in \mathbb{N}$
 - Dual object: an involution $*$: $I \rightarrow I$ such that x_{i^*} is the unique simple object with $N_{i,i^*}^0 = 1$ for every i .



Key data: F-symbols (generalization of 6j-symbols)

- A picture language: Let \mathcal{C} be a fusion category.
 - ▶ Let α be a morphism from $x_{i_1} \otimes x_{i_2} \cdots \otimes x_{i_n}$ to $x_{j_1} \otimes x_{j_2} \cdots \otimes x_{j_m}$.
Let β also be a morphism in \mathcal{C} .



Key data: F-symbols (generalization of 6j-symbols)

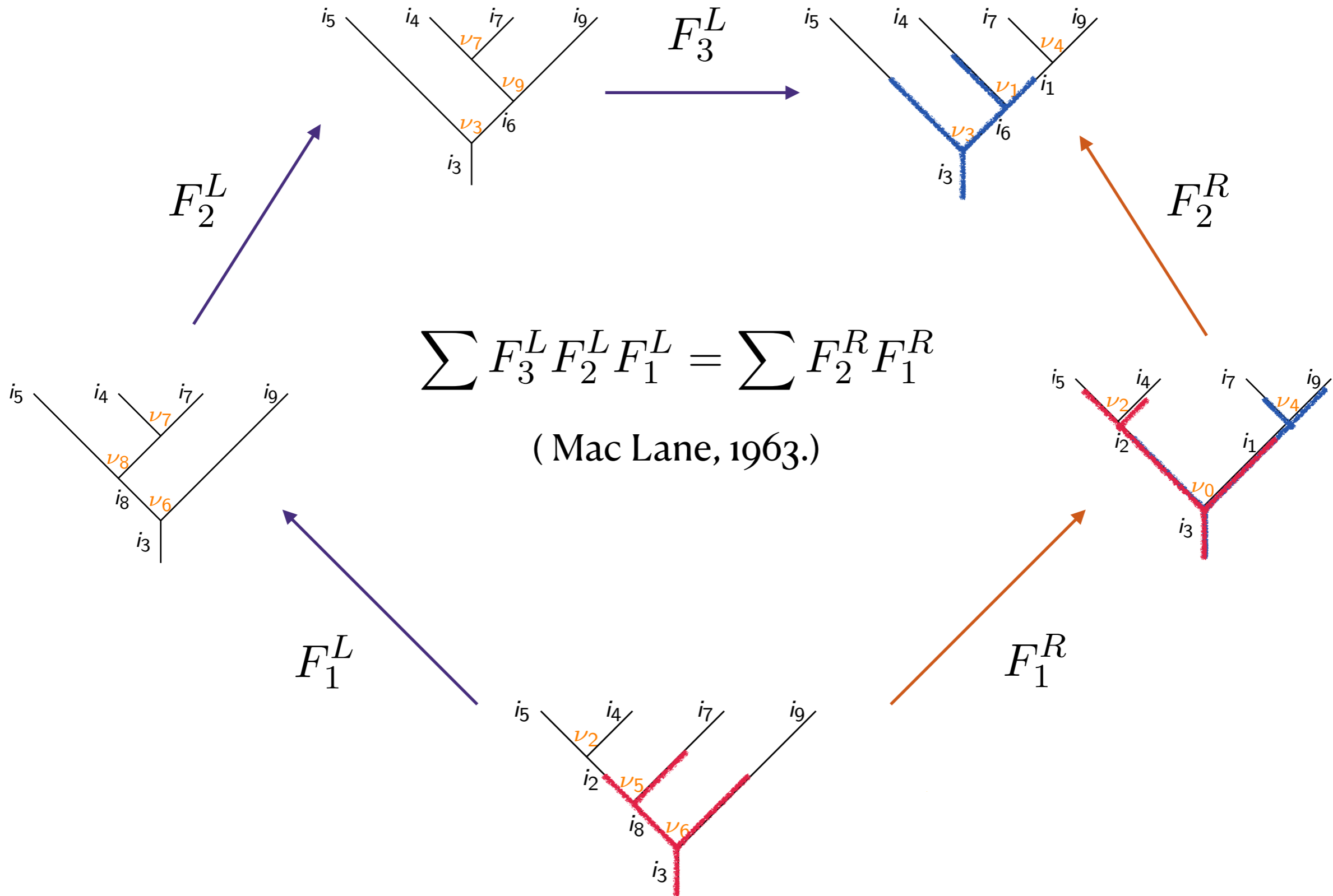
- A F -symbol $\left(\begin{array}{ccc|cc} i_1 & i_2 & i_3 & \nu_1 & \nu_2 \\ i_4 & i_5 & i_6 & \nu_3 & \nu_4 \end{array} \right)$ is defined by

$$\begin{array}{c} i_1 \quad i_2 \quad i_4 \\ \diagdown \quad \diagup \quad / \\ \nu_1 \quad \nu_2 \\ \diagup \quad \diagdown \quad | \\ i_3 \quad i_5 \\ \nu_3 \quad \nu_4 \end{array} = \sum_{i_6} \sum_{\nu_3, \nu_4} \left(\begin{array}{ccc|cc} i_1 & i_2 & i_3 & \nu_1 & \nu_2 \\ i_4 & i_5 & i_6 & \nu_3 & \nu_4 \end{array} \right) \begin{array}{c} i_1 \quad i_2 \quad i_4 \\ \diagdown \quad \diagup \quad / \\ \nu_3 \quad \nu_4 \\ \diagup \quad \diagdown \quad | \\ i_5 \end{array} \cdot$$

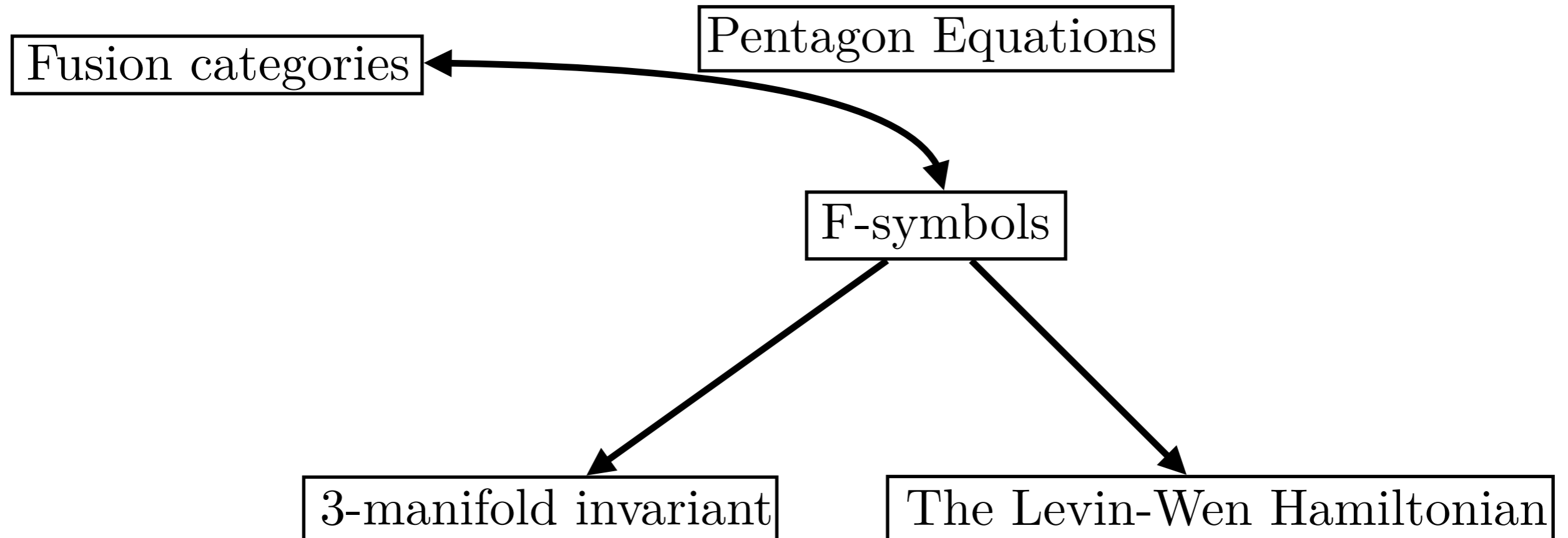
- F-move:

$$\begin{array}{c} i_1 \quad i_2 \quad i_4 \\ \diagdown \quad \diagup \quad / \\ \nu_1 \quad \nu_2 \\ \diagup \quad \diagdown \quad | \\ i_3 \quad i_5 \\ \nu_3 \quad \nu_4 \end{array} \xrightarrow{\left(\begin{array}{ccc|cc} i_1 & i_2 & i_3 & \nu_1 & \nu_2 \\ i_4 & i_5 & i_6 & \nu_3 & \nu_4 \end{array} \right)} \begin{array}{c} i_1 \quad i_2 \quad i_4 \\ \diagdown \quad \diagup \quad / \\ \nu_3 \quad \nu_4 \\ \diagup \quad \diagdown \quad | \\ i_5 \end{array} \cdot$$

Consistency: Pentagon Equations

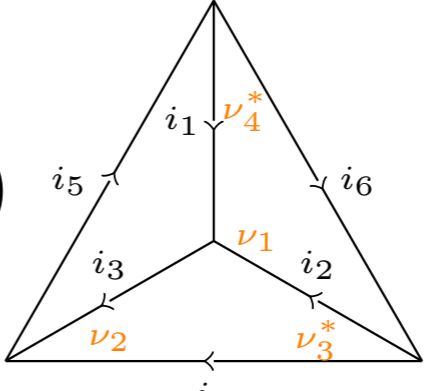


F-symbols and Pentagon equations

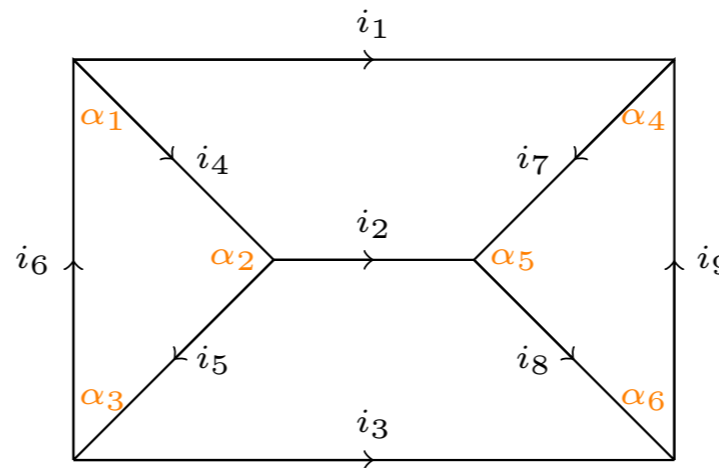


Main idea: Triangular Prism Equations

- First we express F-symbols as tetrahedrons:

$$\left(\begin{array}{ccc|cc} i_1 & i_2 & i_3 & \nu_1 & \nu_2 \\ i_4 & i_5 & i_6 & \nu_3 & \nu_4 \end{array} \right) = d(x_{i_6}) \cdot \text{tetrahedron} \cdot$$


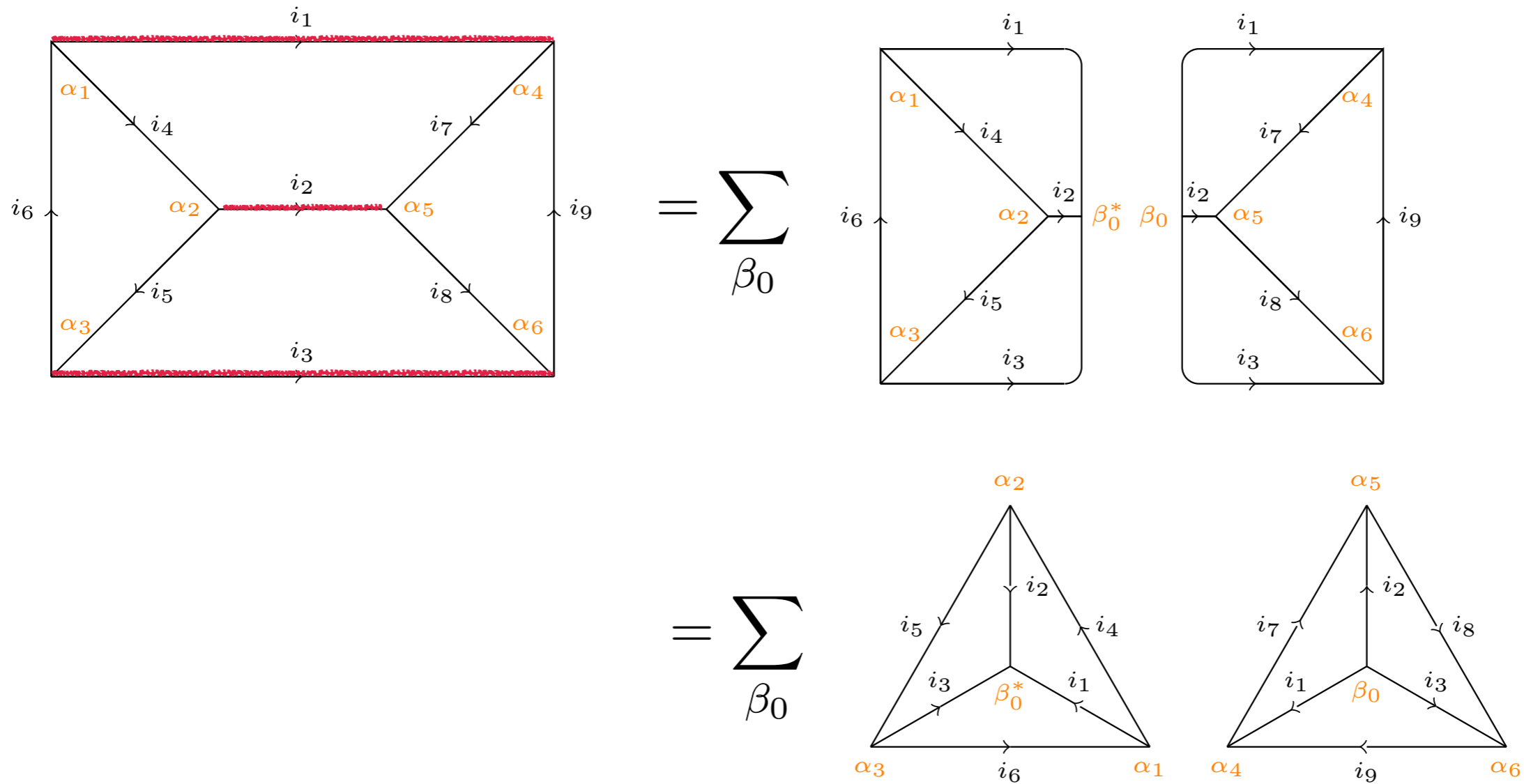
- Given (i_1, \dots, i_9) and $(\alpha_1, \dots, \alpha_6)$, there are two distinct approaches to evaluate the **triangular prism digram**



One can obtain an equation by equating the two values, which we call a *Triangular Prism Equation* (TPE).

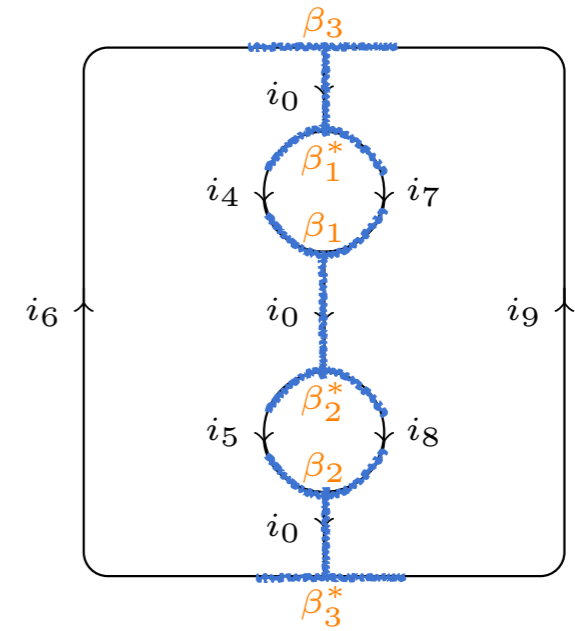
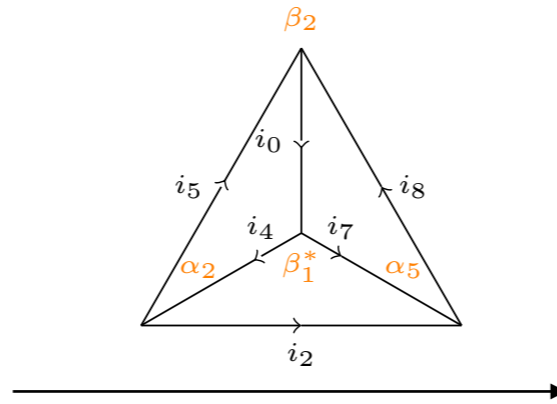
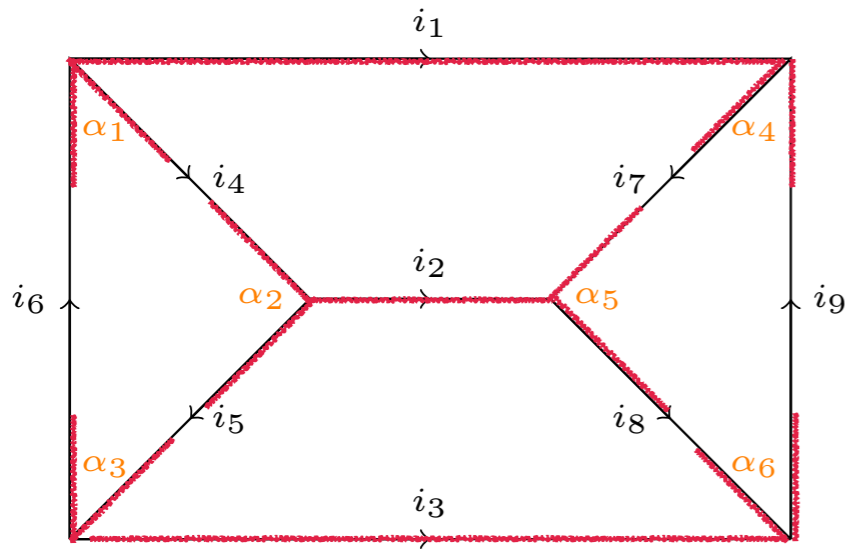
Triangular Prism Equations

► One evaluation is given as



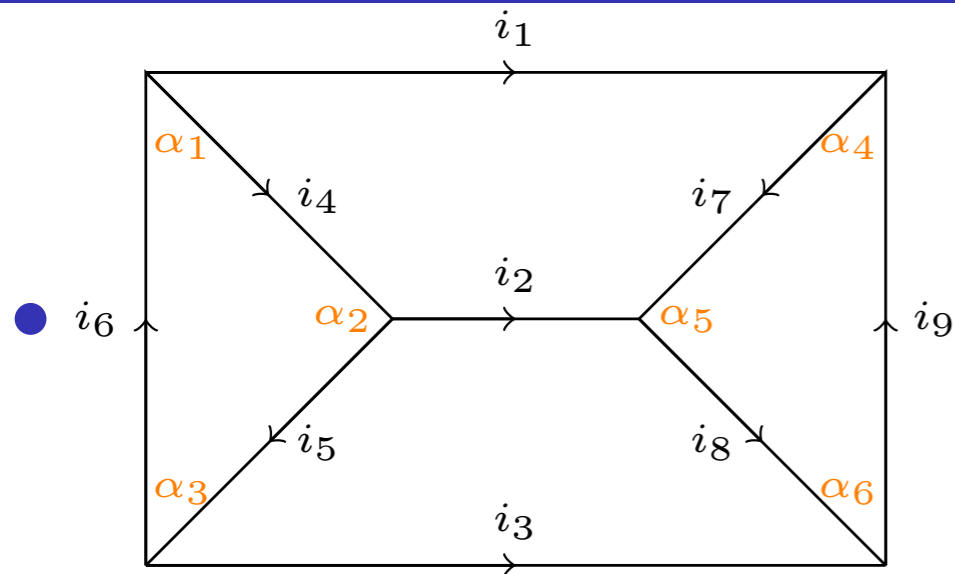
Triangular Prism Equations

► The other evaluation is given as

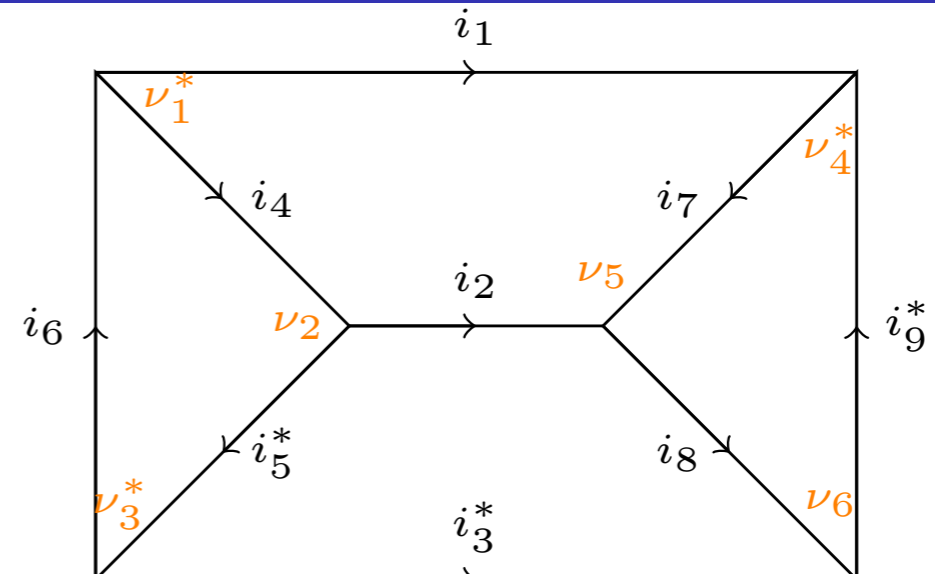


$$= \sum_{i_0 \in I_S} \sum_{\beta_i} d(x_{i_0})$$

Triangular Prism Equations



(Standard TPE)



(PE)

Theorem (Liu, Palcoux, R. 2020)

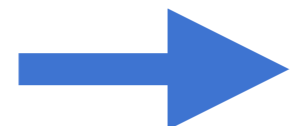
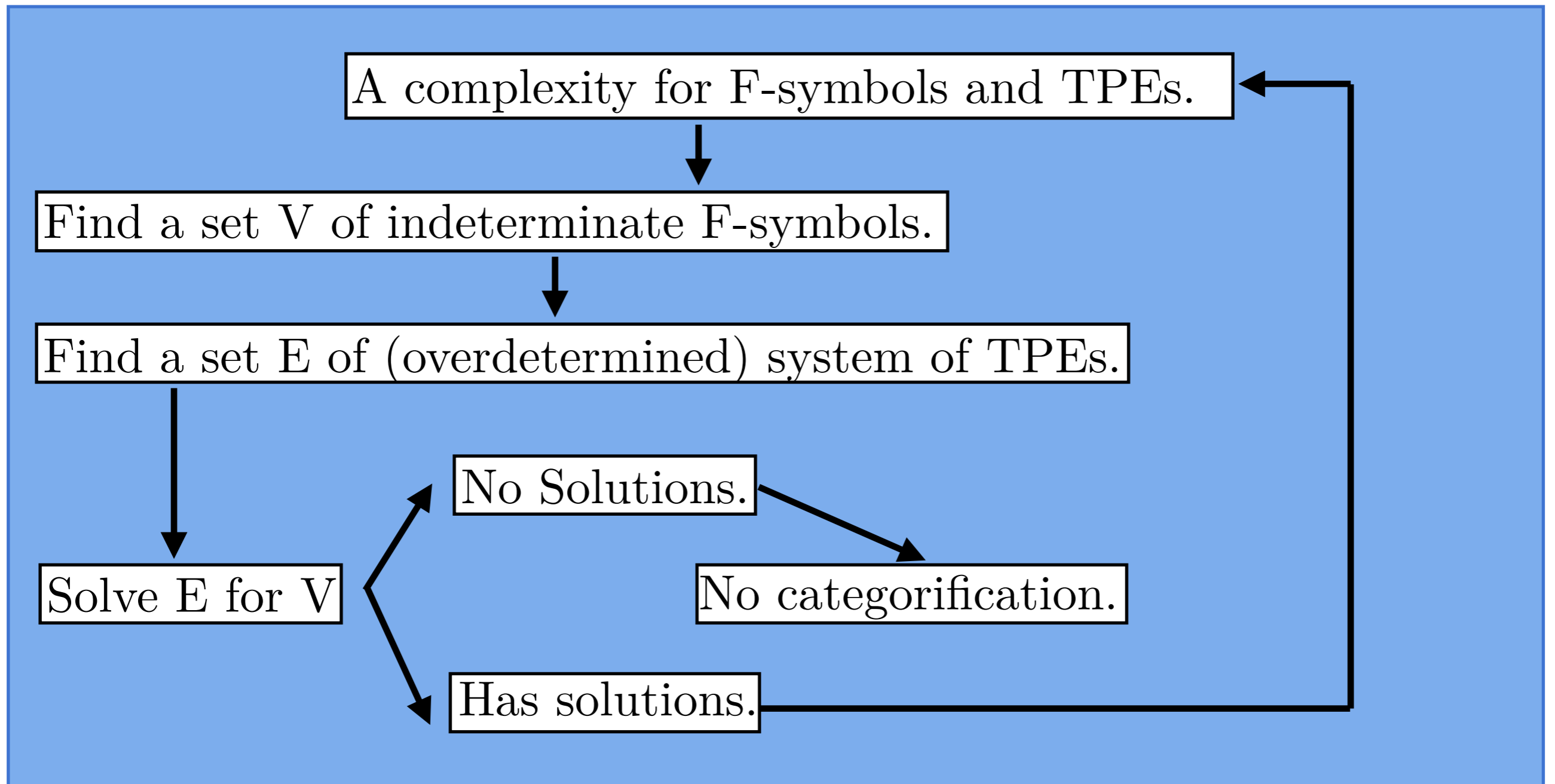
Let \mathcal{C} be a fusion category. The triangular prism equations given by (PE) can be identified as pentagon equations.

Theorem (Liu, Palcoux, R. 2020)

Assume that F -symbols are multilinear functions on the morphisms. Then solving the triangular prism equations from (Standard TPE) is equivalent to solving the ones from (PE).

Localization

We propose a **localization** strategy to solve for F-symbols:



Fusion category

Near-group category

Definition (Near-group category, Izumi, [Izu93])

Let G be a finite abelian group. We say \mathcal{C} is a near-group category $G + m$ if it has simple objects $\{\rho, g : g \in G\}$ with fusion rules:

$$\begin{aligned}\rho g &= g\rho = \rho \\ \rho^2 &= m\rho + \sum_{g \in G} g\end{aligned}$$

Many fusion categories arises as near-group categories. For instance:

- Tambara-Yamagami categories as $G + 0$.
- The $D_5^{(1)}$ subfactor and the module category of group S_3 and A_4 as $\mathbb{Z}_2 + 1, \mathbb{Z}_2 + 1, \mathbb{Z}_3 + 2$.
- The A_4, E_6 and Izumi-Xu subfactors as $\mathbb{Z}_n + n$ for $n = 1, 2, 3$.

Izumi's equations

The hypothetical family of Izumi's: $G + n$, where $n = |G|$.

Theorem (Izumi, [Izu93, Izu01, I⁺17])

Let χ be a non-trivial bicharacter. Consider two functions $a : G \rightarrow \mathbb{T}$, $b : G \rightarrow \mathbb{C}$ and $c \in \mathbb{T}$ satisfying the following conditions:

$$a(0) = 1; \quad a(g) = a(-g); \quad a(g+h)\chi(g,h) = a(g)a(h), \quad g, h \in G,$$

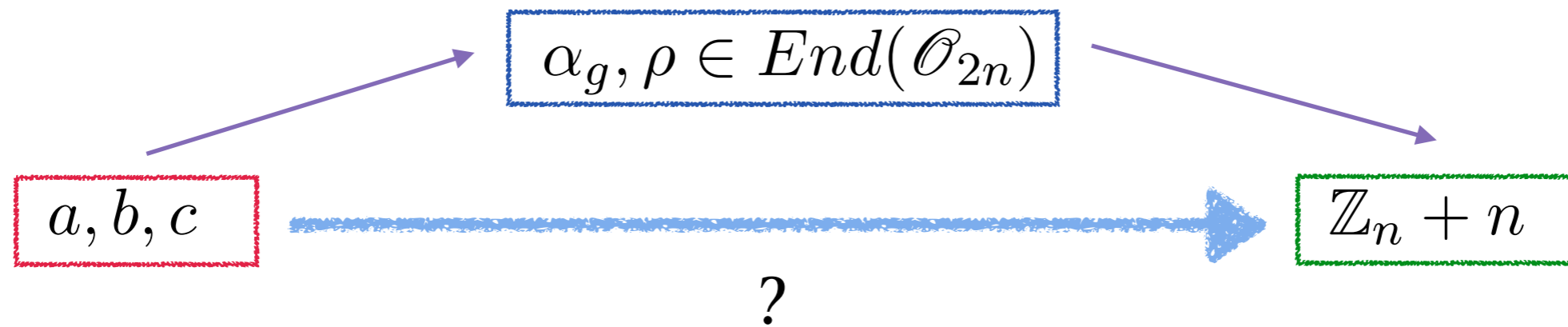
$$a(g)b(-g) = \overline{b(g)},$$

$$\frac{c\sqrt{n}}{d} + \sum_{b \in G} b(g) = 0,$$

$$\sum_{g \in G} b(g+h)\overline{b(g)} = \delta_{h,0} - 1/d$$

Izumi's equations

Let \mathcal{O}_{2n} be the Cuntz algebra generated by $\{S_g, T_h : g, h \in G\}$.



Theorem (Izumi, [Izu93, Izu01, I⁺17])

Mod \dots
 Does there exist a direct construction from Izumi's equations to the near-group category?

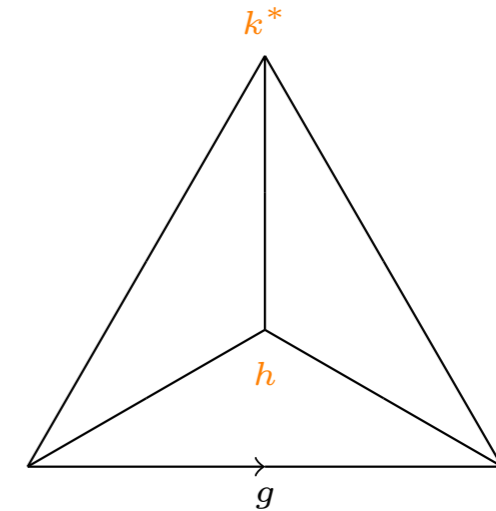
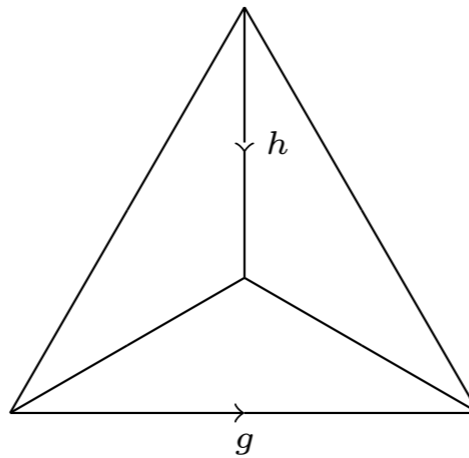
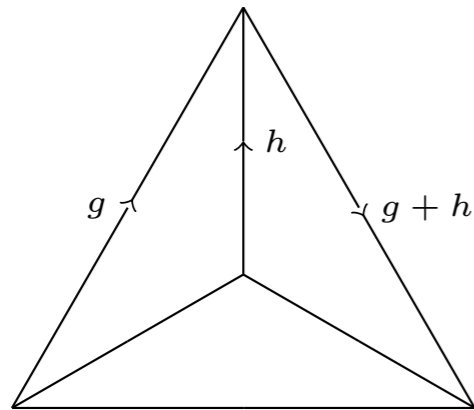
$$b(g) = cb(g)$$

$$\sum_{g \in G} b(g+h)b(g+k)\overline{b(g)} = \overline{\chi(h,k)}b(h)b(k) - \frac{c}{d\sqrt{n}}$$

Then there exists a near-group category $G + n$.

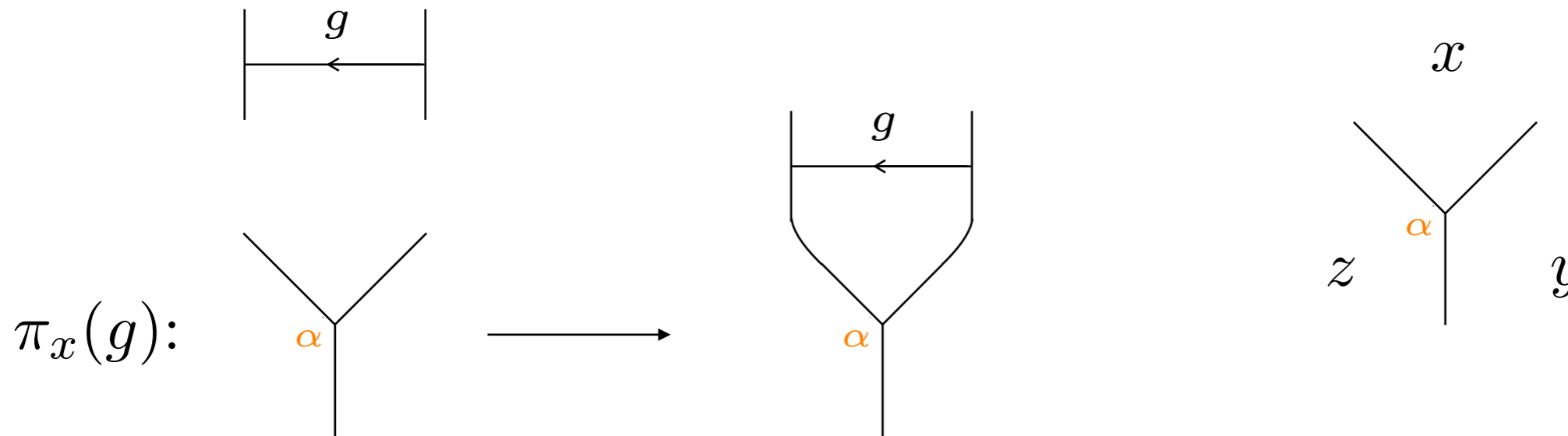
Localization

The complexity $\#F$ of F-symbols, and the complexity $\#E$ of TPEs are the number of invertible objects:



The space $\text{hom}(\rho \otimes \rho, \rho)$

- Let α be a morphism in $\text{hom}(\rho \otimes \rho, \rho)$.



Lemma

There exists unitary representations π_x, π_y, π_z of the group G on $\text{hom}(\rho \otimes \rho, \rho)$. Moreover, we have the commutation relations as below,

$$\pi_z(g)\pi_x(h) = \chi(g, h)\pi_x(h)\pi_z(g);$$

$$\pi_x(g)\pi_y(h) = \chi(g, h)\pi_y(h)\pi_x(g);$$

$$\pi_y(h)\pi_z(g) = \chi(g, h)\pi_z(g)\pi_y(h).$$

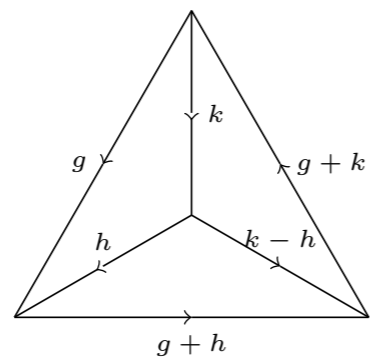
The space $\text{hom}(\rho \otimes \rho, \rho)$

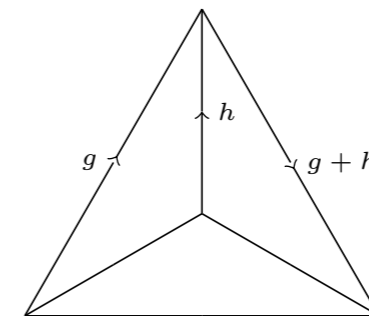
Theorem (Pauli actions)

Let $G = \mathbb{Z}_n$ and $\chi(g, h) = q^{gh}$ be a nontrivial bicharacter. Then there exists a basis $\{v_h : h \in G\}$ such that $\pi_x(1)$, $\pi_y(1)$ and $\pi_z(1)$ are exactly the Pauli matrices x, y, z on the n -dimensional space.

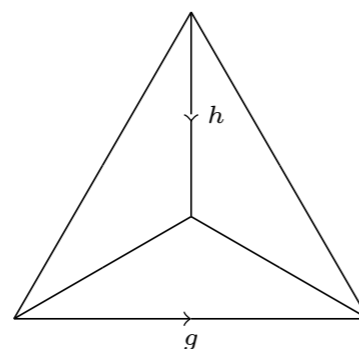
F-symbols with invertible objects

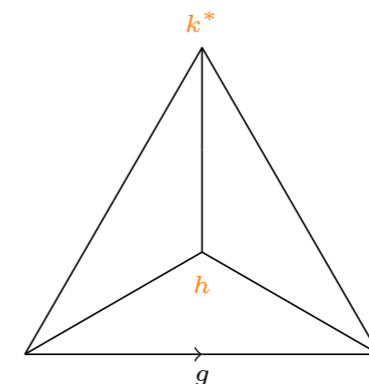
- 3 or more invertible objects:


 $= 1;$


 $= d_\rho.$

- 2 and 1 invertible objects:


 $= \chi(g, h)d_\rho;$


 $= \chi(g, h)\delta_{h,k}.$

a, c

F-symbols with no invertible objects

Lemma

There exists a function $\beta : G \rightarrow \mathbb{C}$ such that

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} = \beta(g) \sum_{h \in G} \left(\overline{\chi(h, g)} \begin{array}{c} \diagup \\ \diagdown \\ h^* \\ \diagup \\ \diagdown \\ h \end{array} \right) + \frac{\delta_{0, g}}{d_\rho^2} \left(\sum_{h \in G} \begin{array}{c} \diagup \\ \diagdown \\ h \\ \diagup \\ \diagdown \end{array} \right).$$

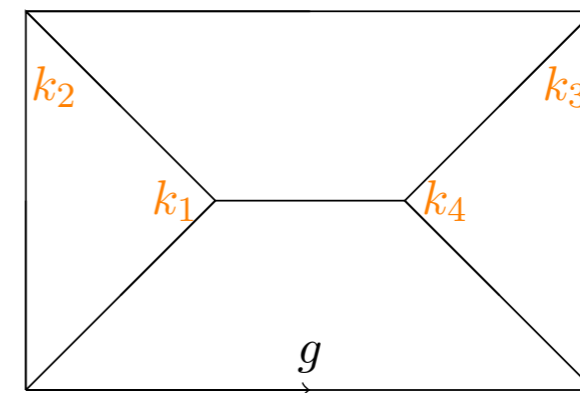
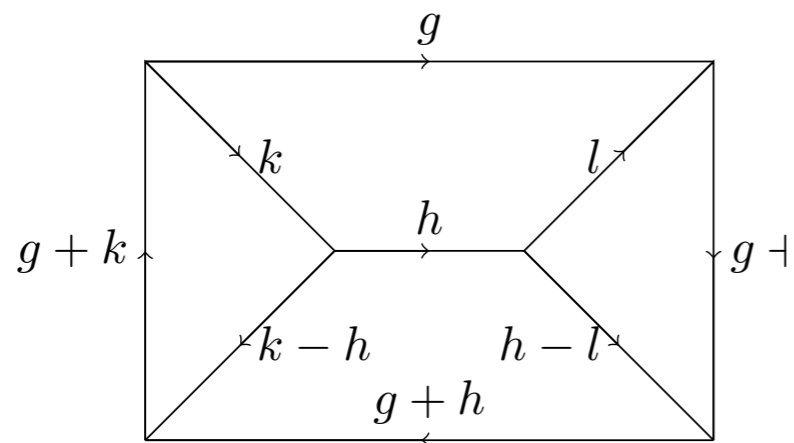
- The F-symbols with no invertible objects:

$$\begin{array}{c} \diagup \\ \diagdown \\ h^* \\ \diagup \\ \diagdown \\ h-g \\ \diagup \\ \diagdown \\ g \\ \diagup \\ \diagdown \\ k-g \end{array} = \frac{a(h-g)a(k-g)}{a(h)a(k)} \overline{\chi(h, k)} \beta(k) \frac{1}{d_\rho}$$

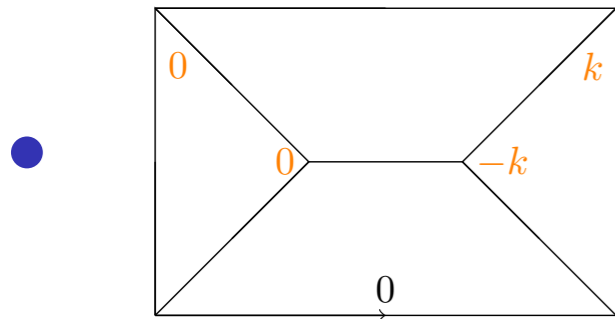
Localization

The complexity $\#F$ of F-symbols, and the complexity $\#E$ of TPEs are the number of invertible objects:

$\#E^{-1}$	>2	TPEs			1	0
F-sym Bulk		Completely determined by β.				
Boundary						

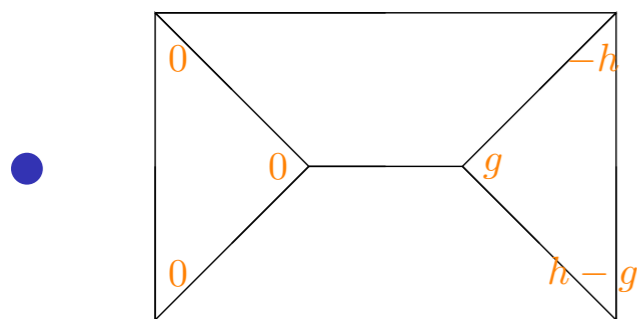


Triangular prism equations



:

$$\beta(g)\beta(-g) + \delta_{g,0} \frac{1}{d_\rho} = \frac{a(g)}{|G|}$$



:

$$\frac{a(h-g)}{|G|} \sum_{k \in G} \frac{\chi(h,k)}{a(k)} \frac{\beta(g-k)\beta(k)}{a(k)}$$

$$= \beta(-h)\beta(g)\beta(h-g) + \delta_{h,0}\delta_{g,0} \frac{1}{d_\rho}$$

Near-group category $\mathbb{Z}_n + n$

Theorem (Z. Liu, S. Palcoux, R. 2020)

Let G be \mathbb{Z}_n . Let χ, ζ, a be as above. Then the near-group category $\mathbb{Z}_n + n$ is completely determined by a complex number $c \in \mathbb{C}$ and a function $\beta : G \rightarrow \mathbb{C}$ satisfying the triangular prism equations [STPE]:

$$c^3 \hat{a}(0) = \frac{1}{n}$$

$$\beta(g)\beta(-g) + \delta_{g,0} \frac{1}{d_\rho} = \frac{a(g)}{|G|}$$

$$\frac{a(h-g)}{|G|} \sum_{k \in G} \frac{\overline{\chi(h,k)} \beta(g-k) \beta(k)}{a(k)} = \beta(-h)\beta(g)\beta(h-g) + \delta_{h,0} \delta_{g,0} \frac{1}{d_\rho}$$

Near-group category $\mathbb{Z}_n + n$

- ▶ Izumi's Equations and $[STPE]$ are both Triangular Prism Equations from a certain configuration.
- ▶ $[STPE]$ has a better performance in the aspect of computation.

Theorem (Evans and Gannon, 2015)

Let G be an abelian group with $|G| \leq 12$. There are numerical solutions to Izumi's equations for the near-group category $G + |G|$.

For $n = 7$, the Groebner basis are given as below,

$$v_1^2 + (p^4 + p^2)v_1 \frac{1}{d} + (2/7p^5 + 1/7p^4 - 2/7p^3 + 1/7p^2 + 2/7p)v_1 \\ + (1/7p^5 - 1/7p^4 + 1/7p^3 - 1/7p^2 + 1/7p - 1/7),$$

$$(\frac{1}{d})^2 + \frac{1}{d} - 1/7,$$

$$v_6 + v_1 + (p^4 + p^2)\frac{1}{d} + (2/7p^5 + 1/7p^4 - 2/7p^3 + 1/7p^2 + 2/7p),$$

$$v_5 + (p^4 - 2p^3 + 3p^2 - 2p + 1)v_1 + (p^3 - p^2 + p - 1)\frac{1}{d} + (-3/7p^5 + 2/7p^3 - 3/7p^2 + 3/7p - 2/7),$$

$$v_4 + (-p^4 + 2p^3 - p^2)v_1 + (p^4 - p^3 + p^2 - p)\frac{1}{d} + (-1/7p^5 + 4/7p^4 - 2/7p^3 + 2/7p^2 - 4/7p + 1/7),$$

$$v_3 + (p^4 - 2p^3 + p^2)v_1 + (p^3 - p^2)\frac{1}{d} + (3/7p^5 - 5/7p^4 + 6/7p^3 - 6/7p^2 + 5/7p - 3/7),$$

$$v_2 + (-p^4 + 2p^3 - 3p^2 + 2p - 1)v_1 + (-p^3 + 2p^2 - 2p + 1)\frac{1}{d} + (-1/7p^5 - 4/7p^3 + 6/7p^2 - 6/7p + 4/7),$$

$$v_0 + \frac{1}{d}.$$

$n.$

Summary

- ▶ Propose **triangular prism equations** as a new method to solve pentagon equations for F-symbols.
- ▶ Provide a categorical approach to construct near-group categories. Produce *non-unitary* near-group categories.
- ▶ Provide a complete localization strategy for \mathcal{F}_{210} .

Thank you for your attention!