Reconstructing (1+1)-χCFTs from (2+1)-TQFTs

Realizing TQFTs/MTCs/Anyon models by χCFTs/VOAs/Edge physics

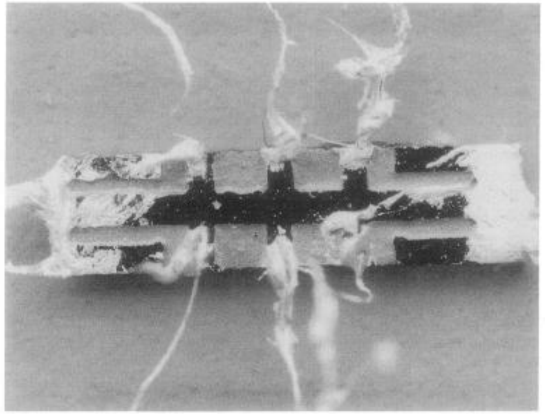
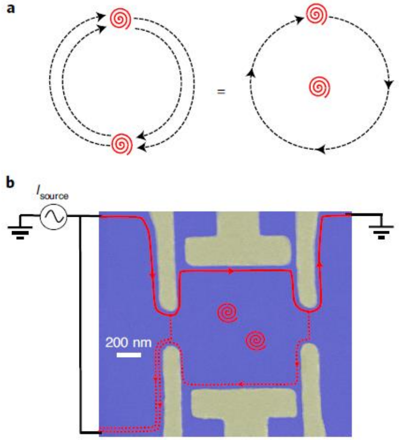


Figure 8. Photograph of a GaAs/AlGaAs sample. The size is about 6 x 1.5 mm. Black area (in reality mirror-like but reflecting the black camera) is the original surface above the 2DES. Gray areas have been scratched away to confine the current path to the center of the sample. White areas are indium blotches used to make contact to the 2DES. Gold wires are attached. Specimens like this one, prepared with little attention to exact dimension nor to tidiness, show quantization of the Hall resistance to an accuracy of a few 10 parts in a billion. The specimen shown is the sample in which the fractional quantum Hall effect (FQHE) was discovered in 1981.



Zhenghan Wang

Microsoft Station Q and UC at Santa Barbara

Harvard, October 6, 2020

Mathematical Quantum Field Theory

- Constructive
- Axiomatic
- Algebraic
- Functional
- …
- **Topological quantum field theory (TQFT)**
- **Chiral conformal field theory (χCFT)**
- …
- (Aspects beyond perturbation and geometry)

Another Motivation:
Quantum Churn-Turing Thesis

Simulation of QFTs efficiently by quantum computers

Inspired by the bulk-edge connection for 2D topological phases of matter, study χCFTs via (2+1)-TQFTs. χCFTs and (2+1)-TQFTs share:

Modular tensor category (MTC)

Classical and quantum conformal field theory
G Moore, N Seiberg - Communications in Mathematical Physics, 1989

(2+1)-TQFT=MTC (Turaev+**)**

(1+1)-χ**CFT=?MTC +** ϵ (Dream)

All CFTs and MTCs are unitary

Why MTC Good Starting?

- **Ubiquitous and abundance:** Quantum groups, von Neuman algebra, low dim topology, M-theory, condensed matter, quantum computing,…
- **Emerging structure theory:** first classify up to Witt equivalence (gapped boundaries), then within each class, reduced to seeds (condensation and gauging), then enumerate for each rank (number theory and representation of modular groups SL(2,Z)),…
- **…**

Anyon model=(2+1)-TQFT=Modular Tensor Category (MTC), all unitary.

Motivations
for Abelian Anyon Models: Besides as examples of reconstruction

- Fractional Quantum Hall States
- Topological Insulators
- Quantum cellular automata
- Fracton physics
- Exotic CFTs: Lattices with exotic symmetries related to potential generalized Moonshine??? Probably tip of an iceberg---lots of exotic anyon models
- …

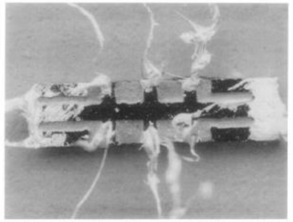
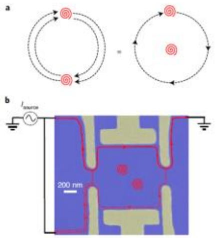


Figure 8. Photograph of a GaAs/AlGaAs sample. The size is about 6 x 1.5 mm. Black area (in reality mirror-like but reflecting the black camera) is the original surface above the 2DES. Gray areas have been scratched away to confine the current path to the center of the sample. White areas are indium blotches used to make contact to the 2DES. Gold wires are attached. Specimens like this one, prepared with little attention to exact dimension nor to tidiness, show quantization of the Hall resistance to an accuracy of a few 10 parts in a billion. The specimen shown is the sample in which the fractional quantum Hall effect (FQHE) was discovered in 1981.



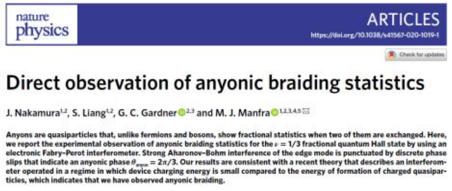
nature > news > article

NEWS · 03 JULY 2020

Welcome anyons! Physicists find best evidence yet for long-sought 2D structures

The 'quasiparticles' defy the categories of ordinary particles and herald a potential way to build quantum computers.

Davide Castelvecchi



nature physics ARTICLES

https://doi.org/10.1038/s41567-020-1019-1

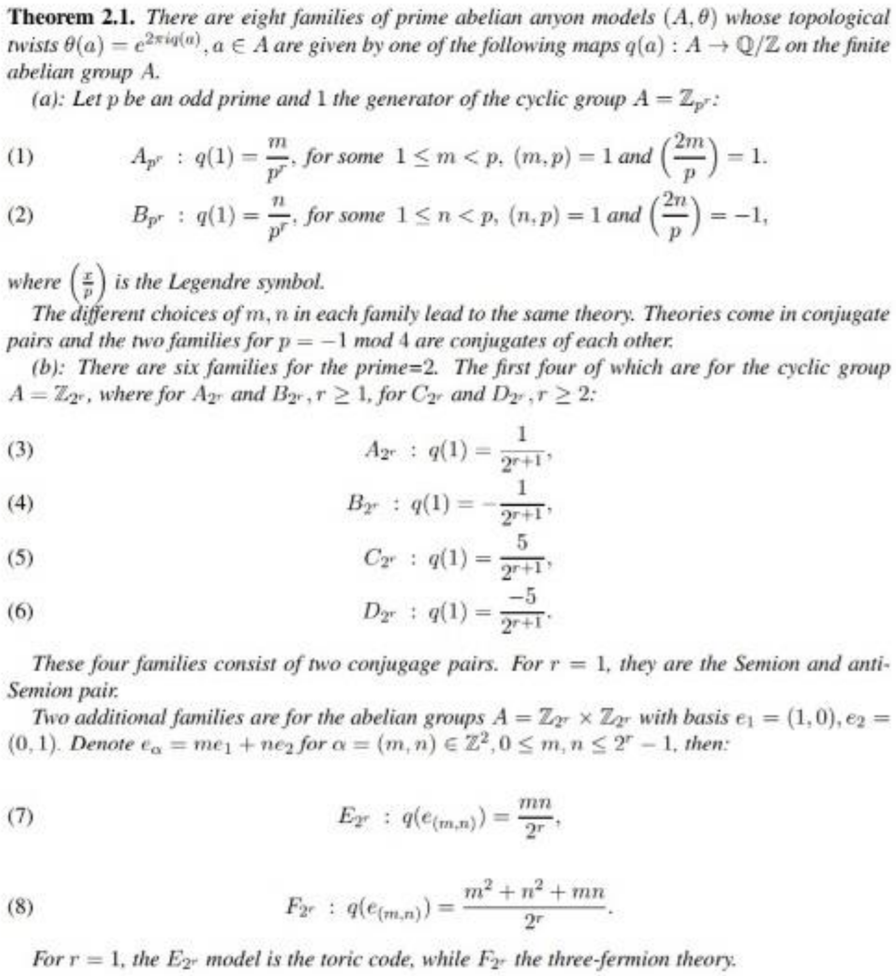
Check for updates

Direct observation of anyonic braiding statistics

J. Nakamura[1,2], S. Liang[1,2], G. C. Gardner[2,3] and M. J. Manfra[1,2,3,4,5]

Anyons are quasiparticles that, unlike fermions and bosons, show fractional statistics when two of them are exchanged. Here, we report the experimental observation of anyonic braiding statistics for the $\nu = 1/3$ fractional quantum Hall state by using an electronic Fabry–Perot interferometer. Strong Aharonov–Bohm interference of the edge mode is punctuated by discrete phase slips that indicate an anyonic phase $\theta_{anyon} = 2\pi/3$. Our results are consistent with a recent theory that describes an interferometer operated in a regime in which device charging energy is small compared to the energy of formation of charged quasiparticles, which indicates that we have observed anyonic braiding.

Prime Abelian Anyon Models



Theorem 2.1. *There are eight families of prime abelian anyon models* (A, θ) *whose topological twists* $\theta(a) = e^{2\pi i q(a)}, a \in A$ *are given by one of the following maps* $q(a) : A \to \mathbb{Q}/\mathbb{Z}$ *on the finite abelian group* A.

(a): Let p *be an odd prime and* 1 *the generator of the cyclic group* $A = \mathbb{Z}_{p^r}$*:*

$$A_{p^r} \;:\; q(1) = \frac{m}{p^r}, \text{ for some } 1 \leq m < p,\ (m,p) = 1 \text{ and } \left(\frac{2m}{p}\right) = 1. \tag{1}$$

$$B_{p^r} \;:\; q(1) = \frac{n}{p^r}, \text{ for some } 1 \leq n < p,\ (n,p) = 1 \text{ and } \left(\frac{2n}{p}\right) = -1, \tag{2}$$

where $\left(\frac{x}{p}\right)$ *is the Legendre symbol.*

The different choices of m, n *in each family lead to the same theory. Theories come in conjugate pairs and the two families for* $p = -1 \bmod 4$ *are conjugates of each other.*

(b): There are six families for the prime=2. The first four of which are for the cyclic group $A = \mathbb{Z}_{2^r}$*, where for* A_{2^r} *and* $B_{2^r}, r \geq 1$*, for* C_{2^r} *and* $D_{2^r}, r \geq 2$*:*

$$A_{2^r} \;:\; q(1) = \frac{1}{2^{r+1}}, \tag{3}$$

$$B_{2^r} \;:\; q(1) = -\frac{1}{2^{r+1}}, \tag{4}$$

$$C_{2^r} \;:\; q(1) = \frac{5}{2^{r+1}}, \tag{5}$$

$$D_{2^r} \;:\; q(1) = \frac{-5}{2^{r+1}}. \tag{6}$$

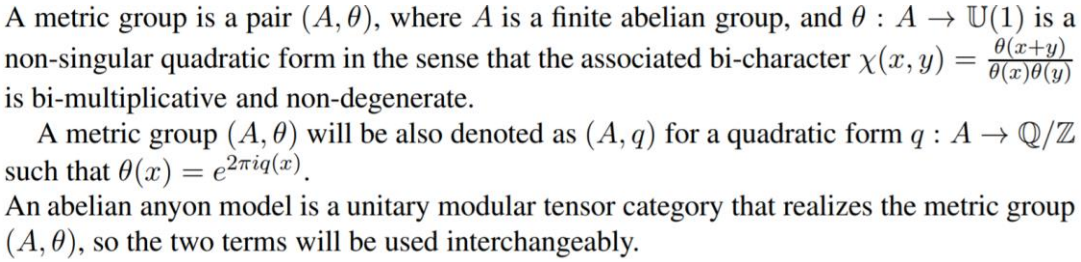
These four families consist of two conjugage pairs. For $r = 1$*, they are the Semion and anti-Semion pair.*

Two additional families are for the abelian groups $A = \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$ *with basis* $e_1 = (1,0), e_2 = (0,1)$*. Denote* $e_\alpha = me_1 + ne_2$ *for* $\alpha = (m,n) \in \mathbb{Z}^2, 0 \leq m, n \leq 2^r - 1$*, then:*

$$E_{2^r} \;:\; q(e_{(m,n)}) = \frac{mn}{2^r}, \tag{7}$$

$$F_{2^r} \;:\; q(e_{(m,n)}) = \frac{m^2 + n^2 + mn}{2^r}. \tag{8}$$

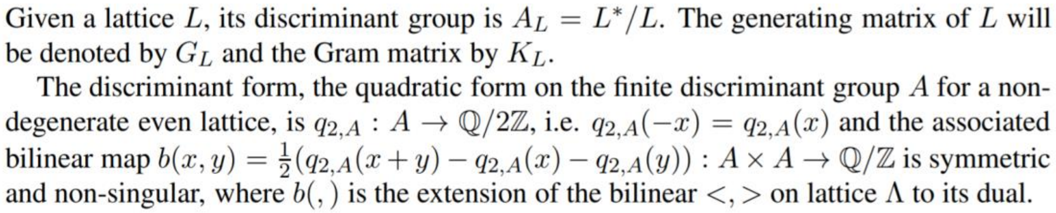
For $r = 1$*, the* E_{2^r} *model is the toric code, while* F_{2^r} *the three-fermion theory.*



A metric group is a pair (A, θ), where A is a finite abelian group, and $\theta : A \to \mathbb{U}(1)$ is a non-singular quadratic form in the sense that the associated bi-character $\chi(x,y) = \frac{\theta(x+y)}{\theta(x)\theta(y)}$ is bi-multiplicative and non-degenerate.

A metric group (A, θ) will be also denoted as (A, q) for a quadratic form $q : A \to \mathbb{Q}/\mathbb{Z}$ such that $\theta(x) = e^{2\pi i q(x)}$.

An abelian anyon model is a unitary modular tensor category that realizes the metric group (A, θ), so the two terms will be used interchangeably.

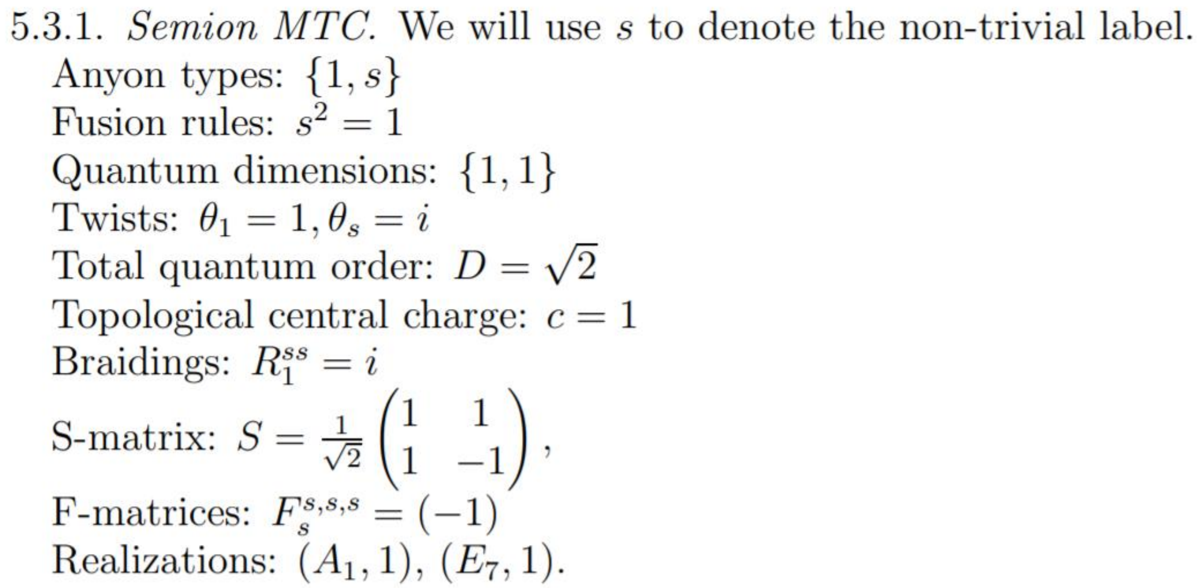


Given a lattice L, its discriminant group is $A_L = L^*/L$. The generating matrix of L will be denoted by G_L and the Gram matrix by K_L.

The discriminant form, the quadratic form on the finite discriminant group A for a non-degenerate even lattice, is $q_{2,A} : A \to \mathbb{Q}/2\mathbb{Z}$, i.e. $q_{2,A}(-x) = q_{2,A}(x)$ and the associated bilinear map $b(x,y) = \frac{1}{2}(q_{2,A}(x+y) - q_{2,A}(x) - q_{2,A}(y)) : A \times A \to \mathbb{Q}/\mathbb{Z}$ is symmetric and non-singular, where $b(,)$ is the extension of the bilinear $<,>$ on lattice Λ to its dual.

Examples of Abelian Anyon models:

Metric finite abelian groups



5.3.1. *Semion MTC.* We will use s to denote the non-trivial label.

Anyon types: $\{1, s\}$

Fusion rules: $s^2 = 1$

Quantum dimensions: $\{1, 1\}$

Twists: $\theta_1 = 1, \theta_s = i$

Total quantum order: $D = \sqrt{2}$

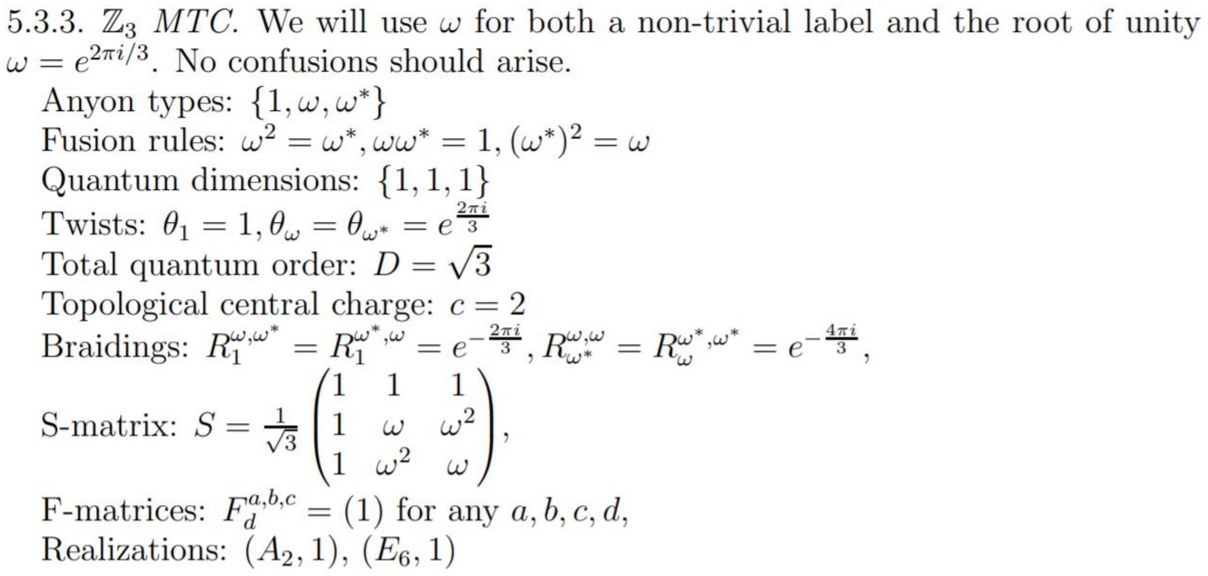
Topological central charge: $c = 1$

Braidings: $R_1^{ss} = i$

S-matrix: $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

F-matrices: $F_s^{s,s,s} = (-1)$

Realizations: $(A_1, 1)$, $(E_7, 1)$.



5.3.3. $\mathbb{Z}_3$ *MTC.* We will use ω for both a non-trivial label and the root of unity $\omega = e^{2\pi i/3}$. No confusions should arise.

Anyon types: $\{1, \omega, \omega^*\}$

Fusion rules: $\omega^2 = \omega^*, \omega\omega^* = 1, (\omega^*)^2 = \omega$

Quantum dimensions: $\{1, 1, 1\}$

Twists: $\theta_1 = 1, \theta_\omega = \theta_{\omega^*} = e^{\frac{2\pi i}{3}}$

Total quantum order: $D = \sqrt{3}$

Topological central charge: $c = 2$

Braidings: $R_1^{\omega,\omega^*} = R_1^{\omega^*,\omega} = e^{-\frac{2\pi i}{3}}, R_{\omega^*}^{\omega,\omega} = R_\omega^{\omega^*,\omega^*} = e^{-\frac{4\pi i}{3}}$,

S-matrix: $S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$,

F-matrices: $F_d^{a,b,c} = (1)$ for any a, b, c, d,

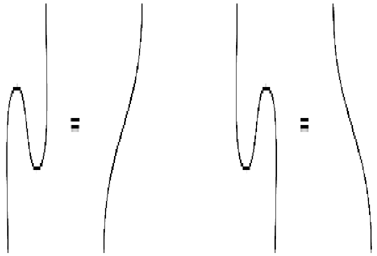
Realizations: $(A_2, 1)$, $(E_6, 1)$

A UMC is a unitary fusion category with a non-degenerate braiding

A fusion category is a categorification of
a based ring $\mathbb{Z}[\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{r-1}]$

finite rigid $\mathbb{C}$-linear semisimple monoidal category with simple unit

monoidal: $(\otimes, \mathbf{1})$,
semisimple: $X \cong \oplus_i m_i X_i$,
linear: $\mathrm{Hom}(X, Y) \in \mathrm{Vec}_{\mathbb{C}}$,
rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$



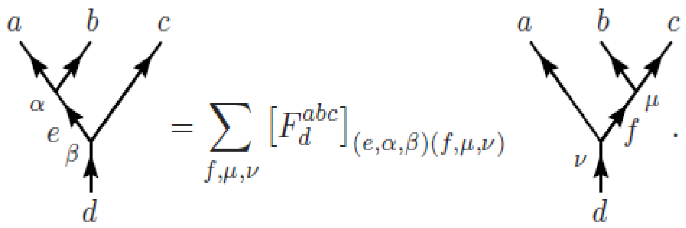
finite rank: $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = X_0, \ldots, X_{r-1}\}$

$\boldsymbol{X}$ simple if $\mathbf{Hom}(\boldsymbol{X}, \boldsymbol{X}) = \mathbb{C}$

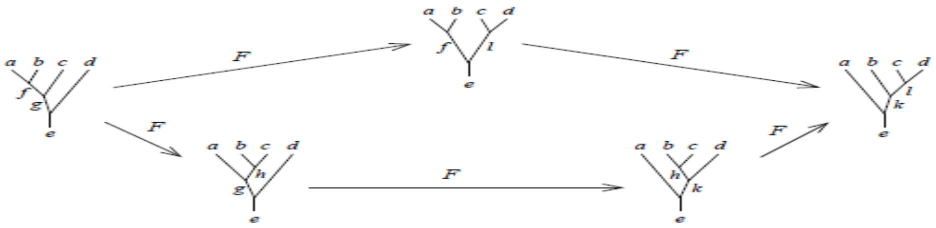
Rank of $\boldsymbol{\mathcal{C}}$: $\mathbf{r}(\boldsymbol{\mathcal{C}}) = \boldsymbol{r} = \mathbf{dim}\, \boldsymbol{V}(\boldsymbol{T}^2)$

UMTC = Anyon Model $\mathcal{C}$

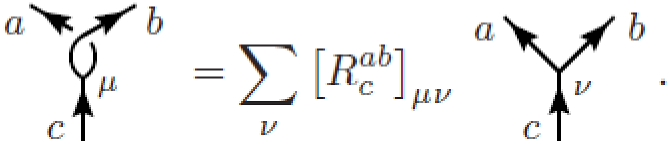
An anyon model: a collection of numbers $\{L, N^{c}_{ab}, F^{abc}_{d;nm}, R^{ab}_{c}, \epsilon_i\}$ that satisfy some polynomial constraint equations including pentagons and hexagons.



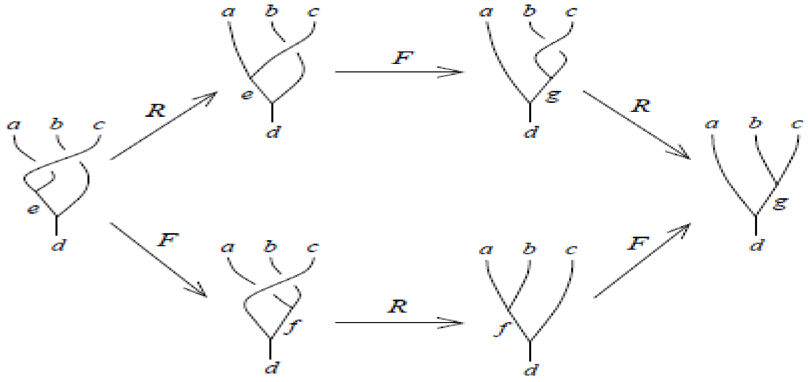
6j symbols for recoupling



Pentagons for 6j symbols



R-symbol for braiding



Hexagons for R-symbols

Invariants of Modular Tensor Category

MTC $\mathcal{C}$ $\rightleftarrows$ RT (2+1)-TQFT (V,Z)

- Pairing $\langle Y^2, \mathcal{C}\rangle = V(Y^2;\mathcal{C}) \in \mathrm{Rep}(\mathcal{M}(Y^2))$ for a surface Y^2, $\mathcal{M}(Y^2) =$ mapping class group
- Pairing $Z_{X,L,\mathcal{C}} = \langle (X^3, L_C), \mathcal{C}\rangle \in \mathbb{C}$ for colored framed oriented links L_c in 3-mfd X^3

fix $\mathcal{C}$, $Z_{X,L,\mathcal{C}}$ invariant of (X^3, L_c)

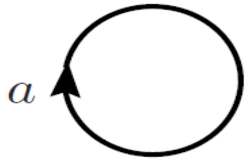
fix (X^3, L_c), $Z_{X,L,\mathcal{C}}$ *invariant of* $\mathcal{C}$

fix Y^2, $V(Y^2;\mathcal{C})$ *invariant of* $\mathcal{C}$

Quantum Dimensions and Twists: Unknot

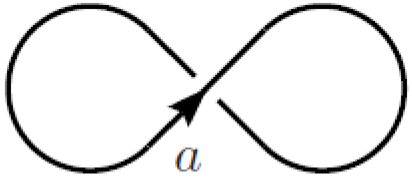
- Label set $L =$ isomorphism classes of simple objects
- Quantum dimension of a simple/label $a \in L$:

$$d_a = d_{\bar{a}} =$$



- Topological twist/spin of a: finite order by Vafa’s thm

$$\theta_a = \theta_{\bar{a}} = \frac{1}{d_a}$$

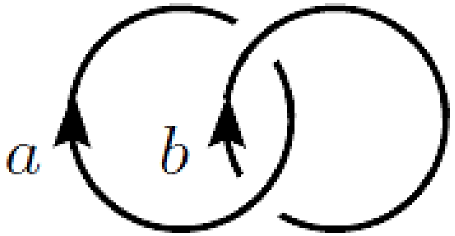


- Dimension D^2 of a modular category (D>0):

$$\dim(\mathcal{C}) = \boldsymbol{D^2} = \boldsymbol{\sum_{a \in L} d_a^2}$$

Modular S-Matrix: Hopf Link

- Modular *S*-matrix: $\tilde{s}_{ab} =$



- Modular *T*-matrix: $\tilde{t}_{ab} = \delta_{ab}\theta_a$-diagonal

- (S, T)-form a projective rep. of $SL(2, \mathbb{Z})$:

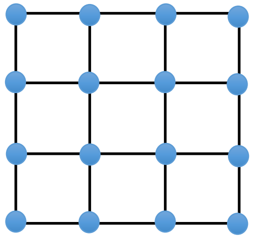
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \tilde{s}$$

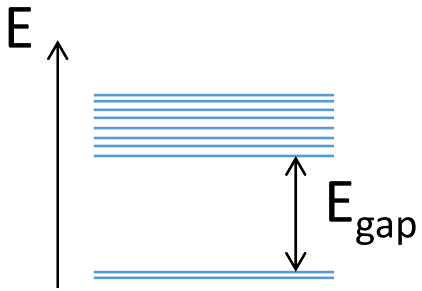
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \tilde{t}$$

Anyons: Topological Phases

Local Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$

Local, Gapped Hamiltonian $H : \mathcal{H} \to \mathcal{H}$



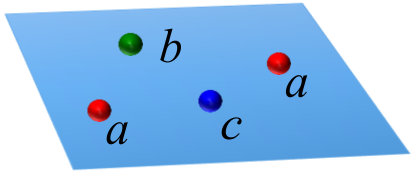


Two gapped Hamiltonians H_1, H_2 realize the same topological phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase is a class of gapped Hamiltonians that realize the same TQFT or anyon model in low energy.

Anyons: Quarsi-particles

Finite-energy topological quasiparticle excitations=anyons

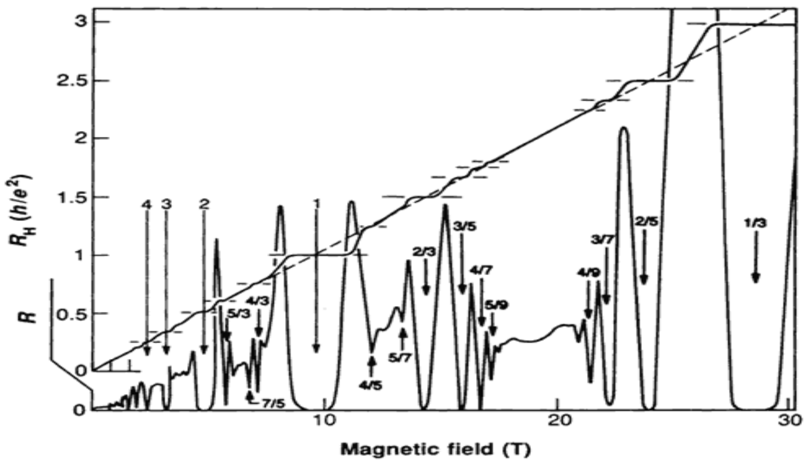


Quasiparticles a, b, c

Two quasiparticles have the same topological charge or anyon type if they differ by local operators

Anyons in 2+1 dimensions described mathematically by a Unitary Modular Tensor Category $\mathcal{C}$

Fractional Quantum Hall Effect: Fractions Have Been Observed ~80
80 Anyon models?



$\nu = \frac{N_e}{N_\phi}$ filling factor or fraction

N_e = # of electrons

N_ϕ =# of flux quanta

How to model the quantum state(s) at a filling fraction?

What are the electrons doing at a plateau?

1/3 2/3 4/3 5/3 7/3 8/3
1/5 2/5 3/5 4/5 6/5 7/5 8/5 11/5 12/5
1/7 2/7 3/7 4/7 5/7 9/7 10/7 12/7 16/7 19/7
1/9 2/9 4/9 5/9 7/9 11/9 13/9 25/9
2/11 3/11 4/11 5/11 6/11 7/11 8/11 16/11 17/11
2/13 3/13 4/13 5/13 6/13 7/13 10/13 20/13
2/15 4/15 7/15 8/15 11/15 22/15 23/15
2/17 3/17 4/17 5/17 6/17 8/17 9/17
3/19 4/19 5/19 9/19 10/19
5/21 10/21
6/23
6/25

m/5, m=14,16, 19

5/2
7/2
19/8

Pan et al (2008)

2D Topological Phases

A genus is a pair $(\mathcal{C}, c)$ of a UMTC $\mathcal{C}$ (=anyon model) and a (rational) number c≥0 (not mod 8) such that

$$\frac{p_+}{D} = e^{\frac{\pi i c}{4}}, p_+ = \sum_i \theta_i\, d_i^2, D^{\mathbf{2}} = \sum_{a\in L} d_a^2, \quad D > 0.$$

Any such c mod 8 is called the chiral central charge of $\mathcal{C}$.

“Conjecture”:

2D topological phases are 1-1 correspondence with genera $(\mathcal{C}$, c).

Bulk-edge Connection of 2D Topological Phases

- Edge physics of fractional quantum Hall liquids:

∂Witten-Chern-Simons theories$\sim$Wen's chiral Luttinger liquids

∂TQFTs/UMTCs $\sim$ χCFTs/Chiral algebras

Chiral algebras $\longrightarrow$ UMTCs=Rep(chiral algebras)

Injective? No. e.g. all holomorphic ones goes to trivial.

Onto?

Conjecture: Yes

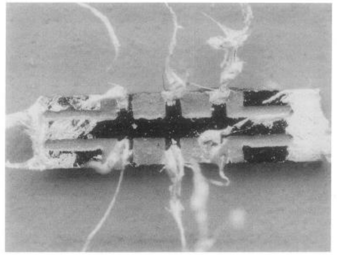


Figure 8. Photograph of a GaAs/AlGaAs sample. The size is about 6 x 1.5 *mm*. Black area (in reality mirror-like but reflecting the black camera) is the original surface above the 2DES. Gray areas have been scratched away to confine the current path to the center of the sample. White areas are indium blotches used to make contact to the 2DES. Gold wires are attached. Specimens like this one, prepared with little attention to exact dimension nor to tidiness, show quantization of the Hall resistance to an accuracy of a few 10 parts in a billion. The specimen shown is the sample in which the fractional quantum Hall effect (FQHE) was discovered in 1981.

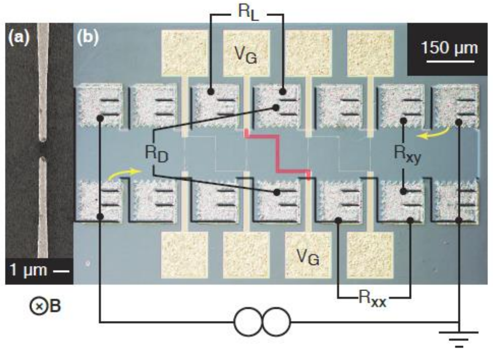


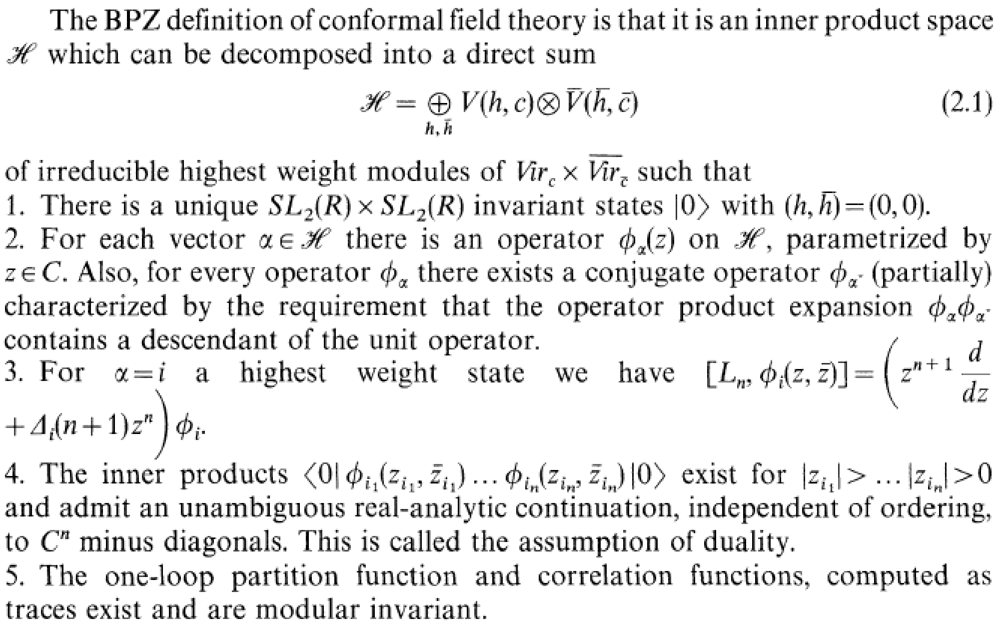
Figure 7.1: (a) SEM micrograph of the 0.5 μm QPC. (b) Optical micrograph of the entire device (the outline of the wet-etched Hall bar has been enhanced for clarity). The measurement circuit for the red-highlighted QPC is drawn schematically, with the direction of the edge-current flow indicated by the yellow arrows.

- Tannaka-Krein duality (Gannon):

Reconstruct chiral algebras from UMTCs=Rep(Chiral algebras)

Symmetric fusion categories are 1-1 correspondence with pairs (G, μ)

Chiral and Full Conformal Field Theories

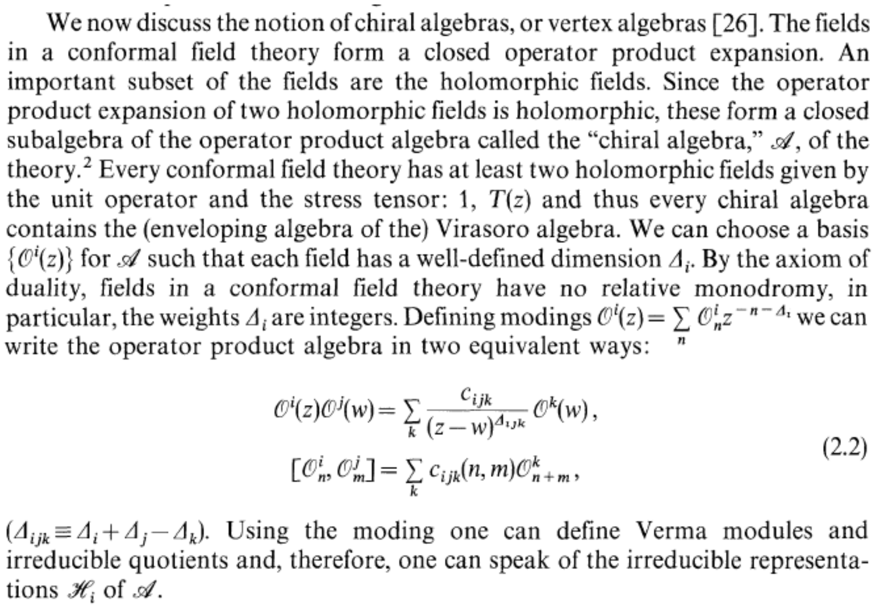


The BPZ definition of conformal field theory is that it is an inner product space $\mathscr{H}$ which can be decomposed into a direct sum

$$\mathscr{H}=\bigoplus_{h,\bar{h}} V(h,c)\otimes\bar{V}(\bar{h},\bar{c}) \tag{2.1}$$

of irreducible highest weight modules of $Vir_c\times\overline{Vir}_{\bar{c}}$ such that

1. There is a unique $SL_2(R)\times SL_2(R)$ invariant states $|0\rangle$ with $(h,\bar{h})=(0,0)$.
2. For each vector $\alpha\in\mathscr{H}$ there is an operator $\phi_\alpha(z)$ on $\mathscr{H}$, parametrized by $z\in C$. Also, for every operator ϕ_α there exists a conjugate operator $\phi_{\alpha^\cdot}$ (partially) characterized by the requirement that the operator product expansion $\phi_\alpha\phi_{\alpha^\cdot}$ contains a descendant of the unit operator.
3. For $\alpha=i$ a highest weight state we have $[L_n,\phi_i(z,\bar{z})]=\left(z^{n+1}\frac{d}{dz}+\Delta_i(n+1)z^n\right)\phi_i$.
4. The inner products $\langle 0|\phi_{i_1}(z_{i_1},\bar{z}_{i_1})\dots\phi_{i_n}(z_{i_n},\bar{z}_{i_n})|0\rangle$ exist for $|z_{i_1}|>\dots|z_{i_n}|>0$ and admit an unambiguous real-analytic continuation, independent of ordering, to C^n minus diagonals. This is called the assumption of duality.
5. The one-loop partition function and correlation functions, computed as traces exist and are modular invariant.



We now discuss the notion of chiral algebras, or vertex algebras [26]. The fields in a conformal field theory form a closed operator product expansion. An important subset of the fields are the holomorphic fields. Since the operator product expansion of two holomorphic fields is holomorphic, these form a closed subalgebra of the operator product algebra called the "chiral algebra," $\mathscr{A}$, of the theory.[2] Every conformal field theory has at least two holomorphic fields given by the unit operator and the stress tensor: 1, $T(z)$ and thus every chiral algebra contains the (enveloping algebra of the) Virasoro algebra. We can choose a basis $\{\mathscr{O}^i(z)\}$ for $\mathscr{A}$ such that each field has a well-defined dimension Δ_i. By the axiom of duality, fields in a conformal field theory have no relative monodromy, in particular, the weights Δ_i are integers. Defining modings $\mathscr{O}^i(z)=\sum_n \mathscr{O}^i_n z^{-n-\Delta_i}$ we can write the operator product algebra in two equivalent ways:

$$\begin{aligned}\mathscr{O}^i(z)\mathscr{O}^j(w)&=\sum_k \frac{c_{ijk}}{(z-w)^{\Delta_{ijk}}}\mathscr{O}^k(w),\\ [\mathscr{O}^i_n,\mathscr{O}^j_m]&=\sum_k c_{ijk}(n,m)\mathscr{O}^k_{n+m},\end{aligned} \tag{2.2}$$

($\Delta_{ijk}\equiv\Delta_i+\Delta_j-\Delta_k$). Using the moding one can define Verma modules and irreducible quotients and, therefore, one can speak of the irreducible representations $\mathscr{H}_i$ of $\mathscr{A}$.

Classical and quantum conformal field theory

G Moore, N Seiberg - Communications in Mathematical Physics, 1989

Chiral CFT (χCFT) = the chiral algebra mathematically=vertex operator algebra (VOA).
A full CFT is determined by a VOA V plus an indecomposable module category over Rep(V).

Vertex Operator Algebra (VOA)

A VOA is a quadruple (V, Y, 1, ω), where $V = \bigoplus_n V_n$ is Z-graded vector space and

$$Y : V \to \mathfrak{F}(V),\ \ v \mapsto Y(v,z) = \sum v_n z^{-n-1}$$

$$\mathbf{1}, \omega \in V,\ \ \mathbf{1} \neq 0.$$

The fields $Y(v,z)$ are mutually local and creative, and the following hold:

$Y(\omega, z) = \sum L_n z^{-n-2}$ *with a constant* c *such that*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} c\mathrm{Id}_V$$

$$V_n = \{v \in V_n \mid L_0 v = nv\}$$

$$\dim V_n < \infty,\ \ V_n = 0 \ \textit{for}\ n \ll 0$$

$$Y(L_{-1}u, z) = \partial Y(u,z)$$

$$\textit{locality} : Y(u,z) \sim Y(v,z)$$

$$\textit{creativity} : Y(u,z)\mathbf{1} = u + O(z)$$

$\mathfrak{F}(V) = \{a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field}\}.$

a(z) is a field if it satisfied the truncation condition,

I.e. for $v \in V$ *there is an integer* N *(depending on* v*) such that* $a_n(v) = 0$ *for all* $n > N$.

Genus of Vertex Operator Algebra

Genus of lattice:

genus of a lattice Λ is equivalent to (q, G, c),

G=discriminant Λ^*/Λ, $q: G \to U(1), c$ =signature.

Genus of VOA:

genus of VOA=the pair $(\mathcal{C}, c)$. Recall $\frac{p_+}{D} = e^{\frac{\pi i c}{4}}$

$\mathcal{C}$=MTC of Rep(Good VOA) (**Huang**), c=central charge.

Conjectures

Given a UMTC $\mathcal{C}$

1. **Existence** (W., Gannon):

There is a genus ($\mathcal{C}$, c) that can be realized by a VOA (χCFT).

e.g. (Toric code, 8) is realized by $SO(16)_1$

$$D_n = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 2 & -1 & 0 & 0 \\ 0 & \ddots & -1 & 2 & -1 & -1 \\ \vdots & \ddots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

2. **Genus finiteness** conjecture (Hoehn):

There are only finitely many different realizations of any genus

Known Realization

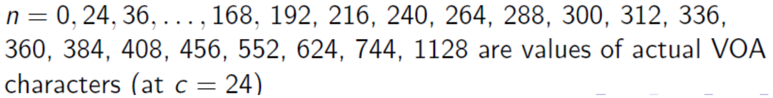
- Quantum groups MTCs: WZW models
- Double of finite groups: Evans, Gannon
- Abelian anyon models
- Low rank ones
- …
- **Exotic ones???** Drinfeld centers from subfactors such as DHaag.

Holomorphic VOAs: Trivial UMTC Vec

- (Vec, 0) trivial
- (Vec, 8) E_8
- (Vec, 16), $E_8 \bigoplus E_8, D_{16}$
- Monster Moonshine VOA has genus (Vec, 24)

  John McKay's remark: 196 884=196883+1

  $J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$
- There are at least 71 VOAs in the genus (Vec, 24)

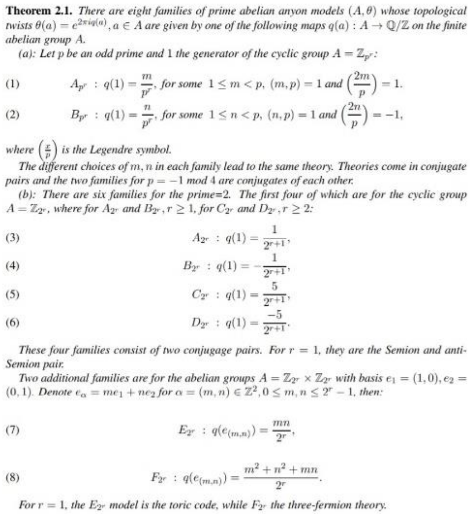


$n = 0, 24, 36, \ldots, 168$, 192, 216, 240, 264, 288, 300, 312, 336, 360, 384, 408, 456, 552, 624, 744, 1128 are values of actual VOA characters (at $c = 24$)

- Classify VOAs modulo lattice ones

Classification of Prime Abelian Anyon Models

- Enumeration
- Central charge
- Symmetry



Theorem 2.1. *There are eight families of prime abelian anyon models (A, θ) whose topological twists $\theta(a) = e^{2\pi i q(a)}, a \in A$ are given by one of the following maps $q(a) : A \to \mathbb{Q}/\mathbb{Z}$ on the finite abelian group A.*

(a): Let p be an odd prime and 1 the generator of the cyclic group $A = \mathbb{Z}_{p^r}$:

$$A_{p^r} : q(1) = \frac{m}{p^r}, \text{ for some } 1 \leq m < p,\ (m,p) = 1 \text{ and } \left(\frac{2m}{p}\right) = 1. \tag{1}$$

$$B_{p^r} : q(1) = \frac{n}{p^r}, \text{ for some } 1 \leq n < p,\ (n,p) = 1 \text{ and } \left(\frac{2n}{p}\right) = -1, \tag{2}$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol.

The different choices of m, n in each family lead to the same theory. Theories come in conjugate pairs and the two families for $p = -1 \bmod 4$ are conjugates of each other.

(b): There are six families for the prime=2. The first four of which are for the cyclic group $A = \mathbb{Z}_{2^r}$, where for A_{2^r} and B_{2^r}, $r \geq 1$, for C_{2^r} and D_{2^r}, $r \geq 2$:

$$A_{2^r} : q(1) = \frac{1}{2^{r+1}}, \tag{3}$$

$$B_{2^r} : q(1) = -\frac{1}{2^{r+1}}, \tag{4}$$

$$C_{2^r} : q(1) = \frac{5}{2^{r+1}}, \tag{5}$$

$$D_{2^r} : q(1) = \frac{-5}{2^{r+1}}. \tag{6}$$

These four families consist of two conjugage pairs. For $r = 1$, they are the Semion and anti-Semion pair.

Two additional families are for the abelian groups $A = \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$ with basis $e_1 = (1,0), e_2 = (0,1)$. Denote $e_\alpha = me_1 + ne_2$ for $\alpha = (m,n) \in \mathbb{Z}^2, 0 \leq m, n \leq 2^r - 1$, then:

$$E_{2^r} : q(e_{(m,n)}) = \frac{mn}{2^r}, \tag{7}$$

$$F_{2^r} : q(e_{(m,n)}) = \frac{m^2 + n^2 + mn}{2^r}. \tag{8}$$

For $r = 1$, the E_{2^r} model is the toric code, while F_{2^r} the three-fermion theory.

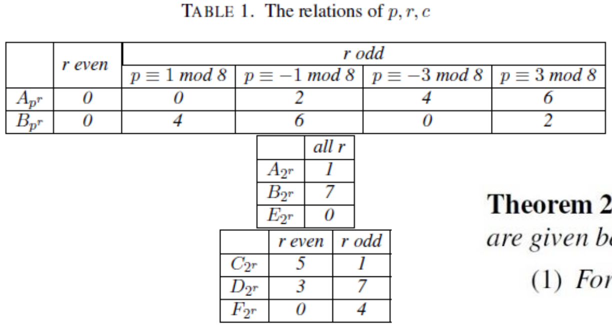
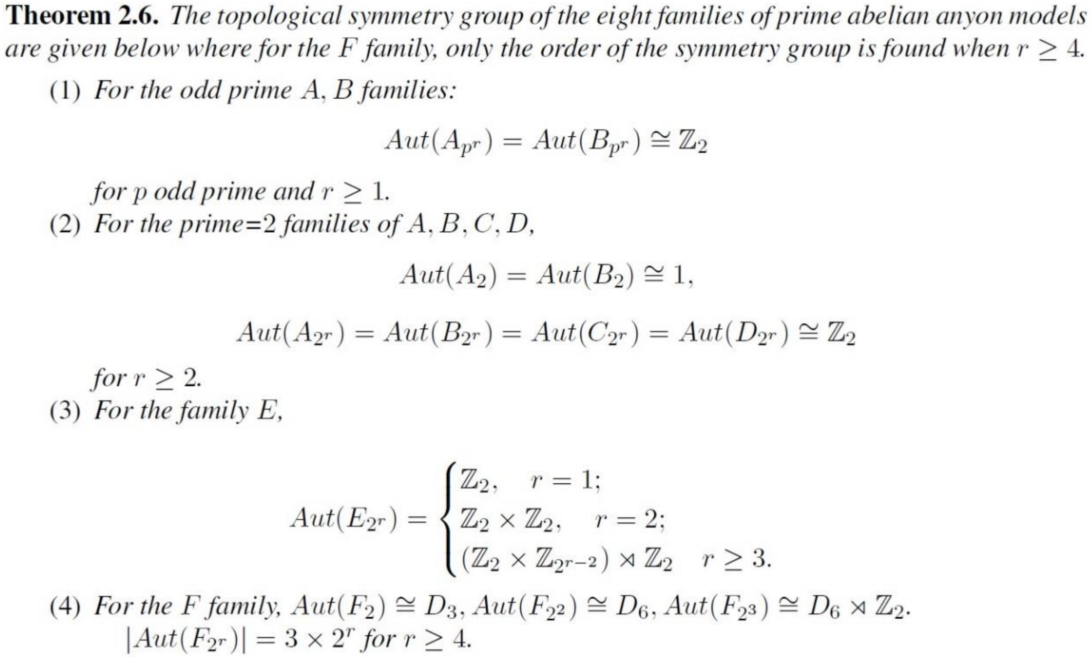


TABLE 1. The relations of p, r, c

	r even	*r odd*			
		$p \equiv 1$ *mod 8*	$p \equiv -1$ *mod 8*	$p \equiv -3$ *mod 8*	$p \equiv 3$ *mod 8*
A_{p^r}	0	0	2	4	6
B_{p^r}	0	4	6	0	2

	all r
A_{2^r}	1
B_{2^r}	7
E_{2^r}	0

	r even	*r odd*
C_{2^r}	5	1
D_{2^r}	3	7
F_{2^r}	0	4



Theorem 2.6. *The topological symmetry group of the eight families of prime abelian anyon models are given below where for the F family, only the order of the symmetry group is found when $r \geq 4$.*

(1) *For the odd prime A, B families:*

$$Aut(A_{p^r}) = Aut(B_{p^r}) \cong \mathbb{Z}_2$$

for p odd prime and $r \geq 1$.

(2) *For the prime=2 families of A, B, C, D,*

$$Aut(A_2) = Aut(B_2) \cong 1,$$

$$Aut(A_{2^r}) = Aut(B_{2^r}) = Aut(C_{2^r}) = Aut(D_{2^r}) \cong \mathbb{Z}_2$$

for $r \geq 2$.

(3) *For the family E,*

$$Aut(E_{2^r}) = \begin{cases} \mathbb{Z}_2, & r = 1; \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & r = 2; \\ (\mathbb{Z}_2 \times \mathbb{Z}_{2^{r-2}}) \rtimes \mathbb{Z}_2 & r \geq 3. \end{cases}$$

(4) *For the F family, $Aut(F_2) \cong D_3$, $Aut(F_{2^2}) \cong D_6$, $Aut(F_{2^3}) \cong D_6 \rtimes \mathbb{Z}_2$. $|Aut(F_{2^r})| = 3 \times 2^r$ for $r \geq 4$.*

Lattice Realization

- **Lattice VOAs:**

+-definite lattices lead to VOAs by Borcherds

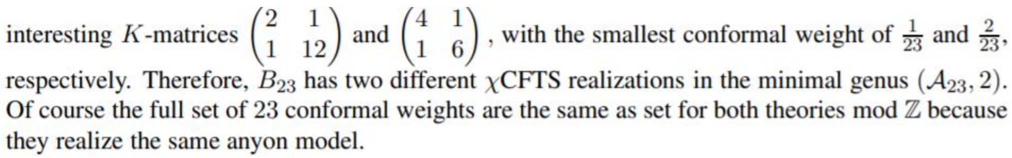
- **+Definite lattices:**

1) Nikulin existence proof : "Integral symmetric bilinear forms and some of their applications." *Mathematics of the USSR-Izvestiya* 14.1 (1980): 103.

2) Evans, Gannon construction: "Tambara-Yamagami, loop groups, bundles and KK-theory." *arXiv preprint arXiv:2003.09672* (2020).

3) Algorithmic approach: "In and around abelian Anyon Models." *arXiv preprint arXiv:2004.12048* (2020).

- Minimal central charge not unique in $(\overline{\boldsymbol{SU(23)}}_{\mathbf{1}}$, 2):



interesting K-matrices $\begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}$ and $\begin{pmatrix} 4 & 1 \\ 1 & 6 \end{pmatrix}$, with the smallest conformal weight of $\frac{1}{23}$ and $\frac{2}{23}$, respectively. Therefore, B_{23} has two different χCFTS realizations in the minimal genus $(\mathcal{A}_{23}, 2)$. Of course the full set of 23 conformal weights are the same as set for both theories mod $\mathbb{Z}$ because they realize the same anyon model.

- ...

Extremal CFTs:

Given Central Charge with Maximal Total Conformal Weights

Given a realizable genus ($\mathcal{C}$, c) with c>0, then χCFTs in the genus ($\mathcal{C}$,c) with ($\mathcal{C}$,c,h_i) s. t.

$$\frac{1}{2}\boldsymbol{r}(\boldsymbol{r}-\mathbf{1})+\frac{\mathbf{1}}{\mathbf{24}}\boldsymbol{rc}-\sum_{\boldsymbol{i}}\boldsymbol{h}_{\boldsymbol{i}} \geq \mathbf{0}$$

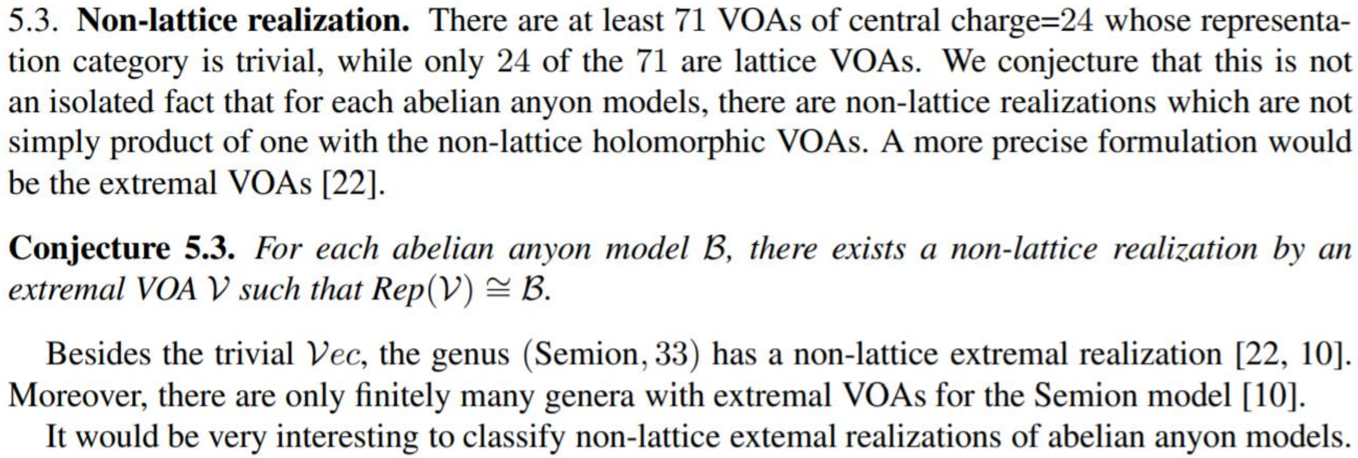
$$\operatorname{Tr}(\Lambda)=\frac{5r}{12}+\frac{1}{4}\operatorname{Tr}(s)+\frac{2}{3\sqrt{3}}\operatorname{Re}(e^{-\frac{\pi i}{6}}\operatorname{Tr}(U))$$

where is r the rank of $\mathcal{C}$, s, t=$e^{2\pi i\Lambda}$ the modular matrices, $U = st^{-1}$, and h_i≥0 conformal weights.

For all $c+8l, l \geq 0$, the extremal solutions, if realizable, are not products with holomorphic ones.

Non-lattice Extremal Realization: Generalization of Moonshine?

- **Any prime anyon model can be realized as rep category of extremal non-lattice VOAs such as the moonshine for the trivial theory.**
- Interesting bulk-edge physics: stability
- Lattice with exotic symmetries:



5.3. **Non-lattice realization.** There are at least 71 VOAs of central charge=24 whose representation category is trivial, while only 24 of the 71 are lattice VOAs. We conjecture that this is not an isolated fact that for each abelian anyon models, there are non-lattice realizations which are not simply product of one with the non-lattice holomorphic VOAs. A more precise formulation would be the extremal VOAs [22].

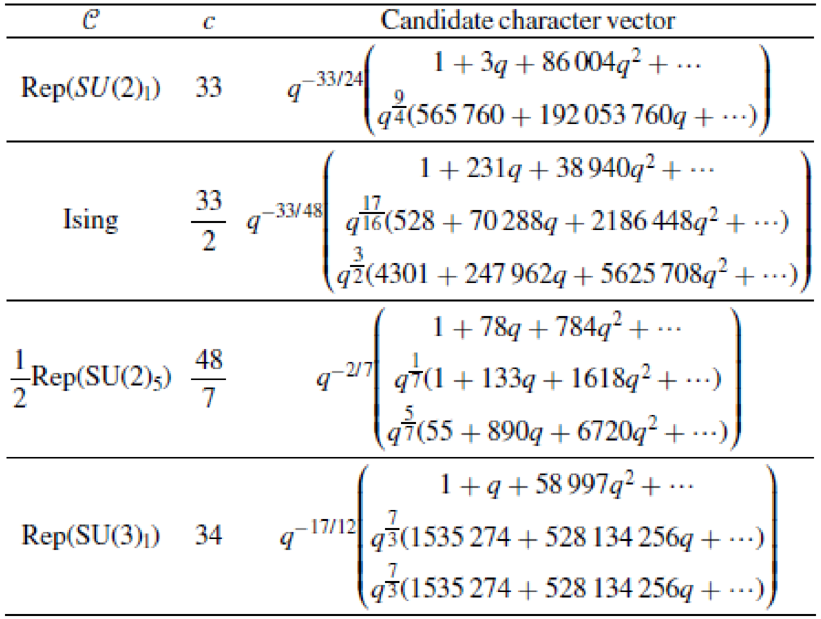
Conjecture 5.3. *For each abelian anyon model $\mathcal{B}$, there exists a non-lattice realization by an extremal VOA $\mathcal{V}$ such that $Rep(\mathcal{V}) \cong \mathcal{B}$.*

Besides the trivial $\mathcal{V}ec$, the genus $(\text{Semion}, 33)$ has a non-lattice extremal realization [22, 10]. Moreover, there are only finitely many genera with extremal VOAs for the Semion model [10].

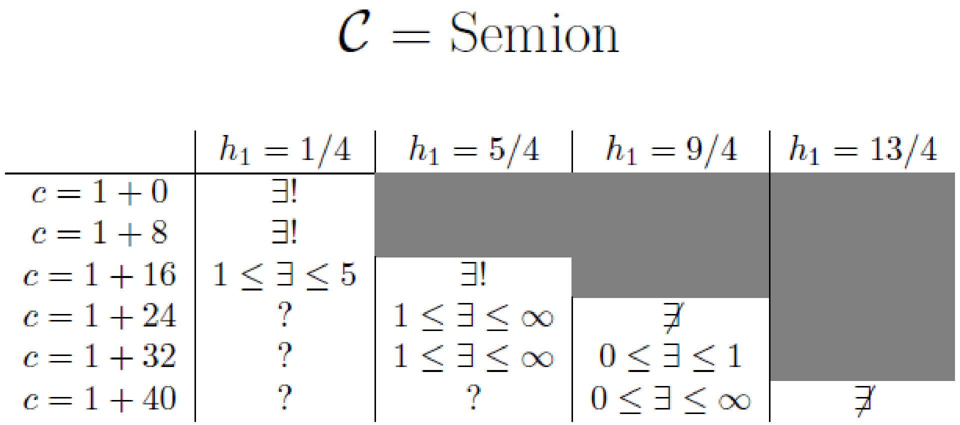
It would be very interesting to classify non-lattice extemal realizations of abelian anyon models.

Semion $SU(2)_1$: Candidate Characters

J. Phys. A: Math. Theor. **50** (2017) 115204 J E Tener and Z Wang

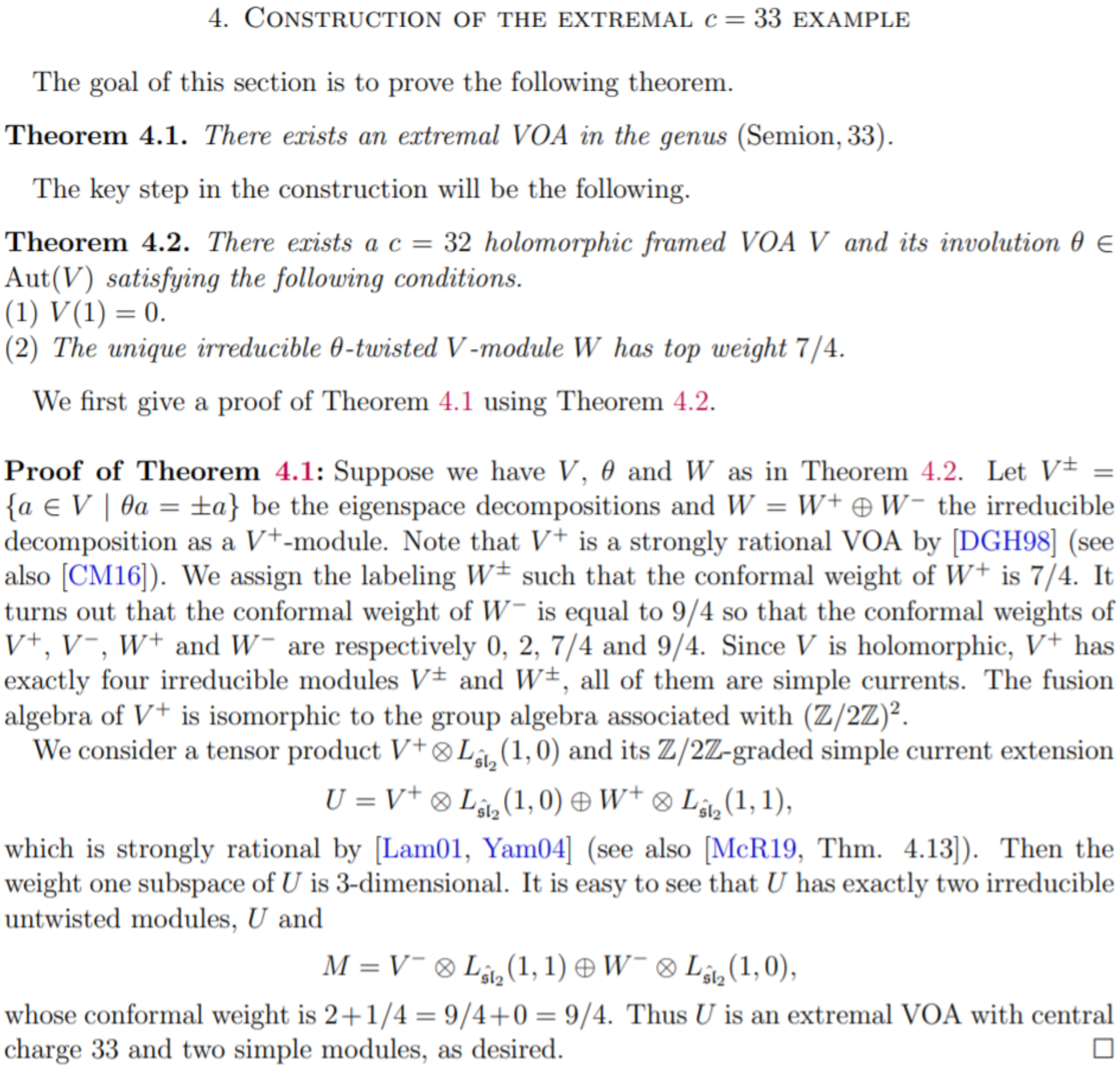


$\mathcal{C}$	c	Candidate character vector
Rep$(SU(2)_1)$	33	$q^{-33/24}\begin{pmatrix} 1+3q+86\,004q^2+\cdots \\ q^{\frac{9}{4}}(565\,760+192\,053\,760q+\cdots) \end{pmatrix}$
Ising	$\frac{33}{2}$	$q^{-33/48}\begin{pmatrix} 1+231q+38\,940q^2+\cdots \\ q^{\frac{17}{16}}(528+70\,288q+2186\,448q^2+\cdots) \\ q^{\frac{3}{2}}(4301+247\,962q+5625\,708q^2+\cdots) \end{pmatrix}$
$\frac{1}{2}$Rep(SU(2)$_5$)	$\frac{48}{7}$	$q^{-2/7}\begin{pmatrix} 1+78q+784q^2+\cdots \\ q^{\frac{1}{7}}(1+133q+1618q^2+\cdots) \\ q^{\frac{5}{7}}(55+890q+6720q^2+\cdots) \end{pmatrix}$
Rep(SU(3)$_1$)	34	$q^{-17/12}\begin{pmatrix} 1+q+58\,997q^2+\cdots \\ q^{\frac{7}{3}}(1535\,274+528\,134\,256q+\cdots) \\ q^{\frac{7}{3}}(1535\,274+528\,134\,256q+\cdots) \end{pmatrix}$



$\mathcal{C}$ = Semion

	$h_1 = 1/4$	$h_1 = 5/4$	$h_1 = 9/4$	$h_1 = 13/4$
$c = 1 + 0$	$\exists!$	▨	▨	▨
$c = 1 + 8$	$\exists!$	▨	▨	▨
$c = 1 + 16$	$1 \leq \exists \leq 5$	$\exists!$	▨	▨
$c = 1 + 24$	?	$1 \leq \exists \leq \infty$	$\nexists$	▨
$c = 1 + 32$	?	$1 \leq \exists \leq \infty$	$0 \leq \exists \leq 1$	▨
$c = 1 + 40$	?	?	$0 \leq \exists \leq \infty$	$\nexists$



4. CONSTRUCTION OF THE EXTREMAL $c = 33$ EXAMPLE

The goal of this section is to prove the following theorem.

Theorem 4.1. *There exists an extremal VOA in the genus* (Semion, 33).

The key step in the construction will be the following.

Theorem 4.2. *There exists a* $c = 32$ *holomorphic framed VOA* V *and its involution* $\theta \in \mathrm{Aut}(V)$ *satisfying the following conditions.*
(1) $V(1) = 0$.
(2) *The unique irreducible* θ*-twisted* V*-module* W *has top weight* $7/4$.

We first give a proof of Theorem 4.1 using Theorem 4.2.

Proof of Theorem 4.1: Suppose we have V, θ and W as in Theorem 4.2. Let $V^{\pm} = \{a \in V \mid \theta a = \pm a\}$ be the eigenspace decompositions and $W = W^{+} \oplus W^{-}$ the irreducible decomposition as a V^{+}-module. Note that V^{+} is a strongly rational VOA by [DGH98] (see also [CM16]). We assign the labeling $W^{\pm}$ such that the conformal weight of W^{+} is $7/4$. It turns out that the conformal weight of W^{-} is equal to $9/4$ so that the conformal weights of V^{+}, V^{-}, W^{+} and W^{-} are respectively 0, 2, $7/4$ and $9/4$. Since V is holomorphic, V^{+} has exactly four irreducible modules $V^{\pm}$ and $W^{\pm}$, all of them are simple currents. The fusion algebra of V^{+} is isomorphic to the group algebra associated with $(\mathbb{Z}/2\mathbb{Z})^2$.

We consider a tensor product $V^{+} \otimes L_{\hat{\mathfrak{sl}}_2}(1,0)$ and its $\mathbb{Z}/2\mathbb{Z}$-graded simple current extension

$$U = V^{+} \otimes L_{\hat{\mathfrak{sl}}_2}(1,0) \oplus W^{+} \otimes L_{\hat{\mathfrak{sl}}_2}(1,1),$$

which is strongly rational by [Lam01, Yam04] (see also [McR19, Thm. 4.13]). Then the weight one subspace of U is 3-dimensional. It is easy to see that U has exactly two irreducible untwisted modules, U and

$$M = V^{-} \otimes L_{\hat{\mathfrak{sl}}_2}(1,1) \oplus W^{-} \otimes L_{\hat{\mathfrak{sl}}_2}(1,0),$$

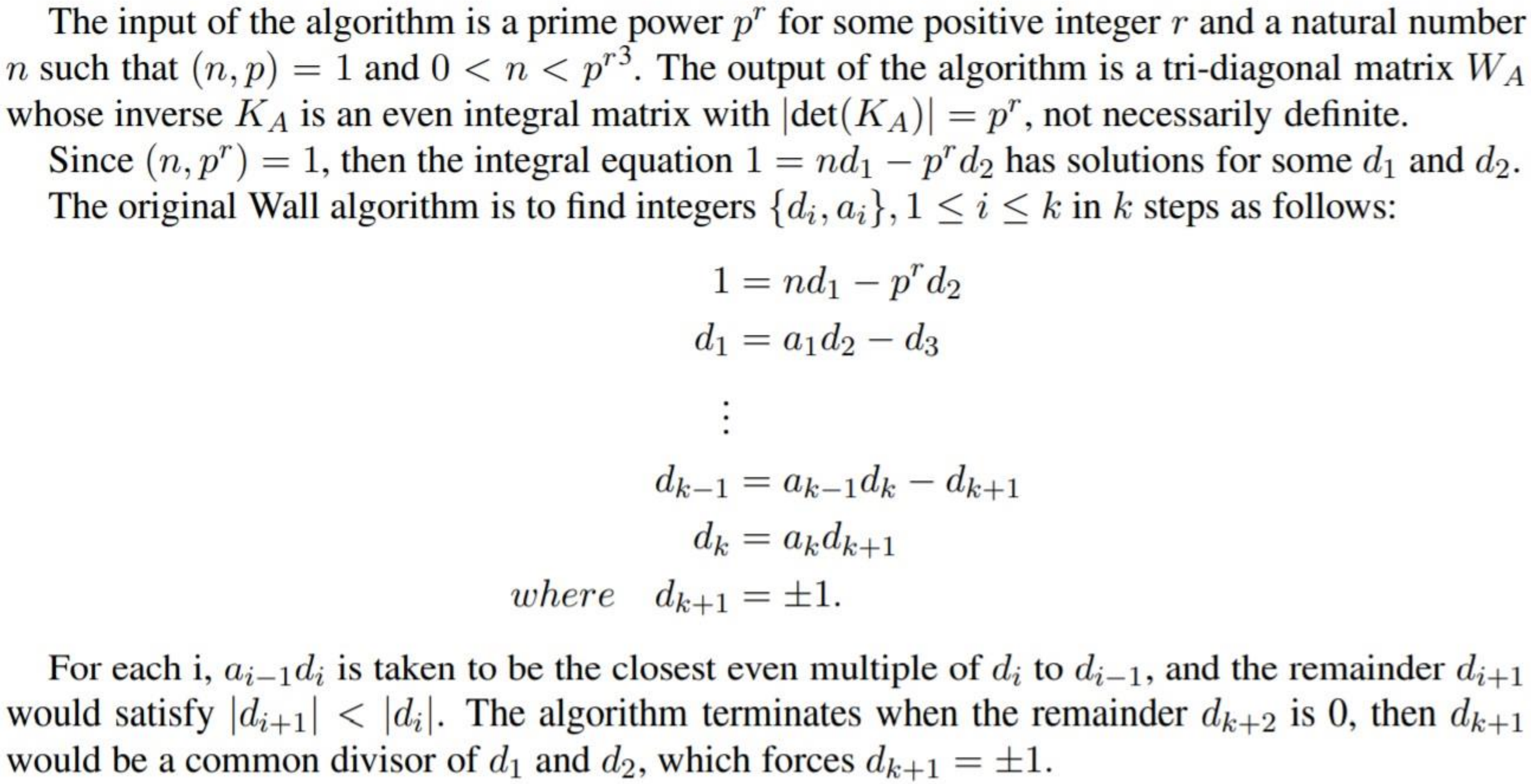
whose conformal weight is $2 + 1/4 = 9/4 + 0 = 9/4$. Thus U is an extremal VOA with central charge 33 and two simple modules, as desired. □

CHING HUNG LAM, HIROSHI YAMAUCHI in Grady, J. Connor, Ching Hung Lam, James E. Tener, and Hiroshi Yamauchi. "Classification of extremal vertex operator algebras with two simple modules." *Journal of Mathematical Physics* 61, no. 5 (2020): 052302.

Proposed generalized Moonshine is about lattices with exotic symmetries.

Algorithmic Lattice Realization: Wall Algorithm

from metric group to lattice



The input of the algorithm is a prime power p^r for some positive integer r and a natural number n such that $(n, p) = 1$ and $0 < n < p^{r3}$. The output of the algorithm is a tri-diagonal matrix W_A whose inverse K_A is an even integral matrix with $|\det(K_A)| = p^r$, not necessarily definite.

Since $(n, p^r) = 1$, then the integral equation $1 = nd_1 - p^r d_2$ has solutions for some d_1 and d_2.

The original Wall algorithm is to find integers $\{d_i, a_i\}, 1 \leq i \leq k$ in k steps as follows:

$$\begin{aligned} 1 &= nd_1 - p^r d_2 \\ d_1 &= a_1 d_2 - d_3 \\ &\vdots \\ d_{k-1} &= a_{k-1} d_k - d_{k+1} \\ d_k &= a_k d_{k+1} \\ where \quad d_{k+1} &= \pm 1. \end{aligned}$$

For each i, $a_{i-1} d_i$ is taken to be the closest even multiple of d_i to d_{i-1}, and the remainder d_{i+1} would satisfy $|d_{i+1}| < |d_i|$. The algorithm terminates when the remainder d_{k+2} is 0, then d_{k+1} would be a common divisor of d_1 and d_2, which forces $d_{k+1} = \pm 1$.

General: Vector-Valued Modular Forms (VVMF)

$\mathcal{C}$—MTC with real s-matrix and irreducible $\rho : SL(2,\mathbb{Z}) \to U(r)$.
$\tilde{s}$ un-normalized s-matrix and $s = \frac{\tilde{s}}{D}$, normalized
$\tilde{t} = (\theta_i)$ un-normalized t-matrix, and $t = e^{-\frac{2\pi i c}{24}}\tilde{t}$, normalized
A vector-valued modular form is a holomorphic function
$X : H \to \mathbb{C}^r$ with finite poles at infinity such that

$$X(\gamma\tau) = \rho(\gamma)X(\tau), \tau \in H, \gamma \in SL(2,\mathbb{Z}),$$

where $H = \{z, Imz > 0\}$.

$\mathcal{M}_\rho$=all VVMFs with ρ as multiplier---infinite dim vector space over C.

We will fix an MTC as above.

Gannon, T. The theory of vector-valued modular forms for the modular group. *Conformal Field Theory, Automorphic Forms and Related Topics*. Springer Berlin Heidelberg, 2014. 247-286.

Zhu’s Theorem

Given a good VOA and an irreducible module M,
the character χ_M of M is

$$\mathrm{Tr}_M\, q^{L_0^M - c/24} = q^{h-c/24} \sum_{n \geq 0} \dim M_{h+n} q^n$$

The vector X=$(\chi_1, \dots, \chi_r)^t$ **is a vector-valued modular form.**

General Reconstruction:

1. VVMFs
2. Anyonic Chains
3. ……

- **Modular group $SL(2,\mathbb{Z})$ in two places in a TQFT:**

  Torus and 4 punctured sphere MCGs (spherical braid group)

- **Torus characters and sphere correlators as VVMFs**
- …